

# Measure of irrationality for certain infinite series

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This is a joint work with Kenji Amano. In 1995, Duverney proved the irrationality of  $\sum_{n=0}^{\infty} r^n / (q^n - r)$  for an integer  $q$  ( $|q| > 1$ ) and  $\mathbb{Q}^\times \setminus \{q^i \mid i = 1, 2, \dots\}$  under the some conditions on  $q$  and  $r$ . But, in applying Duverney's theorem, it is not easy to check that  $q$  and  $r$  satisfy the conditions in the theorem.

We generalized Duverney's theorem and obtained the measure of irrationality for certain infinite series. In this talk, we will describe the following results.

**Theorem 1.** *Let  $\mathbf{K}$  be either  $\mathbb{Q}$  or an imaginary quadratic field,  $q$  ( $|q| > 1$ ) be an integer of  $\mathbf{K}$ , and  $r \in \mathbf{K}^\times$  ( $|r| < |q|$ ). Then*

$$\theta = \sum_{n=1}^{\infty} \frac{r^n}{q^n - r^l} \notin \mathbf{K},$$

where  $l \geq 1$  is positive integer. Moreover, for any  $\varepsilon > 0$ , there exists a constant  $C = C(\varepsilon) > 0$  such that the following inequality

$$|h_1\theta + h_2| \geq \frac{C}{|h_1|^{5l^2-1+\varepsilon}}$$

holds for all  $(h_1, h_2) \in O_{\mathbf{K}}$  ( $h_1 \neq 0$ ).

**Example 1.** *Let  $q = 3$ ,  $r = 2$ , and  $l = 1, 2, 3$ . Then the numbers*

$$\sum_{n=1}^{\infty} \frac{2^n}{3^n - 2}, \quad \sum_{n=1}^{\infty} \frac{2^n}{3^n - 4}, \quad \sum_{n=1}^{\infty} \frac{2^n}{3^n - 8}$$

are irrational.

**Example 2.** *Let  $\omega$  be the root of unity satisfying  $\omega^3 = 1$ . Then*

$$\sum_{n=1}^{\infty} \frac{1}{2^n - \omega^n} \notin \mathbb{Q}(\omega).$$