Rationality problem of two-dimensional quasi-monomial group actions (joint work with H. Kitayama)

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§0. Introduction

Inverse Galois problem (IGP)

Does every finite group occur as a quotient group of the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$?

- Related to rationality problem (Emmy Noether's strategy: 1913)

A finite group $G \curvearrowright k(x_g \mid g \in G)$: rational function field over k by permutation

$k(x_g \mid g \in G)^G$ is k-rational, i.e. $k(x_g \mid g \in G)^G \simeq k(t_1, \ldots, t_n)$ (Noether's problem has an affirmative answer)

$\Longrightarrow k(x_g \mid g \in G)^G$ is retract k-rational (weaker concept)

$\Longleftrightarrow$ $\exists$ generic extension (polynomial) for (G, k) (Saltman's sense)

$\overset{k\text{:Hilbertian}}{\Longrightarrow}$ IGP for (k, G) has an affirmative answer

Rationality problem for quasi-monomial actions

Definition (quasi-monomial action)

Let K/k be a finite field extension and $G \leq \mathrm{Aut}_k(K(x_1, \ldots, x_n))$; finite where $K(x_1, \ldots, x_n)$ is the rational function field of n variables over K.
The action of G on $K(x_1, \ldots, x_n)$ is called quasi-monomial if
(i) $\sigma(K) \subset K$ for any $\sigma \in G$;
(ii) $K^G = k$;
(iii) for any $\sigma \in G$, $$\sigma(x_j) = c_j(\sigma) \prod_{i=1}^{n} x_i^{a_{ij}}$$
where $c_j(\sigma) \in K^\times$, $1 \leq j \leq n$, $[a_{i,j}]_{1 \leq i,j \leq n} \in GL_n(\mathbb{Z})$.

Rationality problem

Under what situation the fixed field $K(x_1, \ldots, x_n)^G$ is k-rational,
i.e. $K(x_1, \ldots, x_n)^G \simeq k(t_1, \ldots, t_n)$ (= purely transcendental over k),
if G acts on $K(x_1, \ldots, x_n)$ by quasi-monomial k-automorphisms.

Rationality problem for quasi-monomial actions

Definition (quasi-monomial action)

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where $c_j(\sigma) \in K^\times$, $1 \leq j \leq n$, $[a_{i,j}]_{1\leq i,j\leq n} \in GL_n(\mathbb{Z})$.

- When $G \curvearrowright K$; trivial (i.e. $K = k$), called (just) monomial action.
- When $G \curvearrowright K$; trivial and permutation $\leftrightarrow$ $\boxed{\text{Noether's problem}}$.
- When $c_j(\sigma) = 1$ $(\forall\sigma \in G, \forall j)$, called purely (quasi-)monomial.
- $G = \mathrm{Gal}(K/k)$ and purely $\leftrightarrow$ $\boxed{\text{Rationality problem for algebraic tori}}$.

Exercises (1/2): Noether's problem

- $S_n \curvearrowright k(x_1, \ldots, x_n)$; permutation
  $\boxed{\text{Q.}}$ Is $k(x_1, \ldots, x_n)^{S_n}$ k-rational? $\boxed{\text{Ans.}}$ Yes!

  $k(x_1, \ldots, x_n)^{S_n} = k(s_1, \ldots, s_n)$; s_i, ith elementary symmetric

  $\Longrightarrow$ IGP for (k, S_n) has affirmative solution.

- $A_n \curvearrowright k(x_1, \ldots, x_n)$; permutation
  $\boxed{\text{Q.}}$ Is $k(x_1, \ldots, x_n)^{A_n}$ k-rational? $\boxed{\text{Ans.}}$ Yes? ?? ??

  $k(x_1, \ldots, x_n)^{A_n} = k(s_1, \ldots, s_n, \Delta)$; but ...

  $\boxed{\text{Open problem}}$ Is $k(x_1, \ldots, x_n)^{A_n}$ k-rational? $(n \geq 6)$

- $k(x_1, \ldots, x_5)^{A_5}$ is k-rational (Maeda, 1989).

Exercises (2/2): Noether's problem

- $k(x_1, x_2, x_3)^{A_3} = \mathbb{Q}(x_1, x_2, x_3)^{C_3} = \mathbb{Q}(t_1, t_2, t_3)$, $\boxed{\text{Q.}}$ t_1, t_2, t_3?
  $(C_3 : x_1 \mapsto x_2 \mapsto x_3 \mapsto x_1)$
- $\boxed{\text{Ans.}}$ $\mathbb{Q}(x_1, x_2, x_3)^{C_3} = \mathbb{Q}(t_1, t_2, t_3)$ where

$$\begin{aligned}
t_1 &= x_1 + x_2 + x_3,\\
t_2 &= \frac{x_1x_2^2 + x_2x_3^2 + x_3x_1^2 - 3x_1x_2x_3}{x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_2x_3 - x_3x_1},\\
t_3 &= \frac{x_1^2x_2 + x_2^2x_3 + x_3^2x_1 - 3x_1x_2x_3}{x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_2x_3 - x_3x_1}.
\end{aligned}$$

- $\mathbb{Q}(x_1, x_2, \ldots, x_8)^{C_8} = \mathbb{Q}(t_1, t_2, \ldots, t_8)$, $\boxed{\text{Q.}}$ $t_1, t_2, \ldots, t_8$?
  $(C_8 : x_1 \mapsto x_2 \mapsto x_3 \mapsto \cdots \mapsto x_8 \mapsto x_1)$
- $\boxed{\text{Ans.}}$ None: $\mathbb{Q}(x_1, x_2, \ldots, x_8)^{C_8}$ is not $\mathbb{Q}$-rational!

Today's talk

Definition (quasi-monomial action)

Let K/k be a finite field extension and $G \leq \mathrm{Aut}_k(K(x_1, \ldots, x_n))$; finite where $K(x_1, \ldots, x_n)$ is the rational function field of n variables over K. The action of G on $K(x_1, \ldots, x_n)$ is called quasi-monomial if
(i) $\sigma(K) \subset K$ for any $\sigma \in G$;
(ii) $K^G = k$;
(iii) for any $\sigma \in G$, $$\sigma(x_j) = c_j(\sigma) \prod_{i=1}^{n} x_i^{a_{ij}}$$
where $c_j(\sigma) \in K^\times$, $1 \leq j \leq n$, $[a_{i,j}]_{1 \leq i,j \leq n} \in GL_n(\mathbb{Z})$.

§1. $G \curvearrowright K$; trivial: monomial action & Noether's problem
§2. Quasi-monomial actions: Known results
§3. Quasi-monomial actions: Main theorem
§4. $G = \mathrm{Gal}(K/k)$ and purely: rationality problem for algebraic tori

Various rationalities: definitions

$k \subset L$; fin. gen. field extension, L is k-rational $\overset{\text{def}}{\iff}$ $L \simeq k(x_1, \ldots, x_n)$.

Definition (stably rational)

L is called stably k-rational $\overset{\text{def}}{\iff}$ $L(y_1, \ldots, y_m)$ is k-rational.

Definition (retract rational)

L is retract k-rational $\overset{\text{def}}{\iff}$ $\exists k$-algebra $R \subset L$ such that
(i) L is the quotient field of R;
(ii) $\exists f \in k[x_1, \ldots, x_n]$ $\exists k$-algebra hom. $\varphi : R \to k[x_1, \ldots, x_n][1/f]$ and $\psi : k[x_1, \ldots, x_n][1/f] \to R$ satisfying $\psi \circ \varphi = 1_R$.

Definition (unirational)

L is k-unirational $\overset{\text{def}}{\iff}$ $L \subset k(t_1, \ldots, t_n)$.

- Assume $L_1(x_1, \ldots, x_n) \simeq L_2(y_1, \ldots, y_m)$; stably isomorphic.
  If L_1 is retract k-rational, then so is L_2.
- "rational" $\Longrightarrow$ "stably rational" $\Longrightarrow$ "retract rational" $\Longrightarrow$ "unirational"

"rational" $\Longrightarrow$ "stably rational" $\Longrightarrow$ "retract rational" $\Longrightarrow$ "unirational"

- The direction of the implication cannot be reversed.
- (Lüroth's problem) "unirational" $\Longrightarrow$ "rational" ? YES if trdeg$= 1$
- (Castelnuovo, 1894)
  L is $\mathbb{C}$-unirational and $\mathrm{trdeg}_{\mathbb{C}} L = 2 \Longrightarrow L$ is $\mathbb{C}$-rational.
- (Zariski, 1958) Let k be an alg. closed field and $k \subset L \subset k(x,y)$. If $k(x,y)$ is separable algebraic over L, then L is k-rational.
- (Zariski cancellation problem) $V_1 \times \mathbb{P}^n \approx V_2 \times \mathbb{P}^n \Longrightarrow V_1 \approx V_2$?
  In particular, "stably rational" $\Longrightarrow$ "rational"?
- (Beauville, Colliot-Thélène, Sansuc, Swinnerton-Dyer, 1985, Ann. Math.)
  $L = \mathbb{Q}(x,y,t)$ with $x^2 + 3y^2 = t^3 - 2$ (Châtelet surface)
  $\Longrightarrow L$ is not rational but stably $\mathbb{Q}$-rational.
  Indeed, $L(y_1,y_2,y_3)$ is $\mathbb{Q}$-rational.
- $L(y_1,y_2)$ is $\mathbb{Q}$-rational (Shepherd-Barron, 2002, Fano Conf.).
- $\mathbb{Q}(x_1,\ldots,x_{47})^{C_{47}}$ is not stably but retract $\mathbb{Q}$-rational.
- $\mathbb{Q}(x_1,\ldots,x_8)^{C_8}$ is not retract but $\mathbb{Q}$-unirational.

Retract rationality and generic extension

Theorem (Saltman, 1982, DeMeyer)

Let k be an infinite field and G be a finite group.
The following are equivalent:
(i) $k(x_g \mid g \in G)^G$ is retract k-rational.
(ii) There is a generic G-Galois extension over k;
(iii) There exists a generic G-polynomial over k.

- related to Inverse Galois Problem (IGP). (i) $\Longrightarrow$ IGP($G/\mathbb{Q}$): true

Definition (generic polynomial)

A polynomial $f(t_1,\ldots,t_n;X) \in k(t_1,\ldots,t_n)[X]$ is generic for G over k if
(1) $\mathrm{Gal}(f/k(t_1,\ldots,t_n)) \simeq G$;
(2) $\forall L/M \supset k$ with $\mathrm{Gal}(L/M) \simeq G$,
$\exists a_1,\ldots,a_n \in M$ such that $L = \mathrm{Spl}(f(a_1,\ldots,a_n;X)/M)$.

- By Hilbert's irreducibility theorem, $\exists L/\mathbb{Q}$ such that $\mathrm{Gal}(L/\mathbb{Q}) \simeq G$.

§1. Monomial action & Noether's problem

Definition (monomial action) $G \curvearrowright K$; trivial, $k = K^G = K$

An action of G on $k(x_1, \ldots, x_n)$ is monomial $\overset{\text{def}}{\Longleftrightarrow}$

$$\sigma(x_j) = c_j(\sigma) \prod_{i=1}^{n} x_i^{a_{i,j}}, \ 1 \leq j \leq n, \forall \sigma \in G$$

where $[a_{i,j}]_{1 \leq i,j \leq n} \in \mathrm{GL}_n(\mathbb{Z})$, $c_j(\sigma) \in k^\times := k \setminus \{0\}$.

If $c_j(\sigma) = 1$ for any $1 \leq j \leq n$ then σ is called purely monomial.

- Application to Noether's problem (permutation action)

Noether's problem (1/3) [$G = A$; abelian case]

- k; field, G; finite group
- $G \curvearrowright k$; trivial, $G \curvearrowright k(x_g \mid g \in G)$; permutation.
- $k(G) := k(x_g \mid g \in G)^G$; invariant field

Noether's problem (Emmy Noether, 1913)

Is $k(G)$ k-rational?, i.e. $k(G) \simeq k(t_1, \ldots, t_n)$?

- Is the quotient variety $\mathbb{A}^n/G$ k-rational?

- Assume $G = A$; abelian group.
- (Fisher, 1915) $\mathbb{C}(A)$ is $\mathbb{C}$-rational.
- (Masuda, 1955, 1968) $\mathbb{Q}(C_p)$ is $\mathbb{Q}$-rational for $p \leq 11$.
- (Swan, 1969, Invent. Math.) $\mathbb{Q}(C_{47}), \mathbb{Q}(C_{113}), \mathbb{Q}(C_{233})$ are not $\mathbb{Q}$-rational.
- S. Endo and T. Miyata (1973), V.E. Voskresenskii (1973), ... e.g. $\mathbb{Q}(C_8)$ is not $\mathbb{Q}$-rational.
- (Lenstra, 1974, Invent. Math.) $k(A)$ is k-rational $\Longleftrightarrow$ $\boxed{\text{some condition}}$;

Noether's problem (2/3) [$G = A$; abelian case]

- (Endo-Miyata, 1973) $\mathbb{Q}(C_{p^r})$ is $\mathbb{Q}$-rational
  $\iff \exists \alpha \in \mathbb{Z}[\zeta_{\varphi(p^r)}]$ such that $N_{\mathbb{Q}(\zeta_{\varphi(p^r)})/\mathbb{Q}}(\alpha) = \pm p$
- $h(\mathbb{Q}(\zeta_m)) = 1$ if $m < 23$
  $\Longrightarrow \mathbb{Q}(C_p)$ is $\mathbb{Q}$-rational for $p \leq 43$ and $p = 61, 67, 71$.
- (Endo-Miyata, 1973) For $p = 47, 79, 113, 137, 167, \ldots,$
  $\mathbb{Q}(C_p)$ is not $\mathbb{Q}$-rational.
- However, for $p = 59, 83, 89, 97, 107, 163, \ldots$, unknown.
  Under the GRH, $\mathbb{Q}(C_p)$ is not rational for the above primes.
  But it was unknown for $p = 251, 347, 587, 2459, \ldots$
- For $p \leq 20000$, see speaker's paper (using PARI/GP):
  Hoshi, Proc. Japan Acad. Ser. A 91 (2015) 39-44.

Theorem (Plans, 2017, Proc. AMS)

$\mathbb{Q}(C_p)$ is $\mathbb{Q}$-rational $\iff p \leq 43$ or $p = 61, 67, 71$.

- Using lower bound of height, $\mathbb{Q}(C_p)$ is rational $\Rightarrow p < 173$.

Noether's problem (3/3) [G; non-abelian case]

Noether's problem (Emmy Noether, 1913)

Is $k(G)$ k-rational?, i.e. $k(G) \simeq k(t_1, \ldots, t_n)$?

- Assume G; non-abelian group.
- (Maeda, 1989) $k(A_5)$ is k-rational;
- (Rikuna, 2003; Plans, 2007)
  $k(GL_2(\mathbb{F}_3))$ and $k(SL_2(\mathbb{F}_3))$ is k-rational;
- (Serre, 2003)
  if 2-Sylow subgroup of $G \simeq C_{8m}$, then $\mathbb{Q}(G)$ is not $\mathbb{Q}$-rational;
  if 2-Sylow subgroup of $G \simeq Q_{16}$, then $\mathbb{Q}(G)$ is not $\mathbb{Q}$-rational;
  e.g. $G = Q_{16}, SL_2(\mathbb{F}_7), SL_2(\mathbb{F}_9)$,
  $SL_2(\mathbb{F}_q)$ with $q \equiv 7$ or $9 \pmod{16}$.

From Noether's problem to monomial actions (1/2)

- $k(G) := k(x_g \mid g \in G)^G$; invariant field

Noether's problem (Emmy Noether, 1913)

Is $k(G)$ k-rational?, i.e. $k(G) \simeq k(t_1, \ldots, t_n)$?

By Hilbert 90, we have:

No-name lemma (e.g. Miyata, 1971, Remark 3)

Let G act faithfully on k-vector space V, $W \subset V$ faithful $k[G]$-submodule. Then $K(V)^G = K(W)^G(t_1, \ldots, t_m)$.

Rationality problem: linear action

Let G act on finite-dimensional k-vector space V and $\rho : G \to GL(V)$ be a representation. Whether $k(V)^G$ is k-rational?

- the quotient variety V/G is k-rational?

From Noether's problem to monomial actions (2/2)

- For $\rho : G \to GL(V)$; monomial representation, i.e. matrix rep. has exactly one non-zero entry in each row and each column, G acts on $k(\mathbb{P}(V)) = k(\frac{w_1}{w_n}, \ldots, \frac{w_{n-1}}{w_n})$ by monomial action.

By Hilbert 90, we have:

Lemma (e.g. Miyata, 1971, Lemma)

$k(V)^G = k(\mathbb{P}(V))^G(t)$.

- $V/G \approx \mathbb{P}(V)/G \times \mathbb{P}^1$ (birational)

- $k(\mathbb{P}(V))^G$ (monomial action) is k-rational
$\Longrightarrow k(V)^G$ (linear action) is k-rational
$\Longrightarrow k(G)$ (permutation action) is k-rational
(Noether's problem has an affirmative answer)

Example: Noether's problem for $GL_2(\mathbb{F}_3)$ and $SL_2(\mathbb{F}_3)$

- $G = GL_2(\mathbb{F}_3) = \langle A, B, C, D \rangle \subset GL_4(\mathbb{Q})$, $|G| = 48$,
- $H = SL_2(\mathbb{F}_3) = \langle A, B, C \rangle \subset GL_4(\mathbb{Q})$, $|H| = 24$, where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, C = \begin{bmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

- G and H act on $k(V) = k(w_1, w_2, w_3, w_4)$ by

$$A : w_1 \mapsto -w_2 \mapsto -w_1 \mapsto w_2 \mapsto w_1,\ w_3 \mapsto -w_4 \mapsto -w_3 \mapsto w_4 \mapsto w_3,$$
$$B : w_1 \mapsto -w_3 \mapsto -w_1 \mapsto w_3 \mapsto w_1,\ w_2 \mapsto w_4 \mapsto -w_2 \mapsto -w_4 \mapsto w_2,$$
$$C : w_1 \mapsto -w_2 \mapsto w_3 \mapsto w_1,\ w_4 \mapsto w_4, \quad D : w_1 \mapsto w_1,\ w_2 \mapsto -w_2,\ w_3 \leftrightarrow w_4.$$

- $k(\mathbb{P}(V)) = k(x, y, z)$, $x = w_1/w_4$, $y = w_2/w_4$, $z = w_3/w_4$.
- G and H act on $k(x, y, z)$ as $G/Z(G) \simeq S_4$ and $H/Z(H) \simeq A_4$:

$$A : x \mapsto \frac{y}{z},\ y \mapsto \frac{-x}{z},\ z \mapsto \frac{-1}{z},\ B : x \mapsto \frac{-z}{y},\ y \mapsto \frac{-1}{y},\ z \mapsto \frac{x}{y},$$
$$C : x \mapsto y \mapsto z \mapsto x,\ D : x \mapsto \frac{x}{z},\ y \mapsto \frac{-y}{z},\ z \mapsto \frac{1}{z}.$$

- $k(\mathbb{P}(V))^G$: k-rational $\Longrightarrow$ $k(V)^G$: k-rational $\Longrightarrow$ $k(G)$: k-rational.

Monomial action (1/3) [3-dim. case]

Theorem (Hajja,1987) 2-dim. monomial action

$k(x_1,x_2)^G$ is k-rational.

Theorem (Hajja-Kang 1994, H-Rikuna 2008) 3-dim. purely monomial

$k(x_1,x_2,x_3)^G$ is k-rational.

Theorem (Prokhorov, 2010) 3-dim. monomial action over $k=\mathbb{C}$

$\mathbb{C}(x_1,x_2,x_3)^G$ is $\mathbb{C}$-rational.

However,
$\mathbb{Q}(x_1,x_2,x_3)^{\langle\sigma\rangle}$, $\sigma : x_1 \mapsto x_2 \mapsto x_3 \mapsto \frac{-1}{x_1x_2x_3}$ is not $\mathbb{Q}$-rational
(Hajja,1983).

Monomial action (2/3) [3-dim. case]

Theorem (Saltman, 2000) char $k \neq 2$

If $[k(\sqrt{a_1}, \sqrt{a_2}, \sqrt{a_3}) : k] = 8$, then $k(x_1, x_2, x_3)^{\langle\sigma\rangle}$,

$$\sigma : x_1 \mapsto \frac{a_1}{x_1}, x_2 \mapsto \frac{a_2}{x_2}, x_3 \mapsto \frac{a_3}{x_3}$$

is not retract k-rational (hence not k-rational).

Theorem (Kang, 2004)

$k(x_1, x_2, x_3)^{\langle\sigma\rangle}$, σ: $x_1 \mapsto x_2 \mapsto x_3 \mapsto \dfrac{c}{x_1x_2x_3} \mapsto x_1$, is k-rational

$\iff$ at least one of the following conditions is satisfied:

(i) char $k = 2$; (ii) $c \in k^2$; (iii) $-4c \in k^4$; (iv) $-1 \in k^2$.

If $k(x, y, z)^{\langle\sigma\rangle}$ is not k-rational, then it is not retract k-rational.

Recall that

- "rational" $\Longrightarrow$ "stably rational" $\Longrightarrow$ "retract rational" $\Longrightarrow$ "unirational"

Monomial action (3/3) [3-dim. case]

Theorem (Yamasaki, 2012) 3-dim. monomial, char $k \neq 2$

$\exists$ 8 cases $G \leq GL_3(\mathbb{Z})$ s.t $k(x_1, x_2, x_3)^G$ is not retract k-rational.
Moreover, the necessary and sufficient conditions are given.

- Two of 8 cases are Saltman's and Kang's cases.
- $\exists G \leq GL_3(\mathbb{Z})$; 73 finite subgroups (up to conjugacy)

Theorem (H-Kitayama-Yamasaki, 2011) 3-dim. monomial, char $k \neq 2$

$k(x_1, x_2, x_3)^G$ is k-rational except for the 8 cases and $G = A_4$.
For $G = A_4$, if $[k(\sqrt{a}, \sqrt{-1}) : k] \leq 2$, then it is k-rational.

Corollary

$\exists L = k(\sqrt{a})$ such that $L(x_1, x_2, x_3)^G$ is L-rational.

- However, $\exists 4$-dim. $\mathbb{C}(x_1, x_2, x_3, x_4)^{C_2 \times C_2}$ is not retract $\mathbb{C}$-rational.

§2. Quasi-monomial actions: Known results

Notion of "quasi-monomial" actions
is defined in Hoshi-Kang-Kitayama [HKK14], J. Algebra (2014). ▸ Definition

Theorem ([HKK14]) 1-dim. quasi-monomial actions

(1) purely quasi-monomial $\Longrightarrow K(x)^G$ is k-rational.
(2) $K(x)^G$ is k-rational excpet for the case: $\exists N \leq G$ such that
(i) $G/N = \langle \sigma \rangle \simeq C_2$;
(ii) $K(x)^N = k(\alpha)(y)$, $\alpha^2 = a \in K^\times$, $\sigma(\alpha) = -\alpha$ (if char k $\neq 2$),
$\alpha^2 + \alpha = a \in K$, $\sigma(\alpha) = \alpha + 1$ (if char $k = 2$);
(iii) $\sigma \cdot y = b/y$ for some $b \in k^\times$.
For the exceptional case, $K(x)^G = k(\alpha)(y)^{G/N}$ is k-rational $\iff$
Hilbert symbol $(a,b)_k = 0$ (if char $k \neq 2$), $[a,b)_k = 0$ (if char $k = 2$).
Moreover, $K(x)^G$ is not k-rational $\Longrightarrow$ not k-unirational.

Theorem ([HKK14]) 2-dim. purely quasi-monomial actions

$N=\{\sigma\in G\mid \sigma(x)=x,\ \sigma(y)=y\}$, $H=\{\sigma\in G\mid \sigma(\alpha)=\alpha\ (\forall\alpha\in K)\}$.
$K(x,y)^G$ is k-rational except for:
(1) char $k\neq 2$ and (2) (i) $(G/N,HN/N)\simeq(C_4,C_2)$ or (ii) (D_4,C_2).
For the exceptional case, we have $k(x,y)=k(u,v)$:
(i) $(G/N,HN/N)\simeq(C_4,C_2)$,
$K^N=k(\sqrt{a})$, $G/N=\langle\sigma\rangle\simeq C_4$, $\sigma:\sqrt{a}\mapsto-\sqrt{a}$, $u\mapsto\frac{1}{u}$, $v\mapsto-\frac{1}{v}$;
(ii) $(G/N,HN/N)\simeq(D_4,C_2)$;
$K^N=k(\sqrt{a},\sqrt{b})$, $G/N=\langle\sigma,\tau\rangle\simeq D_4$, $\sigma:\sqrt{a}\mapsto-\sqrt{a}$, $\sqrt{b}\mapsto\sqrt{b}$,
$u\mapsto\frac{1}{u}$, $v\mapsto-\frac{1}{v}$, $\tau:\sqrt{a}\mapsto\sqrt{a}$, $\sqrt{b}\mapsto-\sqrt{b}$, $u\mapsto u$, $v\mapsto -v$.
Case (i), $K(x,y)^G$ is k-rational $\iff$ Hilbert symbol $(a,-1)_k=0$.
Case (ii), $K(x,y)^G$ is k-rational $\iff$ Hilbert symbol $(a,-b)_k=0$.
Moreover, $K(x,y)^G$ is not k-rational $\Longrightarrow$ $\mathrm{Br}(k)\neq 0$ and
$K(x,y)^G$ is not k-unirational.

Galois-theoretic interpretation:
(i) k-rational $\iff$ $k(\sqrt{a})$ may be embedded into C_4-ext. of k.
(ii) k-rational $\iff$ $k(\sqrt{a},\sqrt{b})$ may be embedded into D_4-ext. of k.

Application to purely monomial actions (1/2)

Theorem ([HKK14]), 4-dim. purely monomial

Let M be a G-lattice with $\mathrm{rank}_{\mathbb{Z}} M = 4$ and G act on $k(M)$ by purely monomial k-automorphisms. If M is decomposable, i.e. $M = M_1 \oplus M_2$ as $\mathbb{Z}[G]$-modules where $1 \leq \mathrm{rank}_{\mathbb{Z}} M_1 \leq 3$, then $k(M)^G$ is k-rational.

- When $\mathrm{rank}_{\mathbb{Z}} M_1 = 1, \mathrm{rank}_{\mathbb{Z}} M_2 = 3$, it is easy to see $k(M)^G$ is k-rational.
- When $\mathrm{rank}_{\mathbb{Z}} M_1 = \mathrm{rank}_{\mathbb{Z}} M_2 = 2$, we may apply Theorem of 2-dim. to $k(M) = k(x_1, x_2, y_1, y_2) = k(x_1, x_2)(y_1, y_2) = K(y_1, y_2)$.

Theorem ([HKK14]) char $k \neq 2$

Let $C_2 = \langle \tau \rangle$ act on the rational function field $k(x_1, x_2, x_3, x_4)$ by k-automorphisms defined as

$$\tau : x_1 \mapsto -x_1,\ x_2 \mapsto \tfrac{x_4}{x_2},\ x_3 \mapsto \tfrac{(x_4-1)(x_4-x_1^2)}{x_3},\ x_4 \mapsto x_4.$$

Then $k(x_1, x_2, x_3, x_4)^{C_2}$ is not retract k-rational.
In particular, it is not k-rational.

Theorem A ([HKK14]) char $k \neq 2$, 5-dim. purely monomial

Let $D_4 = \langle \rho, \tau \rangle$ act on the rational function field $k(x_1, x_2, x_3, x_4, x_5)$ by k-automorphisms defined as

$$\rho : x_1 \mapsto x_2,\ x_2 \mapsto x_1,\ x_3 \mapsto \tfrac{1}{x_1 x_2 x_3},\ x_4 \mapsto x_5,\ x_5 \mapsto \tfrac{1}{x_4},$$
$$\tau : x_1 \mapsto x_3,\ x_2 \mapsto \tfrac{1}{x_1 x_2 x_3},\ x_3 \mapsto x_1,\ x_4 \mapsto x_5,\ x_5 \mapsto x_4.$$

Then $k(x_1, x_2, x_3, x_4, x_5)^{D_4}$ is not retract k-rational.
In particular, it is not k-rational.

Application to purely monomial actions (2/2)

Theorem ([HKK14]), 5-dim. purely monomial

Let M be a G-lattice and G act on $k(M)$ by purely monomial k-automorphisms. Assume that
(i) $M = M_1 \oplus M_2$ as $\mathbb{Z}[G]$-modules where $\mathrm{rank}_{\mathbb{Z}} M_1 = 3$ and $\mathrm{rank}_{\mathbb{Z}} M_2 = 2$,
(ii) either M_1 or M_2 is a faithful G-lattice.
Then $k(M)^G$ is k-rational except for the case as in Theorem A.

- we may apply Theorem of 2-dim. to $k(M) = k(x_1, x_2, x_3, y_1, y_2) = k(x_1, x_2, x_3)(y_1, y_2) = K(y_1, y_2)$.

More recent results

- 3-dim. purely quasi-monomial actions (Hoshi-Kitayama, 2020, Kyoto J. Math.)

- $\exists 13\ G \leq GL_2(\mathbb{Z})$ up to conjugacy.

Cyclic groups:

$$\begin{array}{llll} C_1 := \{I\}, & C_2^{(1)} := \langle -I \rangle, & C_2^{(2)} := \langle \lambda \rangle, & C_2^{(3)} := \langle \tau \rangle, \\ C_3 := \langle \rho^2 \rangle, & C_4 := \langle \sigma \rangle, & C_6 := \langle \rho \rangle & \end{array}$$

where

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ \lambda = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},\ \tau = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ \sigma = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},\ \rho = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}.$$

Non-cyclic groups:

$$\begin{array}{llll} V_4^{(1)} := \langle \lambda, -I \rangle, & V_4^{(2)} := \langle \tau, -I \rangle, & S_3^{(1)} := \langle \rho^2, \tau \rangle, & S_3^{(2)} := \langle \rho^2, -\tau \rangle, \\ D_4 := \langle \sigma, \tau \rangle, & D_6 := \langle \rho, \tau \rangle. & & \end{array}$$

Note that $\lambda^2 = \tau^2 = I$, $\sigma^2 = \rho^3 = -I$ and $\tau\sigma = \lambda$.

Define $H=\{\sigma\in G\mid \sigma(\alpha)=\alpha \text{ for any } \alpha\in K\}\lhd G$.
Then K is a (G/H)-Galois extension of k.
Normal subgroups H of G are given as follows:

$C_2^{(1)}: H=\{1\},\ \langle -I\rangle,$ $\qquad$ $C_2^{(2)}: H=\{1\},\ \langle \lambda\rangle,$

$C_2^{(3)}: H=\{1\},\ \langle \tau\rangle,$ $\qquad$ $C_3: H=\{1\},\ \langle \rho^2\rangle,$

$C_4: H=\{1\},\ \langle \sigma^2\rangle,\ \langle\sigma\rangle,$ $\qquad$ $C_6: H=\{1\},\ \langle -I\rangle,\ \langle\rho^2\rangle,\ \langle\rho\rangle,$

$V_4^{(1)}: H=\{1\},\ \langle -I\rangle,\ \langle\lambda\rangle,\ \langle -\lambda\rangle,\ \langle\lambda,-I\rangle,$

$V_4^{(2)}: H=\{1\},\ \langle -I\rangle,\ \langle\tau\rangle,\ \langle -\tau\rangle,\ \langle\tau,-I\rangle,$

$S_3^{(1)}: H=\{1\},\ \langle\rho^2\rangle,\ \langle\rho^2,\tau\rangle,$

$S_3^{(2)}: H=\{1\},\ \langle\rho^2\rangle,\ \langle\rho^2,-\tau\rangle,$

$D_4: H=\{1\},\ \langle -I\rangle,\ \langle -I,\lambda\rangle,\ \langle -I,\tau\rangle,\ \langle\sigma\rangle,\ \langle\sigma,\tau\rangle,$

$D_6: H=\{1\},\ \langle -I\rangle,\ \langle\rho^2\rangle,\ \langle\rho\rangle,\ \langle\rho^2,\tau\rangle,\ \langle\rho^2,-\tau\rangle,\langle\rho,\tau\rangle.$

Quasi-monomial actions: Main theorem (3/3)

Main theorem solves the rationality problem of two-dimensional quasi-monomial actions under the condition that $c_j(\sigma) \in k \setminus \{0\}$ where $c_j(\sigma)$ is given as in (iii) of Definition ▸ Definition.

- Let $(a,b)_{n,k}$ be the norm residue symbol of degree n over k.
- Let ω be a primitive cubic root of unity in $\overline{k}$.
- The case $G = H$ was already solved affirmatively by Hajja. ▸ Theorem

Main theorem (Hoshi-Kitayama, 2024, Transform. Groups)

Let k be a field with $\mathrm{char}\, k \neq 2, 3$, K/k be a finite extension and $G \leq GL_2(\mathbb{Z})$ acting on $K(x,y)$ by quasi-monomial k-automorphisms.
We assume that $c_j(\sigma) \in k \setminus \{0\}$ $(\forall \sigma \in G, 1 \leq \forall j \leq 2)$ where $c_j(\sigma)$ is given as in (iii) of Definition ▸ Definition and $G \neq H$ where
$H = \{\sigma \in G \mid \sigma(\alpha) = \alpha \text{ for any } \alpha \in K\}$.

Main theorem (Hoshi-Kitayama, 2024, Transform. Groups) cont.

(1) When $G=C_2^{(1)}=\langle -I\rangle$ and $H=\{1\}$, $K=k(\sqrt{a})$ and $-I:\sqrt{a}\mapsto -\sqrt{a}$, $x\mapsto b/x$, $y\mapsto c/y$ for $a,b,c\in k\setminus\{0\}$.
Then $K(x,y)^G$ is k-rational if and only if $(a,b)_{2,k}=0$ and $(a,c)_{2,k}=0$.
(2) When $G=C_2^{(2)}=\langle\lambda\rangle$ and $H=\{1\}$, $K=k(\sqrt{a})$ and $\lambda:\sqrt{a}\mapsto -\sqrt{a}$, $x\mapsto x$, $y\mapsto b/y$ for $a,b\in k\setminus\{0\}$.
Then $K(x,y)^G$ is k-rational if and only if $(a,b)_{2,k}=0$.
(3) When $G=C_2^{(3)}=\langle\tau\rangle$ and $H=\{1\}$, $K=k(\sqrt{a})$ and $\tau:\sqrt{a}\mapsto -\sqrt{a}$, $x\mapsto y$, $y\mapsto x$ for $a\in k\setminus\{0\}$. Then $K(x,y)^G$ is k-rational.
(4) When $G=C_3=\langle\rho^2\rangle$ and $H=\{1\}$, $\rho^2:x\mapsto y, y\mapsto c/(xy)$ for $c\in k\setminus\{0\}$.
$K(\omega)=k(\omega,\sqrt[3]{\alpha})$ for some $\alpha\in k(\omega)$ and $\rho^2:\sqrt[3]{\alpha}\mapsto\omega\sqrt[3]{\alpha}$.
Then $K(x,y)^G$ is k-rational if and only if $(\alpha,c)_{3,k(\omega)}=0$.
(5) When $G=C_4=\langle\sigma\rangle$, $\sigma:x\mapsto y$, $y\mapsto c/x$ for $c\in k\setminus\{0\}$.
(I) if $H=\{1\}$, then $K=k(\alpha,\beta)$ and $\sigma:\alpha\mapsto\beta$, $\beta\mapsto -\alpha$. We have $k(\alpha^2)=K^{\langle\sigma^2\rangle}$ with $[k(\alpha^2):k]=2$.
Then $K(x,y)^G$ is k-rational if and only if $(\alpha^2,c)_{2,k(\alpha^2)}=0$;
(II) if $H=\langle\sigma^2\rangle$, $K=k(\sqrt{a})$ and $\sigma:\sqrt{a}\mapsto -\sqrt{a}$ for $a\in k\setminus\{0\}$. Then $K(x,y)^G$ is k-rational if and only if $(a,c)_{2,k}=0$ and $(a,-c)_{2,k}=0$.

Main theorem (Hoshi-Kitayama, 2024, Transform. Groups) cont.

(6) When $G = C_6 = \langle \rho \rangle$, $\rho : x \mapsto xy$, $y \mapsto 1/x$. Then $K(x,y)^G$ is k-rational.
(7) When $G = V_4^{(1)} = \langle \lambda, -I \rangle = \langle \lambda, -\lambda \rangle$,
(I) if $H = \{1\}$, then $K = k(\sqrt{a}, \sqrt{b})$ and $\lambda : \sqrt{a} \mapsto -\sqrt{a}$, $\sqrt{b} \mapsto \sqrt{b}$, $x \mapsto x$, $y \mapsto d/y$, $-\lambda : \sqrt{a} \mapsto \sqrt{a}$, $\sqrt{b} \mapsto -\sqrt{b}$, $x \mapsto c/x$, $y \mapsto y$ for $a, b, c, d \in k \setminus \{0\}$.
Then $K(x,y)^G$ is k-rational if and only if $(a,d)_{2,k} = 0$ and $(b,c)_{2,k} = 0$;
(II) if $H = \langle -I \rangle$, then $K = k(\sqrt{a})$ and $\lambda : \sqrt{a} \mapsto -\sqrt{a}$, $x \mapsto x$, $y \mapsto d/y$, $-I : \sqrt{a} \mapsto \sqrt{a}$, $x \mapsto c/x$, $y \mapsto d/y$ for $a, c, d \in k \setminus \{0\}$.
Then $K(x,y)^G$ is k-rational if and only if $(a,d)_{2,k(\sqrt{cd})} = 0$;
(III) if $H = \langle \lambda \rangle$, then $K = k(\sqrt{a})$ and $\lambda : \sqrt{a} \mapsto \sqrt{a}$, $x \mapsto \varepsilon_1 x$, $y \mapsto d/y$, $-I : \sqrt{a} \mapsto -\sqrt{a}$, $x \mapsto c/x$, $y \mapsto d/y$ for $a, c, d \in k \setminus \{0\}$, $\varepsilon_1 = \pm 1$.
If $\varepsilon_1 = 1$, then $K(x,y)^G$ is k-rational if and only if $(a,c)_{2,k} = 0$.
If $\varepsilon_1 = -1$, then $K(x,y)^G$ is k-rational if and only if $(a,-c)_{2,k(\sqrt{ad})} = 0$;
(IV) if $H = \langle -\lambda \rangle$, then $K = k(\sqrt{a})$ and $-\lambda : \sqrt{a} \mapsto \sqrt{a}$, $x \mapsto c/x$, $y \mapsto \varepsilon_2 y$, $-I : \sqrt{a} \mapsto -\sqrt{a}$, $x \mapsto c/x$, $y \mapsto d/y$ for $a, c, d \in k \setminus \{0\}$, $\varepsilon_2 = \pm 1$.
If $\varepsilon_2 = 1$, then $K(x,y)^G$ is k-rational if and only if $(a,d)_{2,k} = 0$.
If $\varepsilon_2 = -1$, then $K(x,y)^G$ is k-rational if and only if $(a,-d)_{2,k(\sqrt{ac})} = 0$.

Main theorem (Hoshi-Kitayama, 2024, Transform. Groups) cont.

(8) When $G=V_4^{(2)}=\langle\tau,-I\rangle$, $\tau:x\mapsto y$, $y\mapsto x$, $-I:x\mapsto c/x$, $y\mapsto c/y$ for $c\in k\setminus\{0\}$.
(I) if $H=\{1\}$, then $K=k(\sqrt{a},\sqrt{b})$ and $\tau:\sqrt{a}\mapsto-\sqrt{a}$, $\sqrt{b}\mapsto\sqrt{b}$, $-I:\sqrt{a}\mapsto\sqrt{a}$, $\sqrt{b}\mapsto-\sqrt{b}$ for $a,b\in k\setminus\{0\}$.
Then $K(x,y)^G$ is k-rational if and only if $(b,c)_{2,k(\sqrt{a})}=0$;
(II) if $H=\langle\tau\rangle$, $\langle-I\rangle$ or $\langle-\tau\rangle$, then $K(x,y)^G$ is k-rational.
(9) When $G=S_3^{(1)}=\langle\rho^2,\tau\rangle$, $\rho^2:x\mapsto y,y\mapsto c/(xy)$, $\tau:x\mapsto y,y\mapsto x$ for $c\in k\setminus\{0\}$.
(I) if $H=\{1\}$, then $K(\omega)=F(\omega,\sqrt[3]{\alpha})$ for some $\alpha\in F(\omega)$ and $\rho^2:\sqrt[3]{\alpha}\mapsto\omega\sqrt[3]{\alpha}$ where $F=K^{\langle\rho^2\rangle}$ with $[F:k]=2$ and $F(\omega)=k(\alpha,\omega)$.
Then $K(x,y)^G$ is k-rational if and only if $(\alpha,c)_{3,k(\alpha,\omega)}=0$;
(II) if $H=\langle\rho^2\rangle$, then $K(x,y)^G$ is k-rational.
(10) When $G=S_3^{(2)}=\langle\rho^2,-\tau\rangle$, $\rho^2:x\mapsto y$, $y\mapsto 1/(xy)$, $-\tau:x\mapsto 1/y$, $y\mapsto 1/x$. Then $K(x,y)^G$ is k-rational.

Main theorem (Hoshi-Kitayama, 2024, Transform. Groups) cont.

(11) When $G = D_4 = \langle \sigma, \tau \rangle$, $\sigma : x \mapsto y$, $y \mapsto c/x$, $\tau : x \mapsto \varepsilon y$, $y \mapsto \varepsilon x$ for $c \in k \setminus \{0\}$, $\varepsilon = \pm 1$.
(I) if $H = \{1\}$, then $K = k(\alpha, \beta)$ and $\sigma : \alpha \mapsto \beta$, $\beta \mapsto -\alpha$, $\tau : \alpha \mapsto \beta$, $\beta \mapsto \alpha$.
We have $k(\alpha^2) = K^{\langle \sigma^2, \sigma\tau \rangle}$ with $[k(\alpha^2) : k] = 2$.
If $\varepsilon = 1$, then $K(x,y)^G$ is k-rational if and only if $(\alpha^2, c)_{2,k(\alpha^2)} = 0$.
If $\varepsilon = -1$, then $K(x,y)^G$ is k-rational if and only if $(\alpha^2, -\beta^2 c)_{2,k(\alpha^2)} = 0$;
(II) if $H = \langle -I \rangle$, then $K = k(\sqrt{a}, \sqrt{b})$ and $\sigma : \sqrt{a} \mapsto -\sqrt{a}$, $\sqrt{b} \mapsto \sqrt{b}$, $\tau : \sqrt{a} \mapsto \sqrt{a}$, $\sqrt{b} \mapsto -\sqrt{b}$ for $a, b \in k \setminus \{0\}$.
Then $K(x,y)^G$ is k-rational if and only if $(a, \varepsilon c)_{2,k} = 0$ and $(a, -\varepsilon bc)_{2,k} = 0$;
(III) if $H = \langle -I, \tau \rangle$, then $K = k(\sqrt{a})$ and $\sigma : \sqrt{a} \mapsto -\sqrt{a}$, $\tau : \sqrt{a} \mapsto \sqrt{a}$ for $a \in k \setminus \{0\}$. Then $K(x,y)^G$ is k-rational if and only if $(a, \varepsilon c)_{2,k} = 0$;
(IV) if $H = \langle -I, \tau\sigma \rangle$ or $\langle \sigma \rangle$, then $K(x,y)^G$ is k-rational.
(12) When $G = D_6 = \langle \rho, \tau \rangle$, $\rho : x \mapsto xy$, $y \mapsto 1/x$, $\tau : x \mapsto y$, $y \mapsto x$.
Then $K(x,y)^G$ is k-rational.
Moreover, if $K(x,y)^G$ is not k-rational, then k is an infinite field, the Brauer group $\mathrm{Br}(k)$ is non-trivial, and $K(x,y)^G$ is not k-unirational.

Remark (from the viewpoint of algebraic geometry)

(1) When $H=\{1\}$ and $c_j(\sigma)=1$ $(\forall\sigma\in G, 1\leq\forall j\leq 2)$, i.e. the action of G is faithful on K and purely monomial, $K(x,y)^G\simeq k(T)$ where T is an algebraic k-torus with $\dim T=2$. In this case, $k(T)\simeq K(x,y)^G$ is k-rational by Voskresenskii's theorem (1967).

(2) When $H=\{1\}$, because $K(x,y)^G\otimes_k K=K(x,y)$ is K-rational, one can consider equivariant compactification of the corresponding (split) algebraic K-torus $T_K=T\otimes_k K$ and apply equivariant MMP, then get a certain (split) G-minimal toric surface S_K,

for (1) $C_2^{(1)}$, (2) $C_2^{(2)}$, (7) $V_4^{(1)}$, $S_K\simeq\mathbb{P}^1_K\times\mathbb{P}^1_K$ with the G-invariant Picard number 2;

for (3) $C_2^{(3)}$, (5) C_4, (8) $V_4^{(2)}$, (11) D_4, $S_K\simeq\mathbb{P}^1_K\times\mathbb{P}^1_K$ with the G-invariant Picard number 1,

for (4) C_3, (10) $S_3^{(2)}$, $S_K\simeq\mathbb{P}^2_K$ (Severi-Brauer surface),

for (6) C_6, (9) $S_3^{(1)}$, (12) D_6, S_K is a del Pezzo surface of degree 6 (see Colliot-Thélène, Karpenko and Merkurjev 2008, Xie 2018).

As an application, we obtain some rationality result of $k(M)^G$ up to 5-dimensional cases under the action of monomial k-automorphisms.

Corollary (Hoshi-Kitayama, 2024, Transform. Groups) $\mathrm{char}\, k \neq 2, 3$

Let G be one of the finite subgroups $C_2^{(3)} = \langle \tau \rangle$, $C_6 = \langle \rho \rangle$, $S_3^{(2)} = \langle \rho^2, -\tau \rangle$, $D_6 = \langle \rho, \tau \rangle$ of $GL_2(\mathbb{Z})$ as in Main theorem (3), (6), (10), (12). Let M be a G-lattice with $M = M_1 \oplus M_2$ as $\mathbb{Z}[G]$-modules where $1 \leq \mathrm{rank}_{\mathbb{Z}} M_1 \leq 3$, $\mathrm{rank}_{\mathbb{Z}} M_2 = 2$ and G act on $k(M)$ by monomial k-automorphisms where M_2 is a faithful G-lattice and the action of $G \leq GL_2(\mathbb{Z})$ on $k(M_2)$ is given as in (iii) of Definition ▸ Definition.

(1) If $\mathrm{rank}_{\mathbb{Z}} M_1 = 1$ or 2, then $k(M)^G$ is k-rational;

(2) If $\mathrm{rank}_{\mathbb{Z}} M_1 = 3$, then $k(M)^G$ is k-rational except for the case $G = D_6$ and the action of G on $k(M_1)$ is given as $G_{3,1,1} = \langle \tau_1, \lambda_1 \rangle \simeq V_4$ in [HKiY] (2011).

For proof (1/4)

- Let $(a,b)_{n,k}$ be the norm residue symbol of degree n over k.

Theorem (Hajja-Kang-Ohm 1994, Kang 2007) $\mathrm{char}\ k \neq 2$

Let $K = k(\sqrt{a})$ be a quadratic field extension of k. Take $\mathrm{Gal}(K/k) = \langle\sigma\rangle$ and extend the action of σ to $K(x,y)$ by

$$\sigma : \sqrt{a} \mapsto -\sqrt{a},\ x \mapsto x,\ y \mapsto \frac{f(x)}{y} \quad (f(x) \in k[x]).$$

Then we have $K(x,y)^{\langle\sigma\rangle} = k(z_1, z_2, x)$ with the relation $z_1^2 - az_2^2 = f(x)$ where $z_1 = \frac{1}{2}(y + \frac{f(x)}{y})$, $z_2 = \frac{1}{2\sqrt{a}}(y - \frac{f(x)}{y})$.

(1) When $f(x) = b$, $K(x,y)^{\langle\sigma\rangle}$ is k-rational if and only if $(a,b)_{2,k} = 0$.

(2) When $\deg f(x) = 1$, $K(x,y)^{\langle\sigma\rangle}$ is always k-rational.

(3) When $f(x) = b(x^2 - c)$ for some $b, c \in k \setminus \{0\}$, then $K(x,y)^{\langle\sigma\rangle}$ is k-rational if and only if $(a,b)_{2,k} \in \mathrm{Br}(k(\sqrt{ac})/k)$, i.e. $(a,b)_{2,k(\sqrt{ac})} = 0$.

Moreover, if $K(x,y)^{\langle\sigma\rangle}$ is not k-rational, then k is an infinite field, the Brauer group $\mathrm{Br}(k)$ is non-trivial, and $K(x,y)^{\langle\sigma\rangle}$ is not k-unirational.

For proof (2/4)

Theorem (Hoshi-Kitayama, 2024, Transform. Groups)

Let $n \geq 2$, k be a field with $\gcd\{\mathrm{char}\ k, n\} = 1$ and $\zeta_n \in k$ where ζ_n is a primitive n-th root of unity in $\overline{k}$. Suppose that $k(\sqrt[n]{a})$ is a cyclic extension of k of degree n with $a \in k \setminus \{0\}$. Take $\mathrm{Gal}(k(\sqrt[n]{a})/k) = \langle \sigma \rangle$ and extend the action of σ to $k(\sqrt[n]{a})(x_1, \ldots, x_{n-1})$ by

$$\sigma : x_1 \mapsto \cdots \mapsto x_{n-1} \mapsto \frac{b}{x_1 \cdots x_{n-1}} \mapsto x_1$$

where $b \in k \setminus \{0\}$. Then the following statements are equivalent:

(i) $k(\sqrt[n]{a})(x_1, \ldots, x_{n-1})^{\langle \sigma \rangle}$ is k-rational;
(ii) $k(\sqrt[n]{a})(x_1, \ldots, x_{n-1})^{\langle \sigma \rangle}$ is stably k-rational;
(iii) $k(\sqrt[n]{a})(x_1, \ldots, x_{n-1})^{\langle \sigma \rangle}$ is k-unirational;
(iv) $(a, b)_{n,k} = 0$.

For proof (3/4)

Theorem (Hoshi-Kitayama, 2024, Transform. Groups) char $k \neq 2$

Let $K = k(\sqrt{a})$ be a quadratic field extension of k. Take $\mathrm{Gal}(K/k) = \langle \sigma \rangle$ and extend the action of σ to $K(x_1, \ldots, x_n)$ by

$$\sigma : \sqrt{a} \mapsto -\sqrt{a},\ x_i \mapsto \frac{b_i}{x_i} \quad (1 \leq i \leq n)$$

where $b_i \in k \setminus \{0\}$. Then $K(x_1, \ldots, x_n)^{\langle \sigma \rangle} = k(s_i, t_i : 1 \leq i \leq n)$ where

$$s_i^2 - a t_i^2 = b_i \quad (1 \leq i \leq n) \tag{1}$$

and the following statements are equivalent:
(i) $K(x_1, \ldots, x_n)^{\langle \sigma \rangle}$ is k-rational;
(ii) $K(x_1, \ldots, x_n)^{\langle \sigma \rangle}$ is stably k-rational;
(iii) $K(x_1, \ldots, x_n)^{\langle \sigma \rangle}$ is k-unirational;
(iv) the variety defined by (1) has a k-rational point;
(v) $(a, b_i)_{2,k} = 0$ for any $1 \leq i \leq n$.

For proof (4/4)

Theorem (Hoshi-Kitayama, 2024, Transform. Groups) $\mathrm{char}\ k \neq 2$

Let $K = k(\sqrt{a_1}, \ldots, \sqrt{a_n})$ where $a_i \in K \setminus \{0\}$ $(1 \leq i \leq n)$. Let σ_i $(1 \leq i \leq n)$ be a k-automorphism of $K(x_1, \ldots, x_n)$ defined by

$$\sigma_i : \sqrt{a_j} \mapsto \begin{cases} -\sqrt{a_j} & \text{if} \quad i = j \\ \sqrt{a_j} & \text{if} \quad i \neq j \end{cases}, \ x_j \mapsto \begin{cases} \dfrac{c_j}{x_j} & \text{if} \quad i = j \\ x_j & \text{if} \quad i \neq j \end{cases} \qquad (1 \leq i, j \leq n)$$

where $c_j \in k \setminus \{0\}$. Then $K(x_1, \ldots, x_n)^{\langle \sigma_1, \ldots, \sigma_n \rangle} = k(s_i, t_i : 1 \leq i \leq n)$ where

$$s_i^2 - a_i t_i^2 = c_i \quad (1 \leq i \leq n) \tag{2}$$

and the following statements are equivalent:

(i) $K(x_1, \ldots, x_n)^{\langle \sigma_1, \ldots, \sigma_n \rangle}$ is k-rational;
(ii) $K(x_1, \ldots, x_n)^{\langle \sigma_1, \ldots, \sigma_n \rangle}$ is stably k-rational;
(iii) $K(x_1, \ldots, x_n)^{\langle \sigma_1, \ldots, \sigma_n \rangle}$ is k-unirational;
(iv) the variety defined by (2) has a k-rational point;
(v) $(a_i, c_i)_{2,k} = 0$ for and $1 \leq i \leq n$.

§4. Rationality problem for algebraic tori (2-dim., 3-dim.)

$G \simeq \mathrm{Gal}(K/k) \curvearrowright K(x_1, \ldots, x_n)$: purely quasi-monomial, $K(x_1, \ldots, x_n)^G$ may be regarded as the function field of algebraic torus T over k which splits over K $(T \otimes_k K \simeq \mathbb{G}_m^n)$.

- T is k-unirational, i.e. $K(x_1, \ldots, x_n)^G \subset k(t_1, \ldots, t_n)$.
- $\exists 13$ $\mathbb{Z}$-coujugacy subgroups $G \leq \mathrm{GL}_2(\mathbb{Z})$.

Theorem (Voskresenskii, 1967) 2-dim. algebraic tori T

T is k-rational.

- $\exists 73$ $\mathbb{Z}$-coujugacy subgroups $G \leq \mathrm{GL}_3(\mathbb{Z})$.

Theorem (Kunyavskii, 1990) 3-dim. algebraic tori T

(i) T is k-rational $\iff$ T is stably k-rational $\iff$ T is retract k-rational $\iff$ $\exists G$: 58 groups;
(ii) T is not k-rational $\iff$ T is not stably k-rational $\iff$ T is not retract k-rational $\iff$ $\exists G$: 15 groups.

Rationality of algebraic tori (4-dim., 5-dim.)

- $\exists 710$ $\mathbb{Z}$-coujugacy subgroups $G \leq \mathrm{GL}_4(\mathbb{Z})$.

Theorem (Hoshi-Yamasaki, 2017, Mem. AMS) 4-dim. alg. tori T

(i) T is stably k-rational $\iff$ $\exists G$: 487 groups;
(ii) T is not stably but retract k-rational $\iff$ $\exists G$: 7 groups;
(iii) T is not retract k-rational $\iff$ $\exists G$: 216 groups.

- $\exists 6079$ $\mathbb{Z}$-coujugacy subgroups $G \leq \mathrm{GL}_5(\mathbb{Z})$.

Theorem (Hoshi-Yamasaki, 2017, Mem. AMS) 5-dim. alg. tori T

(i) T is stably k-rational $\iff$ $\exists G$: 3051 groups;
(ii) T is not stably but retract k-rational $\iff$ $\exists G$: 25 groups;
(iii) T is not retract k-rational $\iff$ $\exists G$: 3003 groups.

- (Voskresenskii's conjecture) any stably rational torus is rational.
- $\exists 85308$ $\mathbb{Z}$-coujugacy subgroups $G \leq \mathrm{GL}_6(\mathbb{Z})$!

Special case: $T = R^{(1)}_{K/k}(\mathbb{G}_m)$; norm one tori (1/5)

- Rationality problem for $T = R^{(1)}_{K/k}(\mathbb{G}_m)$ is investigated by S. Endo, Colliot-Thélène and Sansuc, W. Hürlimann, L. Le Bruyn, A. Cortella and B. Kunyavskii, N. Lemire and M. Lorenz, M. Florence, etc.

Theorem (Endo-Miyata, 1974), (Saltman, 1984)

Let K/k be a finite Galois field extension and $G = \mathrm{Gal}(K/k)$.
(i) T is retract k-rational $\iff$ all the Sylow subgroups of G are cyclic;
(ii) T is stably k-rational $\iff$ G is a cyclic group, or a direct product of a cyclic group of order m and a group $\langle \sigma, \tau \,|\, \sigma^n = \tau^{2^d} = 1, \tau\sigma\tau^{-1} = \sigma^{-1} \rangle$, where $d, m \geq 1, n \geq 3, m, n$: odd, and $(m, n) = 1$.

Theorem (Endo, 2011)

Let K/k be a finite non-Galois, separable field extension and L/k be the Galois closure of K/k. Assume that the Galois group of L/k is nilpotent. Then the norm one torus $T = R^{(1)}_{K/k}(\mathbb{G}_m)$ is not retract k-rational.

Special case: $T = R^{(1)}_{K/k}(\mathbb{G}_m)$; norm one tori (2/5)

- Let K/k be a finite non-Galois, separable field extension
- Let L/k be the Galois closure of K/k.
- Let $G = \mathrm{Gal}(L/k)$ and $H = \mathrm{Gal}(L/K) \leq G$.

Theorem (Endo, 2011)

Assume that all the Sylow subgroups of G are cyclic.
Then T is retract k-rational.
$T = R^{(1)}_{K/k}(\mathbb{G}_m)$ is stably k-rational $\iff$ $G = D_n$, n odd $(n \geq 3)$ or $C_m \times D_n$, m, n odd $(m, n \geq 3)$, $(m, n) = 1$, $H \leq D_n$ with $|H| = 2$.

Special case: $T = R^{(1)}_{K/k}(\mathbb{G}_m)$; norm one tori (3/5)

Theorem (Endo, 2011) $\dim T = n-1$

Assume that $\mathrm{Gal}(L/k) = S_n$, $n \geq 3$, and $\mathrm{Gal}(L/K) = S_{n-1}$ is the stabilizer of one of the letters in S_n.
(i) $R^{(1)}_{K/k}(\mathbb{G}_m)$ is retract k-rational $\iff$ n is a prime;
(ii) $R^{(1)}_{K/k}(\mathbb{G}_m)$ is (stably) k-rational $\iff$ $n = 3$.

Theorem (Endo, 2011) $\dim T = n-1$

Assume that $\mathrm{Gal}(L/k) = A_n$, $n \geq 4$, and $\mathrm{Gal}(L/K) = A_{n-1}$ is the stabilizer of one of the letters in A_n.
(i) $R^{(1)}_{K/k}(\mathbb{G}_m)$ is retract k-rational $\iff$ n is a prime;
(ii) $\exists t \in \mathbb{N}$ s.t. $[R^{(1)}_{K/k}(\mathbb{G}_m)]^{(t)}$ is stably k-rational $\iff$ $n = 5$.

- $[R^{(1)}_{K/k}(\mathbb{G}_m)]^{(t)}$: the product of t copies of $R^{(1)}_{K/k}(\mathbb{G}_m)$.

Special case: $T = R^{(1)}_{K/k}(\mathbb{G}_m)$; norm one tori (4/5)

Theorem ([HY17], Rationality for $R^{(1)}_{K/k}(\mathbb{G}_m)$ (dim. 4, $[K:k]=5$))

Let K/k be a separable field extension of degree 5 and L/k be the Galois closure of K/k. Assume that $G = \mathrm{Gal}(L/k)$ is a transitive subgroup of S_5 and $H = \mathrm{Gal}(L/K)$ is the stabilizer of one of the letters in G. Then the rationality of $R^{(1)}_{K/k}(\mathbb{G}_m)$ is given by

G		$L(M) = L(x_1,x_2,x_3,x_4)^G$
$5T1$	C_5	stably k-rational
$5T2$	D_5	stably k-rational
$5T3$	F_{20}	not stably but retract k-rational
$5T4$	A_5	stably k-rational
$5T5$	S_5	not stably but retract k-rational

- This theorem is already known except for the case of A_5 (Endo).
- Stably k-rationality for the case A_5 is asked by S. Endo (2011).

Special case: $T = R^{(1)}_{K/k}(\mathbb{G}_m)$; norm one tori (5/5)

Corollary of (Endo 2011) and [HY17]

Assume that $\mathrm{Gal}(L/k) = A_n$, $n \geq 4$, and $\mathrm{Gal}(L/K) = A_{n-1}$ is the stabilizer of one of the letters in A_n. Then $R^{(1)}_{K/k}(\mathbb{G}_m)$ is stably k-rational $\iff$ $n = 5$.

More recent results on stably/retract k-rational classification for T

- $G \leq S_n$ $(n \leq 10)$ and $G \neq 9T27 \simeq PSL_2(\mathbb{F}_8)$,
  $G \leq S_p$ and $G \neq PSL_2(\mathbb{F}_{2^e})$ $(p = 2^e + 1 \geq 17$; Fermat prime$)$
  (Hoshi-Yamasaki [HY21] Israel J. Math.)
- $G \leq S_n$ $(n = 12, 14, 15)$ $(n = 2^e)$
  (Hasegawa-Hoshi-Yamasaki [HHY20] Math. Comp.)

$\text{Ш}(T)$ and Hasse norm principle over number fields k (see next slides)

- (Hoshi-Kanai-Yamasaki [HKY22] Math. Comp., [HKY23] JNT)

$\text{Ш}(T)$ and HNP for K/k: Ono's theorem (1963)

- T : algebraic k-torus, i.e. $T \times_k \overline{k} \simeq (\mathbb{G}_{m,\overline{k}})^n$.
- $\text{Ш}(T) := \mathrm{Ker}\{H^1(k,T) \xrightarrow{\mathrm{res}} \bigoplus_{v \in V_k} H^1(k_v,T)\}$: Shafarevich-Tate gp.
- $T = R^{(1)}_{K/k}(\mathbb{G}_m)$ is biregularly isomorphic to the norm hyper surface $f(x_1,\ldots,x_n)=1$ where $f \in k[x_1,\ldots,x_n]$ is the norm form of K/k.

Theorem (Ono 1963, Ann. of Math.)

Let K/k be a finite extension of number fields and $T = R^{(1)}_{K/k}(\mathbb{G}_m)$. Then

$$\text{Ш}(T) \simeq (N_{K/k}(\mathbb{A}_K^\times) \cap k^\times)/N_{K/k}(K^\times)$$

where $\mathbb{A}_K^\times$ is the idele group of K. In particular,

$$\text{Ш}(T) = 0 \iff \text{Hasse norm principle holds for } K/k.$$