- Return to Norm1ToriHNP

HNP.gap

Definition of $\mathrm{Obs}(K/k)$ and $\mathrm{Obs}_1(L/K/k)$ (Drakokhrust and Platonov [PD85a, page 350], [DP87, page 300]).

Let k be a number field, $L \supset K \supset k$ be a tower of finite extensions where L is normal over k. We call the group

$$\mathrm{Obs}(K/k) = (N_{K/k}(\mathbb{A}_K^\times) \cap k^\times)/N_{K/k}(K^\times)$$

the total obstruction to the Hasse norm principle for K/k and

$$\mathrm{Obs}_1(L/K/k) = \left(N_{K/k}(\mathbb{A}_K^\times) \cap k^\times\right) / \left((N_{L/k}(\mathbb{A}_L^\times) \cap k^\times)N_{K/k}(K^\times)\right)$$

the first obstruction to the Hasse norm principle for K/k *corresponding to the tower* $L \supset K \supset k$.

Theorem 1 (Drakokhrust and Platonov [PD85a, page 350], [PD85b, pages 789-790], [DP87, Theorem 1]).

Let k be a number field, $L \supset K \supset k$ be a tower of finite extensions where L is normal over k. Let $G = \mathrm{Gal}(L/k)$ and $H = \mathrm{Gal}(L/K)$. Then

$$\mathrm{Obs}_1(L/K/k) \simeq \mathrm{Ker}\,\psi_1/\varphi_1(\mathrm{Ker}\,\psi_2)$$

where

$$\begin{array}{ccc} H/[H,H] & \xrightarrow{\psi_1} & G/[G,G] \\ \uparrow{\scriptstyle \varphi_1} & & \uparrow{\scriptstyle \varphi_2} \\ \bigoplus_{v \in V_k} \left(\bigoplus_{w|v} H_w/[H_w,H_w] \right) & \xrightarrow{\psi_2} & \bigoplus_{v \in V_k} G_v/[G_v,G_v] \end{array}$$

where ψ_1, φ_1 and φ_2 are defined by the inclusions $H \subset G$, $H_w \subset H$ and $G_v \subset G$ respectively, and

$$\psi_2(h[H_w,H_w]) = x^{-1}hx[G_v,G_v]$$

for $h \in H_w = H \cap xG_vx^{-1}$ $(x \in G)$.

Let ψ_2^v be the restriction of ψ_2 to the subgroup $\bigoplus_{w|v} H_w/[H_w,H_w]$ with respect to $v \in V_k$ and ψ_2^{nr} (resp. ψ_2^{r}) be the restriction of ψ_2 to the unramified (resp. the ramified) places v of k.

Proposition 2 (Drakokhrust and Platonov [DP87]).

Let k, $L \supset K \supset k$, G and H be as in Theorem 1.
(i) ([DP87, Lemma 1]) Places $w_i \mid v$ of K are in one-to-one correspondence with the set of double cosets in the decomposition $G = \cup_{i=1}^{r_v} H x_i G_v$ where $H_{w_i} = H \cap x_i G_v x_i^{-1}$;
(ii) ([DP87, Lemma 2]) If $G_{v_1} \leq G_{v_2}$, then $\varphi_1(\mathrm{Ker}\,\psi_2^{v_1}) \subset \varphi_1(\mathrm{Ker}\,\psi_2^{v_2})$;
(iii) ([DP87, Theorem 2]) $\varphi_1(\mathrm{Ker}\,\psi_2^{\mathrm{nr}}) = \Phi^G(H)/[H,H]$ where $\Phi^G(H) = \langle [h,x] \mid h \in H \cap xHx^{-1}, x \in G \rangle$;
(iv) ([DP87, Lemma 8]) If $[K:k] = p^r$ $(r \geq 1)$ and $\mathrm{Obs}(K_p/k_p) = 1$ where $k_p = L^{G_p}$, $K_p = L^{H_p}$, G_p and H_p are p-Sylow subgroups of G and H respectively, then $\mathrm{Obs}(K/k) = 1$.

Theorem 3 (Drakokhrust and Platonov [DP87, Theorem 3, Corollary 1]).

Let k, $L \supset K \supset k$, G and H be as in Theorem 1. Let $H_i \leq G_i \leq G$ $(1 \leq i \leq m)$, $H_i \leq H \cap G_i$, $k_i = L^{G_i}$ and $K_i = L^{H_i}$. If $\mathrm{Obs}(K_i/k_i) = 1$ for any $1 \leq i \leq m$ and

$$\bigoplus_{i=1}^{m} \widehat{H}^{-3}(G_i,\mathbb{Z}) \xrightarrow{\mathrm{cores}} \widehat{H}^{-3}(G,\mathbb{Z})$$

is surjective, then $\mathrm{Obs}(K/k) = \mathrm{Obs}_1(L/K/k)$. In particular, if $[K:k] = n$ is square-free, then $\mathrm{Obs}(K/k) = \mathrm{Obs}_1(L/K/k)$.

Theorem 4 (Drakokhrust [Dra89, Theorem 1], see also Opolka [Opo80, Satz 3]).

Let k, $L \supset K \supset k$, G and H be as in Theorem 1. Let $\widetilde{L} \supset L \supset k$ be a tower of Galois extensions with $\widetilde{G} = \mathrm{Gal}(\widetilde{L}/k)$ and $\widetilde{H} = \mathrm{Gal}(\widetilde{L}/K)$ which correspond to a central extension $1 \to A \to \widetilde{G} \to G \to 1$ with $A \cap [\widetilde{G},\widetilde{G}] \simeq M(G)$; the Schur multiplier of G (this is equivalent to the inflation $M(\widetilde{G}) \to M(G)$ is zero map, see Beyl and Tappe [BT82, Proposition 2.13, page 85]). Then $\mathrm{Obs}(K/k) = \mathrm{Obs}_1(\widetilde{L}/K/k)$. In particular, $\widetilde{G}$ is a Schur cover of G, i.e. $A \simeq M(G)$, then $\mathrm{Obs}(K/k) = \mathrm{Obs}_1(\widetilde{L}/K/k)$.

FirstObstructionN

```
‣ FirstObstructionN(G,H).ker
```

returns the list $[l_1,[l_2,l_3]]$ where l_1 is the abelian invariant of the numerator of the first obstruction $\mathrm{Ker}\,\psi_1 = \langle y_1,\ldots,y_t \rangle$ with respect to G, H as in

Theorem 1, $l_2 = [e_1, \ldots, e_m]$ is the abelian invariant of $H^{ab} = H/[H,H] = \langle x_1, \ldots, x_m \rangle$ with $e_i = \mathrm{order}(x_i)$ and $l_3 = [l_{3,1}, \ldots, l_{3,t}]$, $l_{3,i} = [r_{i,1}, \ldots, r_{i,m}]$ is the list with $y_i = x_1^{r_{i,1}} \cdots x_m^{r_{i,m}}$ for $H \leq G \leq S_n$.

```
‣ FirstObstructionN(G).ker
```

returns the same as FirstObstructionN(*G*,*H*).ker where $H = \mathrm{Stab}_1(G)$ is the stabilizer of 1 in $G \leq S_n$.

FirstObstructionDnr

```
‣ FirstObstructionDnr(G,H).Dnr
```

returns the list $[l_1, [l_2, l_3]]$ where l_1 is the abelian invariant of the unramified part of the denominator of the first obstruction $\varphi_1(\mathrm{Ker}\,\psi_2^{\mathrm{nr}}) = \Phi^G(H)/[H,H] = \langle y_1, \ldots, y_t \rangle$ with respect to G, H as in Proposition 2 (iii), $l_2 = [e_1, \ldots, e_m]$ is the abelian invariant of $H^{ab} = H/[H,H] = \langle x_1, \ldots, x_m \rangle$ with $e_i = \mathrm{order}(x_i)$ and $l_3 = [l_{3,1}, \ldots, l_{3,t}]$, $l_{3,i} = [r_{i,1}, \ldots, r_{i,m}]$ is the list with $y_i = x_1^{r_{i,1}} \cdots x_m^{r_{i,m}}$ for $H \leq G \leq S_n$.

```
‣ FirstObstructionDnr(G).Dnr
```

returns the same as FirstObstructionDnr(*G*,*H*).Dnr where $H = \mathrm{Stab}_1(G)$ is the stabilizer of 1 in $G \leq S_n$.

FirstObstructionDr

```
‣ FirstObstructionDr(G,Gv,H).Dr
```

returns the list $[l_1, [l_2, l_3]]$ where l_1 is the abelian invariant of the ramified part of the denominator of the first obstruction $\varphi_1(\mathrm{Ker}\,\psi_2^{v}) = \langle y_1, \ldots, y_t \rangle$ with respect to G, G_v, H as in Theorem 1, $l_2 = [e_1, \ldots, e_m]$ is the abelian invariant of $H^{ab} = H/[H,H] = \langle x_1, \ldots, x_m \rangle$ with $e_i = \mathrm{order}(x_i)$ and $l_3 = [l_{3,1}, \ldots, l_{3,t}]$, $l_{3,i} = [r_{i,1}, \ldots, r_{i,m}]$ is the list with $y_i = x_1^{r_{i,1}} \cdots x_m^{r_{i,m}}$ for $H \leq G \leq S_n$.

```
‣ FirstObstructionDr(G,Gv).Dr
```

returns the same as FirstObstructionDr(*G*,*Gv*,*H*).Dr where $H = \mathrm{Stab}_1(G)$ is the stabilizer of 1 in $G \leq S_n$.

SchurCoverG

```
‣ SchurCoverG(G).SchurCover
```

returns one of the Schur covers $\widetilde{G}$ of G in a central extension $1 \to A \to \widetilde{G} \xrightarrow{\pi} G \to 1$ with $A \simeq M(G)$; Schur multiplier of G (see

Karpilovsky [Kap87, page 16]). The Schur covers $\widetilde{G}$ are stem extensions, i.e. $A \leq Z(\widetilde{G}) \cap [\widetilde{G}, \widetilde{G}]$, of the maximal size.

```
‣ SchurCoverG(G).epi
```

returns the surjective map π in a central extension $1 \to A \to \widetilde{G} \xrightarrow{\pi} G \to 1$ with $A \simeq M(G)$; Schur multiplier of G (see Karpilovsky [Kap87, page 16]). The Schur covers $\widetilde{G}$ are stem extensions, i.e. $A \leq Z(\widetilde{G}) \cap [\widetilde{G}, \widetilde{G}]$, of the maximal size.
These functions are based on the built-in function EpimorphismSchurCover in GAP.

MinimalStemExtensions

```
‣ MinimalStemExtensions(G)[j].MinimalStemExtension
```

(resp. MinimalStemExtensions(G)[j].epi) returns the j-th minimal stem extension $\overline{G} = \widetilde{G}/A'$, i.e. $\overline{A} \leq Z(\overline{G}) \cap [\overline{G}, \overline{G}]$, of G provided by the Schur cover $\widetilde{G}$ of G via SchurCoverG(G).SchurCover where A' is the j-th maximal subgroups of $A = M(G)$ (resp. the surjective map $\overline{\pi}$) in the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & A = M(G) & \longrightarrow & \widetilde{G} & \xrightarrow{\pi} & G & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & \overline{A} = A/A' & \longrightarrow & \overline{G} = \widetilde{G}/A' & \xrightarrow{\overline{\pi}} & G & \longrightarrow & 1 \end{array}$$

(see Robinson [Rob96, Exercises 11.4]).
This function is based on the built-in function EpimorphismSchurCover in GAP.

StemExtensions

```
‣ StemExtensions(G)[j].StemExtension
```

(resp. StemExtensions(G)[j].epi) returns the j-th stem extension $\overline{G} = \widetilde{G}/A'$, i.e. $\overline{A} \leq Z(\overline{G}) \cap [\overline{G}, \overline{G}]$, of G provided by the Schur cover $\widetilde{G}$ of G via SchurCoverG(G).SchurCover where A' is the j-th maximal subgroups of $A = M(G)$ (resp. the surjective map $\overline{\pi}$) in the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & A = M(G) & \longrightarrow & \widetilde{G} & \xrightarrow{\pi} & G & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & \overline{A} = A/A' & \longrightarrow & \overline{G} = \widetilde{G}/A' & \xrightarrow{\overline{\pi}} & G & \longrightarrow & 1 \end{array}$$

(see Robinson [Rob96, Exercises 11.4]).
This function is based on the built-in function EpimorphismSchurCover in GAP.

Resolution of G

```
‣ ResolutionNormalSeries(LowerCentralSeries(G),n+1)
```

returns a free resolution RG of G when G is nilpotent.

```
‣ ResolutionNormalSeries(DerivedSeries(G),n+1)
```

returns a free resolution RG of G when G is solvable.

```
‣ ResolutionFiniteGroup(G,n+1)
```

returns a free resolution RG of G when G is finite.
These functions are the built-in functions of HAP in GAP.

ResHnZ

```
‣ ResHnZ(RG,RH,n).HnGZ
```

returns the abelian invariants of $H^n(G,\mathbb{Z})$ with respect to Smith normal form, for free resolution RG of G.

```
‣ ResHnZ(RG,RH,n).HnHZ
```

returns the abelian invariants of $H^n(H,\mathbb{Z})$ with respect to Smith normal form, for free resolution RH of H.

```
‣ ResHnZ(RG,RH,n).Res
```

returns the list $L = [l_1, \ldots, l_s]$ where $H^n(G,\mathbb{Z}) = \langle x_1, \ldots, x_s \rangle \xrightarrow{\text{res}} H^n(H,\mathbb{Z}) = \langle y_1, \ldots, y_t \rangle$, $\text{res}(x_i) = \prod_{j=1}^{t} y_j^{l_{i,j}}$ and $l_i = [l_{i,1}, \ldots, l_{i,t}]$ for free resolutions RG and RH of G and H respectively.

```
‣ ResHnZ(RG,RH,n).Ker
```

returns the list $L = [l_1, [l_2, l_3]]$ where l_1 is the abelian invariant of $\text{Ker}\{H^n(G,\mathbb{Z}) \xrightarrow{\text{res}} H^n(H,\mathbb{Z})\} = \langle y_1, \ldots, y_t \rangle$, $l_2 = [d_1, \ldots, d_s]$ is the abelian invariant of $H^n(G,\mathbb{Z}) = \langle x_1, \ldots, x_s \rangle$ with $d_i = \text{ord}(x_i)$ and $l_3 = [l_{3,1}, \ldots, l_{3,t}]$, $l_{3,j} = [r_{j,1}, \ldots, r_{j,s}]$ is the list with $y_j = x_1^{r_{j,1}} \cdots x_s^{r_{j,s}}$ for free resolutions RG and RH of G and H respectively.

```
‣ ResHnZ(RG,RH,n).Coker
```

returns the list $L = [l_1, [l_2, l_3]]$ where $l_1 = [e_1, \ldots, e_t]$ is the abelian invariant of $\text{Coker}\{H^n(G,\mathbb{Z}) \xrightarrow{\text{res}} H^n(H,\mathbb{Z})\} = \langle \overline{y_1}, \ldots, \overline{y_t} \rangle$ with $e_j = \text{ord}(\overline{y_j})$, $l_2 = [d_1, \ldots, d_s]$ is the abelian invariant of $H^n(H,\mathbb{Z}) = \langle x_1, \ldots, x_s \rangle$ with $d_i = \text{ord}(x_i)$ and $l_3 = [l_{3,1}, \ldots, l_{3,t}]$, $l_{3,j} = [r_{j,1}, \ldots, r_{j,s}]$ is the list with

$\overline{y_j} = \overline{x_1}^{r_{j,1}} \cdots \overline{x_s}^{r_{j,s}}$ for free resolutions RG and RH of G and H respectively.

KerResH3Z

KerResH3Z was improved to ver.2019.10.04 from ver.2020.03.19

```
‣ KerResH3Z(G,H)
```

returns the list $L = [l_1, [l_2, l_3]]$ where l_1 is the abelian invariant of $\mathrm{Ker}\{H^3(G,\mathbb{Z}) \xrightarrow{\mathrm{res}} \oplus_{i=1}^{m'} H^3(G_i,\mathbb{Z})\} = \langle y_1, \ldots, y_t \rangle$ where $H_i \leq G_i \leq G$, $H_i \leq H \cap G_i$, $[G_i : H_i] = n$ and the action of G_i on $\mathbb{Z}[G_i/H_i]$ may be regarded as nTm $(n \leq 15)$ which is not in [HKY22, Table 1] or [HKY23, Table 1], $l_2 = [d_1, \ldots, d_s]$ is the abelian invariant of $H^3(G,\mathbb{Z}) = \langle x_1, \ldots, x_s \rangle$ with $d_{i'} = \mathrm{ord}(x_{i'})$ and $l_3 = [l_{3,1}, \ldots, l_{3,t}]$, $l_{3,j} = [r_{j,1}, \ldots, r_{j,s}]$ is the list with $y_j = x_1^{r_{j,1}} \cdots x_s^{r_{j,s}}$ for groups G and H (cf. Theorem 3).

KerResH3Z(G,H:iterator) was added to ver.2019.10.04 from ver.2020.03.19

```
‣ KerResH3Z(G,H:iterator)
```

returns the same as KerResH3Z(G, H) but using the built-in function IteratorByFunctions of GAP in order to run fast (by applying the new function ChooseGiIterator(G, H) to choose suitable $G_i \leq G$ instead of the old one ChooseGi(G, H)).

ConjugacyClassesSubgroupsNGHOrbitRep

ConjugacyClassesSubgroupsNGHOrbitRep was added to ver.2019.10.04 from ver.2020.03.19

```
‣ ConjugacyClassesSubgroupsNGHOrbitRep(ConjugacyClassesSubgroups(G),H)
```

returns the list $L = [l_1, \ldots, l_t]$ where t is the number of subgroups of G up to conjugacy, $l_r = [l_{r,1}, \ldots, l_{r,u_r}]$ $(1 \leq r \leq t)$, $l_{r,s}$ $(1 \leq s \leq u_r)$ is a representative of the orbit $\mathrm{Orb}_{N_G(H)\backslash G/N_G(G_{v_{r,s}})}(G_{v_{r,s}})$ of $G_{v_{r,s}} \leq G$ under the conjugate action of G which corresponds to the double coset $N_G(H)\backslash G/N_G(G_{v_{r,s}})$ with $\mathrm{Orb}_{G/N_G(G_{v_r})}(G_{v_r}) = \bigcup_{s=1}^{u_r} \mathrm{Orb}_{N_G(H)\backslash G/N_G(G_{v_{r,s}})}(G_{v_{r,s}})$ corresponding to r-th subgroup $G_{v_r} \leq G$ up to conjugacy.

MinConjugacyClassesSubgroups

MinConjugacyClassesSubgroups was added to ver.2019.10.04 from ver.2020.03.19

```
‣ MinConjugacyClassesSubgroups(L)
```

returns the minimal elements of the list $l = \{H_i^G\}$ where

$H_i^G = \{x^{-1}H_ix \mid x \in G\}$ for some subgroups $H_i \leq G$ which satisfy the condition that if $H_i^G, H_r^G \in l$ and $H_i \leq H_r$, then $H_j^G \in l$ for any $H_i \leq H_j \leq H_r$.

IsInvariantUnderAutG

IsInvariantUnderAutG was added to ver.2019.10.04 from ver.2020.03.19

```
‣ IsInvariantUnderAutG(l)
```

returns true if the list $l = \{H_i^G\}$ is closed under the action of all the automorphisms $\mathrm{Aut}(G)$ of G where $H_i^G = \{x^{-1}H_ix \mid x \in G\}$ $(H_i \leq G)$. If not, this returns false.

References

[BT82] F. R. Beyl, J. Tappe, *Group extensions, representations, and the Schur multiplicator*, Lecture Notes in Mathematics, 958. Springer-Verlag, Berlin-New York, 1982.
[Dra89] Yu. A. Drakokhrust, *On the complete obstruction to the Hasse principle*, (Russian) Dokl. Akad. Nauk BSSR **30** (1986) 5-8; translation in Amer. Math. Soc. Transl. (2) **143** (1989) 29-34.
[DP87] Yu. A. Drakokhrust, V. P. Platonov, *The Hasse norm principle for algebraic number fields*, (Russian) Izv. Akad. Nauk SSSR Ser. Mat. **50** (1986) 946-968; translation in Math. USSR-Izv. **29** (1987) 299-322.
[HKY22] A. Hoshi, K. Kanai, A. Yamsaki, *Norm one tori and Hasse norm principle*, Math. Comp. **91** (2022) 2431-2458. AMS Extended version: arXiv:1910.01469.
[HKY23] A. Hoshi, K. Kanai, A. Yamsaki, *Norm one tori and Hasse norm principle, II: Degree 12 case*, J. Number Theory **244** (2023) 84-110. ScienceDirect Extended version: arXiv:2003.08253.
[HKY] A. Hoshi, K. Kanai, A. Yamsaki, *Norm one tori and Hasse norm principle, III: Degree 16 case*, arXiv:2404.01362.
[HKY2] A. Hoshi, K. Kanai, A. Yamsaki, *Hasse norm principle for M_{11} and J_1 extensions*, arXiv:2210.09119.
[Kap87] G. Karpilovsky, *The Schur multiplier*, London Mathematical Society Monographs. New Series, 2. The Clarendon Press, Oxford University Press, New York, 1987.
[Opo80] H. Opolka, *Zur Auflösung zahlentheoretischer Knoten*, Math. Z. **173** (1980) 95-103.
[PD85a] V. P. Platonov, Yu. A. Drakokhrust, *On the Hasse principle for algebraic number fields*, (Russian) Dokl. Akad. Nauk SSSR **281** (1985) 793-797; translation in Soviet Math. Dokl. **31** (1985) 349-353.
[PD85b] V. P. Platonov, Yu. A. Drakokhrust, *The Hasse norm principle for primary extensions of algebraic number fields*, (Russian) Dokl. Akad. Nauk SSSR **285** (1985) 812-815; translation in Soviet Math. Dokl. **32** (1985) 789-792.
[Rob96] D. J. S. Robinson, *A course in the theory of groups*, Second edition. Graduate Texts in Mathematics, 80. Springer-Verlag, New York, 1996.

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