Diophantine Analysis and Related Fields 2009

2-3 March, 2009 at Nihon Univ.

Titles and Abstracts

Tapani Matala-aho (Oulu Univ.)

Padé approximations of generalized hypergeometric series

We shall present short proofs for type II Padé approximations of the generalized hypergeometric and q-hypergeometric series

$$F(t) = \sum_{n=0}^{\infty} \frac{\prod_{k=0}^{n-1} P(k)}{\prod_{k=0}^{n-1} Q(k)} t^n, \quad F_q(t) = \sum_{n=0}^{\infty} \frac{\prod_{k=0}^{n-1} P(q^k)}{\prod_{k=0}^{n-1} Q(q^k)} t^n.$$

In a q-exponential case we will discuss how certain modified approximations give sharp linear independence results. Further, a comparison is done between the remainder series approximations of the exponential series (Prévost and Rivoal) and our modified approximations for a q-analogue of the exponential series.

Yoichi Motohashi (Nihon Univ.)

On the spectral decomposition of sums of Kloosterman sums

This is to introduce the principal part of the sixth chapter (an enlargement) of the new edition of my CUP book 'Spectral Theory of the Riemann Zeta-Function', which is now under preparation. Firstly, a relatively elementary approach to the theory of automorphic representations of SL(2,R) will be briefly developed; secondly an account, with my own twist, of the Kirillov realization of each irreducible representation will be presented. Then an 'extremely' quick proof of the spectral decomposition of sums of Kloosterman sums, due originally to N.V. Kuznetsov, will be shown. This is basically the scheme devised by Cogdell and Piatetski-Shapiro in 1991; but our reasoning is quite explicit in contrast to theirs.

Hyun Kwang Kim (Pohang Univ. of Science and Technology, Korea)

Waring type problems for polytope numbers

In 1770, Lagrange proved a theorem which states that every non-negative integer can be written as the sum of four squares. This result is so called the sum of four squares. There are two major generalizations of Lagrange's sum of four squares. The one is a horizontal generalization due to Cauchy which is known as the polygonal number theorem, and the other is a higher dimensional generalization which is known as Waring's problem. In this talk, we generalize Lagrange's sum of four squares further. We propose a subclass of integer valued polynomials which are constructed geometrically from convex polytopes of 'good' shape, and discuss Waring problem for this subclass of integer valued polynomials.

Akinari Hoshi (Rikkyo Univ.)

On correspondence between solutions of a parametric family of cubic Thue equations and isomorphism classes of simplest cubic fields

We give a correspondence between non-trivial solutions to the parametric family of cubic Thue equations $X^3 - mX^2Y - (m+3)XY^2 - Y^3 = k$ where $k \mid m^2 + 3m + 9$ and isomorphism classes of simplest cubic fields. By applying R. Okazaki's result for non-isomorphic simplest cubic fields, we obtain all solutions to the family of cubic Thue equations for $k \mid m^2 + 3m + 9$.

Ryotaro Okazaki (Doshisha Univ.)

Toward a sharp estimate on the number of solutions to the simultaneous Pell equations of indefinite signature

Simultaneous Pell Equations of Indefinite Signature is a Diophantine system of equations in unknown integers x, y, z,

$$|a_1x^2 - a_2z^2| = 4; \quad |b_1y^2 - b_2z^2| = 4,$$

where the parameters a_1, a_2, b_1, b_2 are positive integers such that none of the products a_1a_2, b_1b_2, $a_1a_2b_1b_2$ is a square number. Alternatively, this system can be written

$$a_1x^2 - a_2z^2 = \pm 4; \quad b_1y^2 - b_2z^2 = \pm 4,$$

where the signatures in the right hand side may depend on the triple (x, y, z). This is the origin of the term "Indefinite Signature".

Under some restriction, this system was studied by Ljunggren, Baker, Davenport and other mathematicians. Indeed, this equation is a nice subject for application of the theory of Diophantine approximation, the theory of elliptic Diophantine Equation of degree 4, the theory of Simultaneous Diophantine Approximations as well as the lower bound on the linear forms of logarithms of algebraic numbers. The recent progress in the last method enabled significant progress in the study of Simultaneous Pell Equaions of the form

$$4x^2 - 4az^2 = +4; \quad 4y^2 - 4bz^2 = +4.$$

After Anglin's computation, many authors including Masser, Rickert, Bennett, Yuan, Mignotte, Cipu and the speaker studied this system. We now know this system has at most 2 solutions in positive integes. In this talk, we will discuss how to prove that Simultaneous Pell Equations of Indefinite Signature has at most 2 positive solutions under the assumption $\max\{a_1, a_2, b_1, b_2\}$ is larger than a certain constant. The assumption is neccessary since it does have 3 solutions in some cases.

Shigeki Akiyama (Niigata Univ.)

Topology of tiles generated by number systems

I shall give an introductory account on the study of the topology of tiles generated by number system. A dominant tool is to introduce an automaton which accepts finite and infinite words. This machine works very well in describing the boundary of tiles and the dynamical system associated to this tile, so called odometer.

Benoît Loridant (JSPS post doc. Niigata Univ.)

Buchi automata and application to the topology of number system tiles

Buchi automata are non deterministic finite state automata that read infinite words on a finite alphabet. They form a suitable framework for the study of number system tiles. Indeed, the language of the boundary of the tiles is described by Buchi automata. Some well-known properties of these automata will be given and their use in the topological study of tiles will be presented.

Hajime Kaneko (Kyoto Univ.)

On normal numbers and powers of algebraic numbers

A normal number in an integer base α is a positive number for which all finite words with letters from the alphabet $\{0, 1, \ldots, \alpha - 1\}$ occur with the proper frequency. Borel showed that almost all positive number ξ are normal in every integer base. However, it is generally difficult to check a given number is normal or not. For instance, we even do not know the numbers $\sqrt{2}$, e, and π are normal in base 10.

On the other hand, it is easily checked that a positive ξ is normal in base α if and only if the sequence $\xi\alpha^n$ $(n = 0, 1, \ldots)$ is uniformly distributed modulo 1. In this talk, we introduce recent results about fractional parts of geometric sequences whose common ratios are algebraic numbers.

Noriko Hirata-Kohno and Rina Takada (Nihon Univ.)

Lower bound of linear forms in 3 p-adic elliptic logarithms

We present a new lower bound of linear forms in 3 p-adic elliptic logarithms obtained by the method of M. Mignotte. We also show an estimate of the dependence relation whenever there is an isogeny among 3 elliptic curves.

Boris Adamczewski (Univ. Claude Bernard Lyon 1, France)

The many faces of $\sum_{n\geq 0} \frac{1}{2^{2^n}}$

To prove the transcendence of a real number is often a formidable task. Our knowledge in this area is somewhat limited and (most of) mathematicians would likely be delighted with a proof of the transcendence of $\zeta(3)$ or $e + \pi$.

In contrast, the real number κ mentioned in the title of my talk is known to be transcendental for a long time. It seems that the first proof is due to Kempner in 1916. Furthermore, it is possible to derive the transcendence of κ by various ways; each of them being concerned with one of the many properties enjoyed by κ.

In this talk, I will survey some of these properties and I will explain how these different proofs lead to deeper results. This will involve among other things: basic additive number theory, Mahler's theory, the Schmidt Subspace Theorem, the folding lemma, continued fractions...

Masayoshi Hata (Kyoto Univ.)

A lower estimate for $\|e^n\|$

We discuss the fractional parts of e^n using non-uniform Pade approximations to e^x. The method is based on the classical integral introduced by Hermite. The study is comparable to that of irrationality mesures, but there are some very different things between them.

Carsten Elsner (FHDW, Germany), Shun Shimomura and Iekata Shiokawa (Keio Univ.)

A Fibonacci reciprocal sum and Euler's formula for zeta values

Let $\{F_n\}_{n\geq 0}$ be the Fibonacci numbers defined by

$$F_0 = 0, \quad F_1 = 1, \quad F_{n+2} = F_{n+1} + F_n \quad (n \geq 0).$$

Assume that $\alpha,\ \beta \in \mathbb{C}$ satisfy

$$\alpha\beta = -1, \quad |\beta| < 1.$$

Let $\{U_n\}_{n\geq 0}$ be defined by

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}\ ,$$

which are generalized Fibonacci numbers. Indeed, if $\beta = (1 - \sqrt{5})/2$, then $U_n = F_n$. For $s \in \mathbb{N}$ consider the reciprocal sum

$$h_{2s} := (\alpha - \beta)^{-2s} \sum_{n=1}^{\infty} \frac{1}{U_{2n}^{2s}}\ ,$$

which is holomorphic for $|\beta| < 1$. In this talk we present an asymptotic representation for this sum as $\beta \to -1 + 0$. A degenerate case of our expression coincides with Euler's formula for $\zeta(2s) = \sum_{n-1}^{\infty} n^{-2s}$.

Fuminori Kawamoto (Gakushuin Univ.), Koshi Tomita (Meijo Univ.)

Continued fractions and certain real quadratic fields of minimal type

In this talk, we introduce the notion of real quadratic fields with period ℓ of minimal type in terms of continued fractions. We show that fundamental units of real quadratic fields that are not of minimal type are relatively small. So, we see by a theorem of Siegel that such fields have relatively large class numbers. Also, we show that there exist at most 52 real quadratic fields of class number 1 that are not of minimal type. Therefore we have to study real quadratic fields of minimal type in order to find many real quadratic fields of class number 1. We shall construct an infinite family of real quadratic fields with large even period of minimal type.

References

[1] F. Kawamoto and K. Tomita, *Continued fractions and certain real quadratic fields of minimal type*, J. Math. Soc. Japan **60** (2008), 865–903.

[2] F. Kawamoto and K. Tomita, *Continued fractions with even period and an infinite family of real quadratic fields of minimal type*, to appear in Osaka Journal of Mathematics.

Takao Komatsu (Hirosaki Univ.)

We consider the three-term recurrence formula $Z_n = T(n)Z_{n-1} + U(n)Z_{n-2}$ $(n \geq 2)$ with arbitrary initial values Z_0 and Z_1, where the integer sequences $(T(n))_{n\geq 0}$ and $(U(n))_{n\geq 0}$ are eventually periodic modulo m. Then the sequence $(Z_n)_{n\geq 0}$ is eventually periodic modulo m. Moreover, we can determine the leaping three-term recurrence relation of the form $C_1 Z_{rn+i} = C_2 Z_{r(n-1)+i} + C_3 Z_{r(n-2)+i}$. As an application, we find some continued fraction expansions of so-called Fibonacci Zeta functions.

Tomohiro Yamada (Kyoto Univ.)

Finiteness of odd superperfect numbers

We prove some new results concerning the equation $\sigma(N) = aM, \sigma(M) = bN$, which implies that there are only finitely many odd superperfect numbers with a fixed number of distinct prime factors.

Takeshi Kurosawa (Keio Univ.)

Arithmetical prpperty of reciprocal sums of binary linear recurrences

Duverney and Nishioka [1] showed transcendence of the reciprocal sums

$$\sum_{k\geq 0}{}' \frac{a_k}{F_{r^k} + b_k}, \quad \sum_{k\geq 0}{}' \frac{a_k}{L_{r^k} + b_k}, \tag{1}$$

where $r \geq 2$ is an integer, F_n is n–th Fibonacci number, L_n is n–th Lucas number, $\{a_k\}_{k\geq 0}$ and $\{b_k\}_{k\geq 0}$ are sequences in $\boldsymbol{K}$ and $O_{\boldsymbol{K}}$, respectively with $\log\|a_k\|, \log\|b_k\| = o(r^k)$, and $\sum_{k\geq 0}{}' p_k/q_k$ is a sum taken over those k with $q_k \neq 0$. excepting some cases. Here we denote $\boldsymbol{K}$ by an algebraic number field and $O_{\boldsymbol{K}}$ by a ring of the integers in $\boldsymbol{K}$. For an algebraic number α, we define its house as $\overline{|\alpha|} = \max\{|\alpha^\sigma| \mid \sigma \in \mathrm{Aut}(\overline{\mathbb{Q}}/\mathbb{Q})\}$ and $\mathrm{den}\,(\alpha)$ as the least positive integer such that $\mathrm{den}\,(\alpha)\,\alpha$ is an algebraic integer, and we set $\|\alpha\| = \max\{\overline{|\alpha|}, \mathrm{den}\,(\alpha)\}$. As far, the transcendence of the sum (1) was discussed under the conditions that $\{a_k\}_{k\geq 0}$ is a linear recurrence and $\{b_k\}_{k\geq 0}$ is a constant. Their result changes the conditions of $\{a_k\}_{k\geq 0}, \{b_k\}_{k\geq 0}$ to the weak growth conditions. In this talk we generalize their result as following.

Let $\{R_n\}_{n\geq 0}$ be a binary linear recurrence defined by

$$R_{n+2} = AR_{n+1} + BR_n, \tag{2}$$

where $A, B, R_0, R_1 \in \mathbb{Z}$. We assume $(A,B), (R_0,R_1) \neq (0,0)$ and $\Delta := A^2 + 4B > 0$. We put $P(X) = X^2 - AX - B$ and denote its roots by α, β with $|\alpha| \geq |\beta|$. Then $\{R_n\}_{n\geq 0}$ is expressed by

$$R_n = c\alpha^n + d\beta^n,$$

where $c = (R_1 - \beta R_0)/(\alpha - \beta)$ and $d = (\alpha R_0 - R_1)/(\alpha - \beta)$.

Theorem 1. *Let $\{R_k\}_{k\geq 0}$ be a linear recurrence defined by (2). Assume that $\Delta > 0$ is not a perfect square. Let $\{a_k\}_{k\geq 0}$ and $\{b_k\}_{k\geq 0}$ be sequences in $\boldsymbol{K}$ with $\log\|a_k\| = o(r^k)$, $\log\|b_k\| = o(r^k)$ and $a_k \neq 0$ for infinitely many k. Then the sum*

$$\theta = {\sum_{k\geq 0}}' \frac{a_k}{R_{r^k} + b_k} \tag{3}$$

is transcendental excepting for the following four cases:

1) *$r = 2$, $A = 0$, and B is not a perfect square. There exists a constant c and a root of unity ω such that $a_k = c2^k\omega^{2^k}$, $b_k = R_0\omega^{2^k} \neq 0$ for all large k.*

2) *$r = 2$, $R_0 = 0$, and $|B| = 1$. There exists a constant c such that $a_k = c$, $b_k = 0$ for all large k.*

3) *$r = 2$, $AR_0 = 2R_1 \neq 0$, and $|B| = 1$. There exist a constant c and integers p, $q(\neq 0)$ such that $a_k = c2^k \sin\left(2^k\frac{p}{q}\pi\right)$, $b_k = R_0\cos\left(2^k\frac{p}{q}\pi\right)$ for all large k.*

4) *$r = 2$, $AR_0 = 2R_1 \neq 0$, and $|B| = 1$. There exist a constant c such that $a_k = c4^k$, $b_k = R_0$ for all large k.*

Examples. *Typical examples of the sum (3) in these exceptional cases 1), 2), 3), and 4) are*

$$\sum_{k\geq 0} \frac{2^k}{B^{2^k} + 1} = \frac{1}{B-1}, \quad \sum_{k\geq 0} \frac{1}{F_{2^k}} = \frac{7-\sqrt{5}}{2},$$

$$\sum_{k\geq 1} \frac{(-2)^k}{L_{2^k} - 1} = -\frac{1}{2}, \quad \sum_{k\geq 1} \frac{4^k}{L_{2^k} + 2} = 4,$$

respectively.

References

[1] D. Duverney and Ku. Nishioka, *An inductive method for proving the transcendence of certain series*, Acta Arith. **110** (2003), no. 4, 305-330.

Masaaki Amou (Gunma Univ.)

Algebraic independence of certain infinite products

We present a result on algebraic independence of certain infinite products, whose transcendence was investigated by Tachiya (J Number Theorey 125). To prove the result we need 1) a quantitative irrationality criterion for functions under consideration; 2) transcendence measures of the values of the functions; and 3) an inductive method developed by Duverney (Math. Proc. Cambridge Philos. Soc. 130).