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FlabbyResolutionBC.gap

Definition of M_G

Let G be a finite subgroup of $\mathrm{GL}(n,\mathbb{Z})$. The G-lattice M_G of rank n is defined to be the G-lattice with a $\mathbb{Z}$-basis $\{u_1,\ldots,u_n\}$ on which G acts by

$$\sigma(u_i)=\sum_{j=1}^{n} a_{i,j}u_j \tag{1}$$

for any $\sigma=[a_{i,j}]\in G$.

Hminus1

```
‣ Hminus1(G)
```

returns the Tate cohomology group $\widehat{H}^{-1}(G,M_G)$ for a finite subgroup $G\leq \mathrm{GL}(n,\mathbb{Z})$.

H0

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‣ H0(G)
```

returns the Tate cohomology group $\widehat{H}^{0}(G,M_G)$ for a finite subgroup $G\leq \mathrm{GL}(n,\mathbb{Z})$.

H1

```
‣ H1(G)
```

returns the cohomology group $H^1(G,M_G)$ for a finite subgroup $G\leq \mathrm{GL}(n,\mathbb{Z})$.

Sha1Omega

```
‣ Sha1Omega(G)
```

returns $Sha^1_w(G,M_G)$.

Sha1OmegaTr

```
‣ Sha1OmegaTr(G)
```

returns $Sha_w^1(G,(M_G)^\circ)$.

ShaOmega

```
‣ ShaOmega(G,n)
```

returns $Sha_w^n(G,M_G)$ for G-lattice M_G.
This function needs HAP package in GAP.

ShaOmegaFromGroup

```
‣ ShaOmegaFromGroup(M,n,G)
```

returns $Sha_w^n(G,M)$ for G-lattice M.
This function needs HAP package in GAP.

TorusInvariants

```
‣ TorusInvariants(G)
```

returns $TI_G=[l_1,l_2,l_3,l_4]$ where

$$l_1=\begin{cases}0 & \text{if } [M_G]^{fl}=0,\\ 1 & \text{if } [M_G]^{fl}\neq 0 \text{ but is invertible},\\ 2 & \text{if } [M_G]^{fl} \text{ is not invertible},\end{cases}$$

$l_2=H^1(G,[M_G]^{fl})\simeq Sha_w^1(G,[M_G]^{fl})$,
$l_3=Sha_w^1(G,(M_G)^\circ)\simeq Sha_w^2(G,([M_G]^{fl})^\circ)$,
$l_4=H^1(G,([M_G]^{fl})^{fl})\simeq Sha_w^2(G,[M_G]^{fl})$ via the command H1(G).

TorusInvariantsHAP

```
‣ TorusInvariantsHap(G)
```

returns $TI_G=[l_1,l_2,l_3,l_4]$ where

$$l_1=\begin{cases}0 & \text{if } [M_G]^{fl}=0,\\ 1 & \text{if } [M_G]^{fl}\neq 0 \text{ but is invertible},\\ 2 & \text{if } [M_G]^{fl} \text{ is not invertible},\end{cases}$$

$l_2=H^1(G,[M_G]^{fl})\simeq Sha_w^1(G,[M_G]^{fl})$,
$l_3=Sha_w^1(G,(M_G)^\circ)\simeq Sha_w^2(G,([M_G]^{fl})^\circ)$,
$l_4=Sha_w^2(G,[M_G]^{fl})$ via the command ShaOmegaFromGroup($[M_G]^{fl},2,G$).
This function needs HAP package in GAP.

ConjugacyClassesSubgroups2TorusInvariants

```
▸ ConjugacyClassesSubgroups2TorusInvariants(G)
```

returns the records ConjugacyClassesSubgroups2 and TorusInvariants where ConjugacyClassesSubgroups2 is the list $[g_1, \ldots, g_m]$ of conjugacy classes of subgroups of $G \leq \mathrm{GL}(n, \mathbb{Z})$ with the fixed ordering via the function ConjugacyClassesSubgroups2(*G*) ([HY17, Section 4.1]) and TorusInvariants is the list [TorusInvariants(g_1), $\ldots$,TorusInvariants(g_m)] via the function TorusInvariants(*G*).

PossibilityOfStablyEquivalentSubdirectProducts

```
▸ PossibilityOfStablyEquivalentSubdirectProducts(G,G',
ConjugacyClassesSubgroups2TorusInvariants(G),
ConjugacyClassesSubgroups2TorusInvariants(G'))
```

returns the list l of the subdirect products $\widetilde{H} \leq G \times G'$ of G and G' up to $(\mathrm{GL}(n_1, \mathbb{Z}) \times \mathrm{GL}(n_2, \mathbb{Z}))$-conjugacy which satisfy $TI_{\varphi_1(H)} = TI_{\varphi_2(H)}$ for any $H \leq \widetilde{H}$ where $\widetilde{H} \leq G \times G'$ is a subdirect product of G and G' which acts on M_G and $M_{G'}$ through the surjections $\varphi_1 : \widetilde{H} \to G$ and $\varphi_2 : \widetilde{H} \to G'$ respectively (indeed, this function computes it for H up to conjugacy for the sake of saving time). In particular, if the length of the list l is zero, then we find that $[M_G]^{fl}$ and $[M_{G'}]^{fl}$ are not weak stably k-equivalent.

FlabbyResolutionLowRank

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▸ FlabbyResolutionLowRank(G).actionF
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returns the matrix representation of the action of G on F where F is a suitable flabby class of M_G ($[F] = [M_G]^{fl}$) with low rank by using backtracking techniques (see [HY17, Chapter 5], see also [HHY Algorithm 4.1 (3)]).

Each isomorphism class of irreducible permutation $\widetilde{H}$-lattices corresponds to a conjugacy class of subgroup H of $\widetilde{H}$ by $H \leftrightarrow \mathbb{Z}[\widetilde{H}/H]$. Let $H_1 = \{1\}, \ldots, H_r = \widetilde{H}$ be all conjugacy classes of subgroups of $\widetilde{H}$ whose ordering corresponds to the GAP function ConjugacyClassesSubgroups2($\widetilde{H}$) (see [HY17, Section 4.1, page 42]).

We suppose that $[F]=[F']$ as $\widetilde{H}$-lattices. Then we have

$$\left(\bigoplus_{i=1}^{r}\mathbb{Z}[\widetilde{H}/H_i]^{\oplus x_i}\right)\oplus F^{\oplus b_1}\ \simeq\ \left(\bigoplus_{i=1}^{r}\mathbb{Z}[\widetilde{H}/H_i]^{\oplus y_i}\right)\oplus F'^{\,\oplus b_1}\qquad(2)$$

where $b_1=1$. We write the equation (2) as

$$\bigoplus_{i=1}^{r}\mathbb{Z}[\widetilde{H}/H_i]^{\oplus a_i}\ \simeq\ (F-F')^{\oplus(-b_1)}\qquad(3)$$

formally where $a_i=x_i-y_i\in\mathbb{Z}$. Then we may consider " $F-F'$ " formally in the sene of (2). By computing some $\mathrm{GL}(n,\mathbb{Z})$-conjugacy class invariants, we will give a necessary condition for $[F]=[F']$.

Let $\{c_1,\ldots,c_r\}$ be a set of complete representatives of the conjugacy classes of $\widetilde{H}$. Let $A_i(c_j)$ be the matrix representation of the factor coset action of $c_j\in\widetilde{H}$ on $\mathbb{Z}[\widetilde{H}/H_i]$ and $B(c_j)$ be the matrix representation of the action of $c_j\in\widetilde{H}$ on $F-F'$.

By (3), for each $c_j\in\widetilde{H}$, we have

$$\sum_{i=1}^{r}a_i\,\mathrm{tr}\,A_i(c_j)+b_1\,\mathrm{tr}\,B(c_j)=0\qquad(4)$$

where $\mathrm{tr}\,A$ is the trace of the matrix A. Similarly, we consider the rank of $H^0=\widehat{Z}^0$. For each H_j, we get

$$\sum_{i=1}^{r}a_i\,\mathrm{rank}\,\widehat{Z}^0(H_j,\mathbb{Z}[\widetilde{H}/H_i])+b_1\,\mathrm{rank}\,\widehat{Z}^0(H_j,F-F')=0.\qquad(5)$$

Finally, we compute $\widehat{H}^0$. Let $Sy_p(A)$ be a p-Sylow subgroup of an abelian group A. $Sy_p(A)$ can be written as a direct product of cyclic groups uniquely. Let $n_{p,e}(Sy_p(A))$ be the number of direct summands of cyclic groups of order p^e. For each H_j,p,e, we get

$$\sum_{i=1}^{r}a_i\,n_{p,e}(Sy_p(\widehat{H}^0(H_j,\mathbb{Z}[\widetilde{H}/H_i])))+b_1\,n_{p,e}(Sy_p(\widehat{H}^0(H_j,F-F')))=$$

By the equalities (4), (5) and (6), we may get a system of linear equations in $a_1,\ldots,a_r,b_1$ over $\mathbb{Z}$. Namely, we have that $[F]=[F']$ as $\widetilde{H}$-lattices $\Longrightarrow$ there exist $a_1,\ldots,a_r\in\mathbb{Z}$ and $b_1=\pm1$ which satisfy (3) $\Longrightarrow$ this system of linear equations has an integer solution in $a_1,\ldots,a_r$ with $b_1=\pm1$.

In particular, if this system of linear equations has no integer solutions, then we conclude that $[F] \neq [F']$ as $\widetilde{H}$-lattices.

PossibilityOfStablyEquivalentFSubdirectProduct

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‣ PossibilityOfStablyEquivalentFSubdirectProduct(H~)
```

returns a basis $\mathcal{L} = \{l_1, \ldots, l_s\}$ of the solution space $\{[a_1, \ldots, a_r, b_1] \mid a_i, b_1 \in \mathbb{Z}\}$ of the system of linear equations which is obtained by the equalities (4), (5) and (6) and gives all possibilities that establish the equation (3) for a subdirect product $\widetilde{H} \leq G \times G'$ of G and G'.

PossibilityOfStablyEquivalentMSubdirectProduct

```
‣ PossibilityOfStablyEquivalentMSubdirectProduct(H~)
```

returns the same as PossibilityOfStablyEquivalentFSubdirectProduct(*H~*) but with respect to M_G and $M_{G'}$ instead of F and F'.

PossibilityOfStablyEquivalentFSubdirectProduct with "H2" option

```
‣ PossibilityOfStablyEquivalentFSubdirectProduct(H~:H2)
```

returns the same as PossibilityOfStablyEquivalentFSubdirectProduct(*H~*) but using also the additional equality

$$\sum_{i=1}^{r} a_i\, n_{p,e}(Sy_p(H^2(\widetilde{H}, \mathbb{Z}[\widetilde{H}/H_i]))) + b_1\, n_{p,e}(Sy_p(H^2(\widetilde{H}, F - F'))) = 0$$

and the equalities (4), (5) and (6).

PossibilityOfStablyEquivalentMSubdirectProduct with "H2" option

```
‣ PossibilityOfStablyEquivalentMSubdirectProduct(H~:H2)
```

returns the same as PossibilityOfStablyEquivalentFSubdirectProduct(*H~*:H2) but with respect to M_G and $M_{G'}$ instead of F and F'.

In general, we will provide a method in order to confirm the isomorphism

$$\left(\bigoplus_{i=1}^{r} \mathbb{Z}[\widetilde{H}/H_i]^{\oplus a_i}\right) \oplus F^{\oplus b_1} \simeq \left(\bigoplus_{i=1}^{r} \mathbb{Z}[\widetilde{H}/H_i]^{\oplus a_i'}\right) \oplus F'^{\,\oplus b_1'} \qquad (8)$$

with $a_i, a_i' \geq 0$, $b_1, b_1' \geq 1$, although it is needed by trial and error.

Let G_1 (resp. G_2) be the matrix representation group of the action of $\widetilde{H}$ on the left-hand side $(\oplus_{i=1}^r \mathbb{Z}[\widetilde{H}/H_i]^{\oplus a_i}) \oplus F^{\oplus b_1}$ (resp. the right-hand side $(\oplus_{i=1}^r \mathbb{Z}[\widetilde{H}/H_i]^{\oplus a_i'}) \oplus F^{\oplus b_1'}$) of the isomorphism (8). Let $\mathcal{P} = \{P_1, \ldots, P_m\}$ be a basis of the solution space of $G_1 P = P G_2$ where $m = \mathrm{rank}_{\mathbb{Z}}\ \mathrm{Hom}(G_1, G_2) = \mathrm{rank}_{\mathbb{Z}}\ \mathrm{Hom}_{\widetilde{H}}(M_{G_1}, M_{G_2})$. Our aim is to find the matrix P which satisfies $G_1 P = P G_2$ by using computer effectively. If we can get a matrix P with det $P = \pm 1$, then G_1 and G_2 are $\mathrm{GL}(n, \mathbb{Z})$-conjugate where n is the rank of both sides of (8) and hence the isomorphism (8) established. This implies that the flabby class $[F^{\oplus b_1}] = [F'^{\oplus b_1'}]$ as $\widetilde{H}$-lattices.

StablyEquivalentFCheckPSubdirectProduct

```
‣ StablyEquivalentFCheckPSubdirectProduct(H~,l1,l2)
```

returns a basis $\mathcal{P} = \{P_1, \ldots, P_m\}$ of the solution space of $G_1 P = P G_2$ where $m = \mathrm{rank}_{\mathbb{Z}}\ \mathrm{Hom}(G_1, G_2)$ and G_1 (resp. G_2) is the matrix representation group of the action of $\widetilde{H}$ on $(\oplus_{i=1}^r \mathbb{Z}[\widetilde{H}/H_i]^{\oplus a_i}) \oplus F^{\oplus b_1}$ (resp. $(\oplus_{i=1}^r \mathbb{Z}[\widetilde{H}/H_i]^{\oplus a_i'}) \oplus F'^{\oplus b_1'}$) with the isomorphism (8) for a subdirect product $\widetilde{H} \leq G \times G'$ of G and G', and lists $l_1 = [a_1, \ldots, a_r, b_1]$, $l_2 = [a_1', \ldots, a_r', b_1']$, if P exists. If such P does not exist, this returns [ ].

StablyEquivalentMCheckPSubdirectProduct

```
‣ StablyEquivalentMCheckPSubdirectProduct(H~,l1,l2)
```

returns the same as StablyEquivalentFCheckPSubdirectProduct(*H~*,*l1*,*l2*) but with respect to M_G and $M_{G'}$ instead of F and F'.

StablyEquivalentFCheckMatSubdirectProduct

```
‣ StablyEquivalentFCheckMatSubdirectProduct(H~,l1,l2,P)
```

returns true if $G_1 P = P G_2$ and det $P = \pm 1$ where G_1 (resp. G_2) is the matrix representation group of the action of $\widetilde{H}$ on $(\oplus_{i=1}^r \mathbb{Z}[\widetilde{H}/H_i]^{\oplus a_i}) \oplus F^{\oplus b_1}$ (resp. $(\oplus_{i=1}^r \mathbb{Z}[\widetilde{H}/H_i]^{\oplus a_i'}) \oplus F'^{\oplus b_1'}$) with the isomorphism (8) for a subdirect product $\widetilde{H} \leq G \times G'$ of G and G', and lists $l_1 = [a_1, \ldots, a_r, b_1]$, $l_2 = [a_1', \ldots, a_r', b_1']$. If not, this returns false.

StablyEquivalentMCheckMatSubdirectProduct

```
‣ StablyEquivalentMCheckMatSubdirectProduct(H~,l1,l2,P)
```

returns the same as StablyEquivalentFCheckMatSubdirectProduct(*H~*,*l1*,*l2*,*P*) but with respect to M_G and $M_{G'}$ instead of F and F'.

StablyEquivalentFCheckGenSubdirectProduct

```
‣ StablyEquivalentFCheckGenSubdirectProduct(H~,l1,l2)
```

returns the list $[\mathcal{M}_1, \mathcal{M}_2]$ where $\mathcal{M}_1 = [g_1, \ldots, g_t]$ (resp. $\mathcal{M}_2 = [g_1', \ldots, g_t']$) is a list of the generators of G_1 (resp. G_2) which is the matrix representation group of the action of $\widetilde{H}$ on $(\oplus_{i=1}^r \mathbb{Z}[\widetilde{H}/H_i]^{\oplus a_i}) \oplus F^{\oplus b_1}$ (resp. $(\oplus_{i=1}^r \mathbb{Z}[\widetilde{H}/H_i]^{\oplus a_i'}) \oplus F'^{\oplus b_1'}$) with the isomorphism (8) for a subdirect product $\widetilde{H} \leq G \times G'$ of G and G', and lists $l_1 = [a_1, \ldots, a_r, b_1]$, $l_2 = [a_1', \ldots, a_r', b_1']$.

StablyEquivalentMCheckGenSubdirectProduct

```
‣ StablyEquivalentMCheckGenSubdirectProduct(H~,l1,l2)
```

returns the same as StablyEquivalentMCheckGenSubdirectProduct(*H~*,*l1*,*l2*) but with respect to M_G and $M_{G'}$ instead of F and F'.

By applying the function StablyEquivalentFCheckPSubdirectProduct, we get a basis $\mathcal{P} = \{P_1, \ldots, P_m\}$ of the solution space of $G_1 P = P G_2$ with det $P_i = \pm 1$ for some $1 \leq i \leq m$ where G_1 (resp. G_2) is the matrix representation group of the action of $\widetilde{H}$ on the left-hand side $(\oplus_{i=1}^r \mathbb{Z}[\widetilde{H}/H_i]^{\oplus a_i}) \oplus F^{\oplus b_1}$ (resp. the right-hand side $(\oplus_{i=1}^r \mathbb{Z}[\widetilde{H}/H_i]^{\oplus a_i'}) \oplus F'^{\oplus b_1'}$) of the isomorphism (8) and $m = \mathrm{rank}_{\mathbb{Z}} \mathrm{Hom}(G_1, G_2)$.

However, in general, we have that det $P_i \neq \pm 1$ for any $1 \leq i \leq m$. In the general case, we should seek a matrix P with det $P = \pm 1$ which is given as a linear combination $P = \sum_{i=1}^m c_i P_i$. This task is important for us and not easy in general even if we use a computer.

We made the following GAP algorithms which may find a matrix $P = \sum_{i=1}^m c_i P_i$ with $G_1 P = P G_2$ and det $P = \pm 1$.

We will explain the algorithms below when the input $\mathcal{P}$ is obtained by StablyEquivalentFCheckPSubdirectProduct($\widetilde{H}, l_1, l_2$) *although it works in more general situations.*

SearchPRowBlocks

```
‣ SearchPRowBlocks(P)
```

returns the records bpBlocks and rowBlocks where bpBlocks (resp. rowBlocks) is the decomposition of the list $l = [1, \ldots, m]$ (resp. $l = [1, \ldots, n]$) with $m = \mathrm{rank}_{\mathbb{Z}} \mathrm{Hom}(G_1, G_2)$ (resp. $n = \mathrm{size}\ G_1$) according to the direct sum decomposition of M_{G_1} for a basis $\mathcal{P} = \{P_1, \ldots, P_m\}$ of the solution space of $G_1 P = P G_2$ where G_1 (resp. G_2) is the matrix representation group of the action of $\widetilde{H}$ on the left-hand side $(\oplus_{i=1}^r \mathbb{Z}[\widetilde{H}/H_i]^{\oplus a_i}) \oplus F^{\oplus b_1}$ (resp. the right-hand side $(\oplus_{i=1}^r \mathbb{Z}[\widetilde{H}/H_i]^{\oplus a_i'}) \oplus F'^{\oplus b_1'})$ of the isomorphism (8).

We write $B[t] =$ SearchPRowBlocks($\mathcal{P}$).bpBlocks[t] and $R[t] =$ SearchPRowBlocks($\mathcal{P}$).rowBlocks[t].

SearchPFilterRowBlocks

```
‣ SearchPFilterRowBlocks(P,B[t],R[t],j)
```

returns the lists $\{M_s\}$ where M_s is the $n_t \times n$ matrix with all invariant factors 1 which is of the form $M_s = \sum_{i\,\in\,B[t]} c_i P_i'$ $(c_i \in \{0,1\})$ at most j non-zero c_i's and P_i' is the submatrix of P_i consists of $R[t]$ rows with $n_t = \mathrm{length}(R[t]$ $)$ for a basis $\mathcal{P} = \{P_1, \ldots, P_m\}$ of the solution space of $G_1 P = P G_2$ where G_1 (resp. G_2) is the matrix representation group of the action of $\widetilde{H}$ on the left-hand side $(\oplus_{i=1}^r \mathbb{Z}[\widetilde{H}/H_i]^{\oplus a_i}) \oplus F^{\oplus b_1}$ (resp. the right-hand side $(\oplus_{i=1}^r \mathbb{Z}[\widetilde{H}/H_i]^{\oplus a_i'}) \oplus F'^{\oplus b_1'})$ of the isomorphism (8), $B[t] =$ SearchPRowBlocks($\mathcal{P}$).bpBlocks[t], $R[t] =$ SearchPRowBlocks($\mathcal{P}$).rowBlocks[t]) and $j \geq 1$.

```
‣ SearchPFilterRowBlocks(P,B[t],R[t],j,C)
```

returns the same as SearchPFilterRowBlocks(*P*,*B*[*t*],*R*[*t*],*j*) but with respect to $c_i \in C$ instead of $c_i \in \{0,1\}$ for the list C of integers.

SearchPFilterRowBlocksRandomMT

```
‣ SearchPFilterRowBlocksRandomMT(P,B[t],R[t],u)
```

returns the same as SearchPFilterRowBlocks(*P*,*B*[*t*],*R*[*t*],*j*) but with respect to random u c_i's via Mersenne Twister instead of at most j non-zero c_i's for integer $u \geq 1$.

```
‣ SearchPFilterRowBlocksRandomMT(P,B[t],R[t],u,C)
```

returns the same as SearchPFilterRowBlocksRandomMT(*P*,*B*[*t*],*R*[*t*],*u*) but with respect to $c_i \in C$ instead of $c_i \in \{0,1\}$ for the list C of integers.

SearchPMergeRowBlock

```
‣ SearchPMergeRowBlock(m1,m2)
```

returns all concatenations of the matrices M_s and M_t vertically with all invariant factors 1 (resp. a concatenation of the matrices M_s and M_t vertically with determinant ± 1) for $m_1 = \{M_s\}$ and $m_2 = \{M_t\}$ where M_s are $n_1 \times n$ matrices and M_t are $n_2 \times n$ matrices with $n_1 + n_2 < n$ (resp. $n_1 + n_2 = n$).

When there exists $t \in \mathbb{Z}$ such that $R[t] = \{j\}$, we can use:

SearchPLinear

```
▸ SearchPLinear(M,P1)
```

returns the list $\{\det(M+P_i)\}_{i\,\in\,B[t]}$ of integers for an $n\times n$ matrix M which is obtained by inserting the zero row into the j-th row of $(n-1)\times n$ matrix $M_s=\sum_{i\,\notin\,B[t]} c_iP_i'$ with all invariant factors 1 and $\mathcal{P}_1=\{P_i\}_{i\,\in\,B[t]}$ where $B[t]=$ SearchPRowBlocks($\mathcal{P}$).bpBlocks[t], P_i' is the submatrix of P_i deleting the j-th row, and $\mathcal{P}=\{P_1,\ldots,P_m\}$ is obtained by StablyEquivalentFCheckPSubdirectProduct($\widetilde{H}$,l_1,l_2) under the assumption that there exists $t\in\mathbb{Z}$ such that $R[t]=\{j\}$.

When there exist $t_1,t_2\in\mathbb{Z}$ such that $R[t_1]=\{j_1\}$, $R[t_2]=\{j_2\}$, we can use:

SearchPBilinear

```
▸ SearchPBilinear(M,P1,P2)
```

returns the matrix $[\det(M+P_{i_1}+P_{i_2})]_{i_1\,\in\,B[t_1],i_2\,\in\,B[t_2]}$ for an $n\times n$ matrix M which is obtained by inserting the two zero rows into the j_1-th row and the j_2-th row of $(n-2)\times n$ matrix $M_s=\sum_{i\,\notin\,B[t_1]\,\cup\,B[t_2]} c_iP_i'$ with all invariant factors 1 and $\mathcal{P}_1=\{P_{i_1}\}_{i_1\in B[t_1]}$, $\mathcal{P}_2=\{P_{i_2}\}_{i_2\in B[t_2]}$, where $B[t_1]=$ SearchPRowBlocks($\mathcal{P}$).bpBlocks[t_1], $B[t_2]=$ SearchPRowBlocks($\mathcal{P}$).bpBlocks[t_2], P_i' is the submatrix of P_i deleting the j_1-th and the j_2-th rows, and $\mathcal{P}=\{P_1,\ldots,P_m\}$ is obtained by StablyEquivalentFCheckPSubdirectProduct($\widetilde{H}$,l_1,l_2) under the assumption that there exist $t_1,t_2\in\mathbb{Z}$ such that $R[t_1]=\{j_1\}$ and $R[t_2]=\{j_2\}$.

When there exists $t\in\mathbb{Z}$ such that $R[t]=\{j_1,j_2\}$, we can use:

SearchPQuadratic

```
▸ SearchPQuadratic(M,P1)
```

returns the matrix $[\frac{1}{2}(\det(M+P_{i_1}+P_{i_2})-\det(M+P_{i_1})-\det(M+P_{i_2}))]_{i_1,i_2\,\in\,B[t]}$ for an $n\times n$ matrix M which is obtained by inserting the two zero rows into the j_1-th row and the j_2-th row of $(n-2)\times n$ matrix $M_s=\sum_{i\,\notin\,B[t]} c_iP_i'$ with all invariant factors 1 and $\mathcal{P}_1=\{P_i\}_{i\in B[t]}$, where $B[t]=$ SearchPRowBlocks($\mathcal{P}$).bpBlocks[t], P_i' is the submatrix of P_i deleting the j_1-th and j_2-th rows and $\mathcal{P}=\{P_1,\ldots,P_m\}$ is obtained by StablyEquivalentFCheckPSubdirectProduct($\widetilde{H}$,l_1,l_2) under the assumption that there exists $t\in\mathbb{Z}$ such that $R[t]=\{j_1,j_2\}$.

When $R[1] = \{1, \ldots, m\}$, we can use:

SearchP1

```
‣ SearchP1(P)
```

returns a matrix $P = \sum_{i=1}^{m} c_i P_i$ with $c_i \in \{0, 1\}$, $G_1 P = P G_2$ and det $P = \pm 1$ where G_1 (resp. G_2) is the matrix representation group of the action of $\widetilde{H}$ on the left-hand side $(\oplus_{i=1}^{r} \mathbb{Z}[\widetilde{H}/H_i]^{\oplus a_i}) \oplus F^{\oplus b_1}$ (resp. the right-hand side $(\oplus_{i=1}^{r} \mathbb{Z}[\widetilde{H}/H_i]^{\oplus a_i'}) \oplus F'^{\oplus b_1'}$) of the isomorphism (8) for $\mathcal{P} = \{P_1, \ldots, P_m\}$ which is obtained by StablyEquivalentFCheckPSubdirectProduct($\widetilde{H}$,l_1,l_2) under the assumption that R[1]$= \{1, \ldots, m\}$.

```
‣ SearchP1(P,C)
```

returns the same as SearchP1(P) but with respect to $c_i \in C$ instead of $c_i \in \{0, 1\}$ for the list C of integers.

Endomorphismring

```
‣ Endomorphismring(G)
```

returns a $\mathbb{Z}$-basis of $\mathrm{End}_{\mathbb{Z}[G]}(M_G)$ for a finite subgroup G of $\mathrm{GL}(n, \mathbb{Z})$.

IsCodimJacobsonEnd1

```
‣ IsCodimJacobsonEnd1(G,p)
```

returns true (resp. false) if $\dim_{\mathbb{Z}/p\mathbb{Z}}(E/pE)/J(E/pE) = 1$ (resp. $\neq 1$) where $E = \mathrm{End}_{\mathbb{Z}[G]}(M_G)$ for a finite subgroup G of $\mathrm{GL}(n, \mathbb{Z})$ and prime number p. If this returns true, then $M_G \otimes_{\mathbb{Z}} \mathbb{Z}_p$ is an indecomposable $\mathbb{Z}_p[G]$-lattice. In particular, M_G is an indecomposable G-lattice (see [HY, Lemma 6.11]).

IdempotentsModp

```
‣ IdempotentsModp(B,p)
```

returns all idempotents of R/pR for a $\mathbb{Z}$-basis B of a subring R of $n \times n$ matrices $M(n, \mathbb{Z})$ over $\mathbb{Z}$ and prime number p. If this returns only the zero and the identity matrices when $R = \mathrm{End}_{\mathbb{Z}[G]}(M_G)$, then $M_G \otimes_{\mathbb{Z}} \mathbb{Z}_p$ is an indecomposable $\mathbb{Z}_p[G]$-lattice. In particular, M_G is an indecomposable G-lattice (see [HY, Lemma 6.10]).

ConjugacyClassesSubgroups2WSEC

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‣ ConjugacyClassesSubgroups2WSEC(G)
```

returns the records ConjugacyClassesSubgroups2 and WSEC where ConjugacyClassesSubgroups2 is the list $[g_1, \ldots, g_m]$ of conjugacy classes of subgroups of $G \leq \mathrm{GL}(n, \mathbb{Z})$ $(n = 3, 4)$ with the fixed ordering via the function ConjugacyClassesSubgroups2(G) (see [HY17, Section 4.1]) and WSEC is the list $[w_1, \ldots, w_m]$ where g_i is in the w_i-th weak stably k-equivalent class WSEC_{w_i} in dimension n.

MaximalInvariantNormalSubgroup

```
‣ MaximalInvariantNormalSubgroup(G,ConjugacyClassesSubgroups2WSEC(G))
```

returns the maximal normal subgroup N of G which satisfies that $\pi(H_1) = \pi(H_2)$ implies $\psi(H_1) = \psi(H_2)$ for any $H_1, H_2 \leq G$ where $\pi : G \to G/N$ is the natural homomorphism, $\psi : H_i \mapsto w_i$, and H_i is in the w_i-th weak stably k-equivalent class WSEC_{w_i} in dimension n.

PossibilityOfStablyEquivalentSubdirectProducts with "WSEC" option

```
‣ PossibilityOfStablyEquivalentSubdirectProducts(G,G',
ConjugacyClassesSubgroups2WSEC(G),
ConjugacyClassesSubgroups2WSEC(G'),["WSEC"])
```

returns the list l of the subdirect products $\widetilde{H} \leq G \times G'$ of G and G' up to $(\mathrm{GL}(n_1, \mathbb{Z}) \times \mathrm{GL}(n_2, \mathbb{Z}))$-conjugacy which satisfy $w_1 = w_2$ for any $H \leq \widetilde{H}$ where $\varphi_i(H)$ is in the w_i-th weak stably k-equivalent class WSEC_{w_i} in dimension n $(n = 3, 4)$ and $\widetilde{H} \leq G \times G'$ is a subdirect product of G and G' which acts on M_G and $M_{G'}$ through the surjections $\varphi_1 : \widetilde{H} \to G$ and $\varphi_2 : \widetilde{H} \to G'$ respectively (indeed, this function computes it for H up to conjugacy for the sake of saving time).

IsomorphismFromSubdirectProduct

```
‣ IsomorphismFromSubdirectProduct(H~)
```

returns the isomorphism $\sigma : G/N \to G'/N'$ which satisfies $\sigma(\varphi_1(h)N) = \varphi_2(h)N'$ for any $h \in \widetilde{H}$ where $N = \varphi_1(\mathrm{Ker}(\varphi_2))$ and $N' = \varphi_2(\mathrm{Ker}(\varphi_1))$ for a subdirect product $\widetilde{H} \leq G \times G'$ of G and G' with surjections $\varphi_1 : \widetilde{H} \to G$ and $\varphi_2 : \widetilde{H} \to G'$.

AutGSubdirectProductsWSECInvariant

```
‣ AutGSubdirectProductsWSECInvariant(G)
```

returns subdirect products $\widetilde{H}_m = \{(g, g^{\sigma_m}) \mid g \in G, g^{\sigma_m} \in G^{\sigma_m}\}$

$(1 \leq m \leq s)$ of G and G^{σ_m} where $\{\sigma_1, \ldots, \sigma_s\}$ is a complete set of representatives of the double coset $X \backslash Z / X$,

$$\mathrm{Inn}(G) \leq X \leq Y \leq Z \leq \mathrm{Aut}(G),$$

$$X = \mathrm{Aut}_{\mathrm{GL}(n,\mathbb{Z})}(G) = \{\sigma \in \mathrm{Aut}(G) \mid G \text{ and } G^{\sigma} \text{ are conjugate in} \mathrm{GL}(n,\mathbb{Z})\}$$

$$Y = \{\sigma \in \mathrm{Aut}(G) \mid [M_G]^{fl} = [M_{G^{\sigma}}]^{fl} \text{ as } \widetilde{H}\text{-lattices where } \widetilde{H} = \{(g, g^{\sigma}) \mid g$$

$$Z = \{\sigma \in \mathrm{Aut}(G) \mid [M_H]^{fl} \sim [M_{H^{\sigma}}]^{fl} \text{ for any } H \leq G\},$$

$\mathrm{Inn}(G)$ is the group of inner automorphisms on G, $\mathrm{Aut}(G)$ is the group of automorphisms on G, $N_{\mathrm{GL}(n,\mathbb{Z})}(G)$ is the normalizer of G in $\mathrm{GL}(n,\mathbb{Z})$ and $Z_{\mathrm{GL}(n,\mathbb{Z})}(G)$ is the centralizer of G in $\mathrm{GL}(n,\mathbb{Z})$.

AutGSubdirectProductsWSECInvariantGen

```
‣ AutGSubdirectProductsWSECInvariantGen(G)
```

returns the same as AutGSubdirectProductsWSECInvariant(G) but with respect to $\{\sigma_1, \ldots, \sigma_t\}$ where $\sigma_1, \ldots, \sigma_t \in Z$ are some minimal number of generators of the double cosets of $X \backslash Z / X$, i.e. minimal number of elements $\sigma_1, \ldots, \sigma_t \in Z$ which satisfy $\langle \sigma_1, \ldots, \sigma_t, x \mid x \in X \rangle = Z$, instead of a complete set of representatives of the double coset $X \backslash Z / X$. If this returns [], then we get $X = Y = Z$.

AutGLnZ

```
‣ AutGLnZ(G)
```

returns

$$X = \mathrm{Aut}_{\mathrm{GL}(n,\mathbb{Z})}(G) = \{\sigma \in \mathrm{Aut}(G) \mid G \text{ and } G^{\sigma} \text{ are conjugate in } \mathrm{GL}(n,\mathbb{Z})\}$$

N3WSECMembersTable

```
‣ N3WSECMembersTable[r][i]
```

returns an integer j which satisfies that $N_{3,j}$ is the i-th group in the weak stably k-equivalent class WSEC_r.

N4WSECMembersTable

```
‣ N4WSECMembersTable[r][i]
```

is the same as N3WSECMembersTable[*r*][*i*] but using $N_{4,j}$ instead of $N_{3,j}$.

I4WSECMembersTable

```
‣ I4WSECMembersTable[r][i]
```

is the same as N3WSECMembersTable[*r*][*i*] but using $I_{4,j}$ instead of $N_{3,j}$.

AutGWSECINvariantSmallDegreeTest

```
‣ AutGWSECINvariantSmallDegreeTest(G)
```

returns the list $l = [l_1, \ldots, l_s]$ $(l_1 \leq \cdots \leq l_s)$ of integers with the minimal $l_s, \ldots, l_1$ which satisfies $Z = Z'$ where

$$Z = \{\sigma \in \mathrm{Aut}(G) \mid [M_H]^{fl} \sim [M_{H^\sigma}]^{fl} \text{ for any } H \leq G\},$$
$$Z' = \{\sigma \in \mathrm{Aut}(G) \mid [M_H]^{fl} \sim [M_{H^\sigma}]^{fl} \text{ for any } H \leq G \text{ with } [G:H] \in l\}$$

for $G \leq \mathrm{GL}(n, \mathbb{Z})$ $(n = 3, 4)$.

References

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[HY17] Akinari Hoshi and Aiichi Yamasaki, Rationality problem for algebraic tori, Mem. Amer. Math. Soc. **248** (2017) no. 1176, v+215 pp. AMS Preprint version: arXiv:1210.4525.
[HY] Akinari Hoshi and Aiichi Yamasaki, Birational classification for algebraic tori, arXiv:2112.02280.