

# Expansion of real numbers by algebraic numbers

Hajime Kaneko

In this talk we discuss fractional parts of powers of algebraic numbers. Koksma proved that when we take any real  $\alpha > 1$  then the sequence  $(\xi\alpha^n)_{n=0}^\infty$  is uniformly distributed modulo 1 for almost all  $\xi$ . Similarly if  $\xi$  is a non-zero real number, the sequence  $(\xi\alpha^n)_{n=0}^\infty$  is uniformly distributed modulo 1 for almost all  $\alpha > 1$ . On the contrary, it is difficult to determine whether the sequence  $(\xi\alpha^n)_{n=0}^\infty$  is uniformly distributed modulo 1 for fixed  $\xi$  and  $\alpha$ . Moreover, it isn't generally known that the fractional parts of  $(\xi\alpha^n)_{n=0}^\infty$  is dense on the torus  $\mathbb{R}/\mathbb{Z}$ .

When  $N$  is a natural number with  $N \geq 2$ , the fractional parts of  $\xi N^n$  is determined by the  $N$ -ary expansion of  $\xi$ . Mahler introduced the "3/2-ary" expansion to estimate the values  $\xi(3/2)^n$  modulo 1. He studied the set of  $Z$ -numbers by using this numerical system.

We now introduce the " $\alpha$ -ary" expansion of  $\xi$ , where  $\alpha > 1$  is an algebraic number whose Galois conjugates have absolute values greater than 1. We can express the integer parts and fractional parts of  $\xi\alpha^n$  by this system. When  $\alpha$  is not a natural number, digits of " $\alpha$ -ary" expansion have some interesting properties. By studying the patterns of digits, we can estimate the fractional parts of  $\xi\alpha^n$ .