

Rational approximation of $(1+x)^a$ and applications to the Thue inequality

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The Thue equation (inequality) is defined by

$$F(X, Y) = k, \quad (|F(X, Y)| \leq k),$$

where $F \in \mathbb{Z}[X, Y]$ is a homogeneous polynomial of degree $n \geq 3$ and k is a positive integer. We have two fundamental theorems on them:

Theorem T (A. Thue, 1909) *Thue equation has only finitely many solutions $(X, Y) \in \mathbb{Z}^2$.*

This theorem is not effective. In 1968 A. Baker gave the upper bound of the solutions (X, Y) :

Theorem B (A. Baker, 1968) *Let $\kappa > n + 1$ and $(X, Y) \in \mathbb{Z}^2$ be a solution of the Thue equation. Then*

$$\max\{|X|, |Y|\} < Ce^{(\log k)^\kappa},$$

where $C = C(n, \kappa, F)$ is an effectively computable number.

But generally this upper bound is too big to determine the exact solutions for a given equation (inequality).

Let α and x be rational numbers in the open interval $(0, 1)$. We denote $x = x_1/x_2$ and $a = \alpha/n = a_1/a_2$ where $x_1, x_2, a_1, a_2 \in \mathbb{Z}^+$ and $\gcd(x_1, x_2) = \gcd(a_1, a_2) = 1$. By considering the integral

$$I_{in}(x) = \frac{1}{2\pi\sqrt{-1}} \int_{\Gamma} \frac{z^i(1+zx)^{n+a}}{(z(z-1))^{n+1}} dz,$$

we have the Padé approximations to $(1+x)^a$. By the Padé approximations we obtain

Theorem 1 *Let $d = a_2^{1-(a_2 \bmod 2)}$, $D = x_2$, $Q = a_2^{\frac{3}{2}}$, $p = \frac{(1+\varepsilon)(1+x+\varepsilon x)}{1-\varepsilon}$, $P = \frac{1+x+\varepsilon x}{\varepsilon(1-\varepsilon)}$, $\varepsilon = \frac{-1-x+\sqrt{1+3x+2x^2}}{x}$, $l = \frac{\sin a\pi}{\pi}$, $L = \frac{1}{x^2}$ and*

$$\lambda = \frac{\log V}{\log U}, \quad c^{-1} = 2pdVC^\lambda,$$

where $V = PDQ$, $U = \frac{L}{DQ}$ and $C = \max\{1, 2dl\}$. If $x_2 > x_1^2 a_2^{\frac{3}{2}}$ then

$$\left| (1+x)^a - \frac{X}{Y} \right| > \frac{c}{Y^{1+\lambda}}$$

for arbitrary positive integers (X, Y) .

Using Theorem 1 we obtain

Theorem 2 *Let k be a positive real number. Suppose $x_2 > x_1^2 a_2^{\frac{3}{2}}$, then any solution $(X, Y) \in (\mathbb{Z}^+)^2$ of the Thue inequality*

$$|X^n - (1+x)^\alpha Y^n| \leq k$$

satisfies

$$Y < \begin{cases} \left(\frac{k}{nc}\right)^{\frac{1}{n-1-\lambda}} & (X \geq Y), \\ \left(\frac{k}{(1+x)^\alpha - 1}\right)^{\frac{1}{n}} & (X < Y). \end{cases}$$

This result depends on the method of G. V. Chudnovsky, J. H. Rickert and I. Wakabayashi. In the talk we shall show several examples.