

はじめに (数学基礎 B2)

数学基礎 B = 線形代数

教科書 「要点明解 線形数学」 培風館

(第1章 行列)

(第2章 連立1次方程式)

▶ 第3章 行列式

▶ 第4章 行列の対角化

講義の情報

<http://mathweb.sc.niigata-u.ac.jp/~hoshi/teaching-j.html>

シラバス

LINK

▶ ノートを取りながら講義を聴くこと.

(ノートを回収して確認する可能性があります)

▶ 講義 → 小テスト (理解度確認テスト, 学務情報システム内)

3.3 行列式の展開

定理 3.9 (重要)

n 次正方行列 A, B に対して, $|AB| = |A| |B|$.

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$$|AB| = (a_{11}b_{11} + a_{12}b_{21})(a_{21}b_{12} + a_{22}b_{22}) - (a_{11}b_{12} + a_{12}b_{22})(a_{21}b_{11} + a_{22}b_{21})$$

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$$= \cancel{a_{11}b_{11}a_{21}b_{12}} + a_{11}b_{11}a_{22}b_{22} + a_{12}b_{21}a_{21}b_{12} \cancel{+ a_{12}b_{21}a_{22}b_{22}}$$

$$\cancel{- a_{11}b_{12}a_{21}b_{11}} - a_{11}b_{12}a_{22}b_{21} - a_{12}b_{22}a_{21}b_{11} \cancel{- a_{12}b_{22}a_{22}b_{21}}$$

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$$= (a_{11}a_{22} - a_{12}a_{21})(b_{11}b_{22} - b_{12}b_{21})$$

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注意

一般に, $|A + B| \neq |A| + |B|$.

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注意

一般に, $|A + B| \neq |A| + |B|$. 例えば, $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$
 $\Rightarrow |A| + |B| = 1 + 1 = 2 \neq 0 = |A + B|$.

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$$A_{ij} = \begin{pmatrix} a_{11} & \cdots & a_{1,j-1} & \cancel{a_{1j}} & a_{1,j+1} & \cdots & a_{1n} \\ \vdots & & \vdots & \cancel{\quad} & \vdots & & \vdots \\ \cancel{a_{i1}} & \cancel{\quad} & \cancel{a_{i,j-1}} & \cancel{a_{ij}} & \cancel{a_{i,j+1}} & \cancel{\quad} & \cancel{a_{in}} \\ \vdots & & \vdots & \cancel{\quad} & \vdots & & \vdots \\ a_{n1} & \cdots & a_{n,j-1} & \cancel{a_{nj}} & a_{n,j+1} & \cdots & a_{nn} \end{pmatrix}$$

(A_{ij} は A から第 i 行と第 j 列を取り除いた $(n-1) \times (n-1)$ 行列)

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行列式の定義より, $|A| = a_{11}\tilde{a}_{11} + \cdots + a_{1n}\tilde{a}_{1n}$.

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$A = (a_{ij}) : n$ 次正方行列.

$$(1) \quad a_{i1}\tilde{a}_{j1} + \cdots + a_{in}\tilde{a}_{jn} = \sum_{k=1}^n a_{ik}\tilde{a}_{jk} = \begin{cases} |A| & (i = j) \\ 0 & (i \neq j); \end{cases}$$

$$(2) \quad a_{1i}\tilde{a}_{1j} + \cdots + a_{ni}\tilde{a}_{nj} = \sum_{k=1}^n a_{ki}\tilde{a}_{kj} = \begin{cases} |A| & (i = j) \\ 0 & (i \neq j). \end{cases}$$

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▶ (2) で $i = j$ のとき, $|A| = a_{1i}\tilde{a}_{1i} + \cdots + a_{ni}\tilde{a}_{ni} = \sum_{k=1}^n a_{ki}\tilde{a}_{ki}$ を
第 i 列に関する $|A|$ の余因子展開 という.

例 ($n = 3$) 定理 3.10 (1)

$i = j = 1$. 第 1 行に関する $|A|$ の余因子展開.

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$i = j = 2$. 第 2 行に関する $|A|$ の余因子展開.

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例 ($n = 3$) 定理 3.10 (2)

$i = j = 1$. 第 1 列に関する $|A|$ の余因子展開.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}.$$

$i = j = 2$. 第 2 列に関する $|A|$ の余因子展開.

例 ($n = 3$) 定理 3.10 (2)

$i = j = 1$. 第 1 列に関する $|A|$ の余因子展開.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}.$$

$i = j = 2$. 第 2 列に関する $|A|$ の余因子展開.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = -a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{22} \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} - a_{32} \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}.$$

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定理 3.11 (重要)

$A : n$ 次正方行列.

(1) $A : \text{正則} \Leftrightarrow |A| \neq 0$;

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$\therefore A : \text{正則} \Rightarrow AA^{-1} = E_n \Rightarrow |A||A^{-1}| \underset{3.9}{=} |AA^{-1}| = 1$ より, $|A| \neq 0$.

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$$|A| \neq 0 \Rightarrow A\tilde{A} = \tilde{A}A = |A| E_n$$

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$|A| \neq 0 \Rightarrow A\tilde{A} = \tilde{A}A = |A| E_n \Rightarrow A^{-1} = \frac{1}{|A|} \tilde{A}$ で A は正則.

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$|A| \neq 0 \Rightarrow A\tilde{A} = \tilde{A}A = |A| E_n \Rightarrow A^{-1} = \frac{1}{|A|} \tilde{A}$ で A は正則. (2) も OK.

例

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

例

$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$. A の (i, j) 余因子 $\tilde{a}_{ij}$ は,

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定理 3.11 より, A : 正則 $\Leftrightarrow |A| \neq 0 \Leftrightarrow a_{11}a_{22} - a_{12}a_{21} \neq 0$.

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$|A| \neq 0$ のとき,

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$$|A| \neq 0 \text{ のとき, } A^{-1} = \frac{1}{|A|} \tilde{A} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}.$$

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… 第 1 章の内容と一致している

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… 第 1 章の内容と一致している

▶ 各自, 第 3 章の章末問題 (教 p.71) をやっておく!

例

$$A = \begin{pmatrix} 1 & 0 & -1 \\ -2 & -1 & 0 \\ 0 & -1 & 2 \end{pmatrix}.$$

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例

$$A = \begin{pmatrix} 1 & 0 & -1 \\ -2 & -1 & 0 \\ 0 & -1 & 2 \end{pmatrix}. A \text{ の } (i, j) \text{ 余因子 } \tilde{a}_{ij} \text{ は,}$$

$$\tilde{a}_{11} = + \begin{vmatrix} -1 & 0 \\ -1 & 2 \end{vmatrix} = -2, \tilde{a}_{12} = - \begin{vmatrix} -2 & 0 \\ 0 & 2 \end{vmatrix} = 4, \tilde{a}_{13} = + \begin{vmatrix} -2 & -1 \\ 0 & -1 \end{vmatrix} = 2,$$

$$\tilde{a}_{21} = - \begin{vmatrix} 0 & -1 \\ -1 & 2 \end{vmatrix} = 1, \tilde{a}_{22} = + \begin{vmatrix} 1 & -1 \\ 0 & 2 \end{vmatrix} = 2, \tilde{a}_{23} = - \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} = 1,$$

$$\tilde{a}_{31} = + \begin{vmatrix} 0 & -1 \\ -1 & 0 \end{vmatrix} = -1, \tilde{a}_{32} = - \begin{vmatrix} 1 & -1 \\ -2 & 0 \end{vmatrix} = 2, \tilde{a}_{33} = + \begin{vmatrix} 1 & 0 \\ -2 & -1 \end{vmatrix} = -1.$$

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$$|A| = a_{11}\tilde{a}_{11} + a_{12}\tilde{a}_{12} + a_{13}\tilde{a}_{13} = 1 \cdot (-2) + 0 \cdot 4 + (-1) \cdot 2 = -4.$$

例

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$$\therefore A^{-1} = \frac{1}{|A|} \tilde{A} = \frac{1}{-4} \begin{pmatrix} -2 & 1 & -1 \\ 4 & 2 & 2 \\ 2 & 1 & -1 \end{pmatrix}.$$