

# NEW SERIES TRANSFORMATIONS FOR EULER'S CONSTANT

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## Abstract

Let  $s_n = 1 + 1/2 + \dots + 1/(n-1) - \log n$ . In 1995, the author has found a series transformation of the type  $\sum_{k=0}^n \mu_{n,k,\tau} s_{k+\tau}$  with integer coefficients  $\mu_{n,k,\tau}$ , from which geometric convergence to Euler's constant  $\gamma$  for  $\tau = \mathcal{O}(n)$  results. In recently published papers *T.Rivoal* and *Kh. and T.Hessami Pilehrood* have generalized this result. In the lecture a series transformation  $\sum_{k=0}^n \mu_{n,k,\tau_1} s_{k+\tau_2}$  with two parameters  $\tau_1$  and  $\tau_2$  satisfying  $\tau_1 + 1 \leq \tau_2 \leq n + \tau_1 + 1$  and integer coefficients  $\mu_{n,k,\tau_1}$  will be introduced. By applying the Mellin-Barnes integral representation of the  ${}_3F_2$ -function, combinatorial identities and the analysis of the  $\psi$ -function, for  $n = 2m$ ,  $\tau_1 = m - 1$  and  $\tau_2 = 2m$  it is shown that  $S := |\sum_{k=0}^n \mu_{n,k,\tau_1} s_{k+\tau_2} - \gamma| \leq m/2 \cdot |\zeta(2) - q_m|$ , where  $q_m$  are explicitly given rational numbers. Finally,  $\zeta(2) - q_m$  can be expressed in terms of Legendre-type integrals, which give upper bounds for  $S$ . In particular, for  $n = 2m$ ,  $\tau_1 = m - 1$  and  $\tau_2 = 2m$  this bound equals to  $2m \cdot 64^{-m}$ . By the way, we find a linear three-term recurrence formula for a specific weighted sum on a  ${}_3F_2$ -function.