Applications of measure theory to digital sum problems

Zenji KOBAYASHI[1], Tatsuya OKADA[2], Takeshi SEKIGUCHI[1]and Yasunobu SHIOTA[1]
[1]*Faculty of Liberal Arts and Sciences, Tohoku Gakin University*,[2] *Fukushima Medical College*

Let $n=\sum_{i\geq 0}\alpha_i(n)2^i$ ($\alpha_i(n)\in\{0,1\}$) be a binary expansion of $n\in\mathbf{N}$ and define the binary digital sum $s(n)$ by $s(n)=\sum_{i\geq 0}\alpha_i(n)$. Let $I=I_{0,0}=[0,1]$ and $I_{n,j}=\left[\frac{j}{2^n},\frac{j+1}{2^n}\right)$, $j=0,1,\ldots,2^n-2$, $I_{n,2^n-1}=\left[\frac{2^n-1}{2^n},1\right]$ for $n=1,2,3,\ldots$ Define the binomial measure μ_r ($0<r<1$) by a probability measure on I such that

$$\mu_r(I_{n+1,2j})=r\mu_r(I_{n,j}),\qquad \mu_r(I_{n+1,2j+1})=(1-r)\mu_r(I_{n,j})$$

for $n=0,1,2,\ldots,j=0,1,\ldots,2^n-1$. We denote the distribution function of μ_r by $L(r,x)=\mu_r([0,x])$. It is well-known that $L(r,.)$ is a strictly increasing continuous singular function except for $r=1/2$ and $\frac{\partial}{\partial r}L(r,\cdot)\big|_{r=1/2}$ is the Takagi function.

We gave an explicit formula of the exponential sum of digital sums $F(\xi,N)=\sum_{n=0}^{N-1}e^{\xi s(n)}$by use of $L(r,x)$. On the other hand we studied the recursive construction of $\frac{\partial^k}{\partial r^k}L(r,\cdot)$. Combining these results we can immediately derive a formula of the power sums of digital sums $S_k(N)=\sum_{n=0}^{N-1}s(n)^k$.

Theorem 1 *Let $t=\log_2 N$ for $N\in\mathbf{N}$ and denote its decimal part by $\{t\}$. We have*

$$\begin{aligned} F(\xi,N) &= N^{\log_2(1+e^\xi)}2^{(1-\{t\})\log_2(1+e^\xi)}L\Big(\frac{1}{1+e^\xi},\frac{1}{2^{1-\{t\}}}\Big) \\ S_k(N) &= \frac{\partial^k}{\partial\xi^k}F(\xi,N)\Big|_{\xi=0} \end{aligned}$$

for $\xi\in\mathbf{R}$ and $k=1,2,3,\ldots$

This is the genralization of the results of Trollope, Delange, Coquet, Osbaldestin, Grabner, Kirschenhofer, Prodinger and Tichy.

We generalize these results to the sum of p-adic digits by use of multinomial measures, the digital sum of the Gray code representation of natural numbers by use of Gray measures and the digital sum of the block of binary numbers by use of probability measures which have Markov property.

Let $p\geq 2$ be a positive integer. Let $J_{n,j}=\left[\frac{j}{p^n},\frac{j+1}{p^n}\right)$, $j=0,1,\ldots,p^n-2$, $J_{n,p^n-1}=\left[\frac{p^n-1}{p^n},1\right]$ for $n=1,2,3,\ldots$ Set $\mathbf{r}=(r_0,r_1,\ldots,r_{p-2})$ be a vector such that $0<r_k<1$ for $k=0,1,\ldots,p-2$ and $0<\sum_{k=0}^{p-2}r_k<1$. We also set $r_{p-1}=1-\sum_{k=0}^{p-2}r_k$. Define the multinomial measure $\mu_{p,\mathbf{r}}$ by a probability measure on I such that

$$\mu_{p,\mathbf{r}}(J_{n+1,pj+k})=r_k\mu_{p,\mathbf{r}}(J_{n,j}),$$

for $n=0,1,2,\ldots,j=0,1,\ldots,p^n-1,k=0,1,\ldots,p-1$ and the Gray measure $\tilde{\mu}_r$ ($0<r<1$) by a probability measure on I such that

$$\begin{aligned} \tilde{\mu}_r(I_{n+1,2j}) &= \begin{cases} r\tilde{\mu}_r(I_{n,j}) & j\text{:even} \\ (1-r)\tilde{\mu}_r(I_{n,j}) & j\text{:odd} \end{cases} \\ \tilde{\mu}_r(I_{n+1,2j+1}) &= \begin{cases} (1-r)\tilde{\mu}_r(I_{n,j}) & j\text{:even} \\ r\tilde{\mu}_r(I_{n,j}) & j\text{:odd} \end{cases} \end{aligned}$$

for $n=0,1,2,\ldots,j=0,1,\ldots,2^n-1$.