Algebraic relations for reciprocal sums of even terms in Fibonacci numbers

Carsten Elsner   carsten.elsner@fhdw.de

Fachhochschule für die Wirtschaft, University of Applied Sciences, Freundallee 15, D-30173 Hannover, Germany

Shun Shimomura   shimomur@math.keio.ac.jp

Iekata Shiokawa   shiokawa@beige.ocn.ne.jp

Department of Mathematics, Keio University, 3-14-1 Hiyoshi, Kohoku-ku, Yokohama 223-8522 Japan

Let $\{F_n\}_{n\geq 0}$ and $\{L_n\}_{n\geq 0}$ be Fibonacci numbers and Lucas numbers defined by

$$\begin{aligned} &F_0 = 0, \quad F_1 = 1, \quad F_{n+2} = F_{n+1} + F_n \quad (n \geq 0), \\ &L_0 = 2, \quad L_1 = 1, \quad L_{n+2} = L_{n+1} + L_n \quad (n \geq 0). \end{aligned}$$

Duverney, Ke. Nishioka, Ku. Nishioka, and the last named author [1] proved the transcendence of the numbers

$$\sum_{n=1}^{\infty} \frac{1}{F_n^{2s}}, \quad \sum_{n=1}^{\infty} \frac{1}{L_n^{2s}}, \quad \sum_{n=1}^{\infty} \frac{1}{F_{2n-1}^{s}}, \quad \sum_{n=1}^{\infty} \frac{1}{L_{2n}^{s}} \quad (s = 1, 2, 3, \dots)$$

by using Nesterenko's theorem ([3]) on Ramanujan functions $P(q)$, $Q(q)$, and $R(q)$.

Let us set

$$\zeta_{\mathrm{F}}(s) := \sum_{n=1}^{\infty} \frac{1}{F_n^s} \quad (s = 1, 2, 3, \dots).$$

In [2], we proved that the values $\zeta_{\mathrm{F}}(2)$, $\zeta_{\mathrm{F}}(4)$, $\zeta_{\mathrm{F}}(6)$ are algebraically independent, and that for any integer $s \geq 4$,

$$\zeta_{\mathrm{F}}(2s) - 5^{s-2} r_s \zeta_{\mathrm{F}}(4) \in \mathbb{Q}(u, v), \quad u = \zeta_{\mathrm{F}}(2), \quad v = \zeta_{\mathrm{F}}(6)$$

with some $r_s \in \mathbb{Q}$ ($r_s = 0$ if and only if s is odd), where the rational function of u and v is explicit; for example,

$$\zeta_{\mathrm{F}}(8) - \frac{15}{14}\zeta_{\mathrm{F}}(4) = \frac{1}{378(4u+5)^2}\Big(256u^6 - 3456u^5 + 2880u^4 + 1792u^3 v - 11100u^3 + 20160u^2 v - 10125u^2 + 7560uv + 3136v^2 - 1050v\Big).$$

Similar results were obtained for $\sum_{n=1}^{\infty}(-1)^{n+1}F_n^{-2s}$, $\sum_{n=1}^{\infty}L_n^{-2s}$, and $\sum_{n=1}^{\infty}(-1)^{n+1}L_n^{-2s}$.

In this talk, we discuss the algebraic independence and algebraic relations for reciprocal sums of even terms including

$$\sum_{n=1}^{\infty}\frac{1}{F_{2n}^{2s}},\quad \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{F_{2n}^{2s}},\quad \sum_{n=1}^{\infty}\frac{1}{L_{2n}^{p}},\quad \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{L_{2n}^{p}}.$$

Furthermore, combining two different even sums, we obtain reciprocal sums of evenly even terms and of unevenly even terms. For example, from $\sum_{n=1}^{\infty}F_{2n}^{-2s}$ and $\sum_{n=1}^{\infty}(-1)^{n+1}F_{2n}^{-2s}$, we derive

$$\Theta(2s)=\sum_{n=1}^{\infty}\frac{1}{F_{4n}^{2s}},\qquad \Theta^{\sharp}(2s)=\sum_{n=1}^{\infty}\frac{1}{F_{4n-2}^{2s}}.$$

We show that the numbers $\Theta(2)$, $\Theta(4)$, $\Theta(6)$ (respectively, $\Theta^{\sharp}(2)$, $\Theta^{\sharp}(4)$, $\Theta^{\sharp}(6)$) are algebraically independent, and for each $s\geq 4$, $\Theta(2s)$ (respectively, $\Theta^{\sharp}(2s)$) is written as an algebraic function of them. In particular, the algebraic relations corresponding to $\Theta^{\sharp}(2s)$ are rational; for example,

$$\Theta^{\sharp}(8)=\frac{1}{70(5x+6y)}\Big(125x^3+450x^2y+540xy^2+216y^3 - 125xy-500xz+50y^2-200yz+200z^2\Big),$$

where $x=\Theta^{\sharp}(2)$, $y=\Theta^{\sharp}(4)$, $z=\Theta^{\sharp}(6)$.

References

[1] D. Duverney, Ke. Nishioka, Ku. Nishioka, and I. Shiokawa, “Transcendence of Rogers-Ramanujan continued fraction and reciprocal sums of Fibonacci numbers,” *Proc. Japan Acad. Ser. A Math. Sci.* **73** (1997), 140–142.

[2] C. Elsner, S. Shimomura, and I. Shiokawa, “Algebraic relations for reciprocal sums of Fibonacci numbers,” *Preprint* (2006).

[3] Nesterenko, Yu. V.: Modular functions and transcendence questions. Mat. Sb. **187**, 65–96 (1996); English transl. Sb. Math. **187**, 1319–1348 (1996)