Birational classification for algebraic tori (I)

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§1. Rationality problem for algebraic tori T (1/3)

- k: a base field which is NOT algebraically closed! (TODAY)
- T: algebraic k-torus, i.e. k-form of a split torus;
  an algebraic group over k (group k-scheme) with $T \times_k \overline{k} \simeq (\mathbb{G}_{m,\overline{k}})^n$.

Rationality problem for algebraic tori

Whether T is k-rational?, i.e. $T \approx \mathbb{P}^n$? (birationally k-equivalent)

Let $R^{(1)}_{K/k}(\mathbb{G}_m)$ be the norm one torus of K/k, i.e. the kernel of the norm map $N_{K/k} : R_{K/k}(\mathbb{G}_m) \to \mathbb{G}_m$ where $R_{K/k}$ is the Weil restriction:

$$1 \longrightarrow R^{(1)}_{K/k}(\mathbb{G}_m) \longrightarrow R_{K/k}(\mathbb{G}_m) \xrightarrow{N_{K/k}} \mathbb{G}_m \longrightarrow 1.$$

$$\dim \qquad n-1 \qquad\qquad n \qquad\qquad 1$$

- $\exists 2$ algebraic k-tori T with $\dim(T) = 1$;
  the trivial torus $\mathbb{G}_m$ and $R^{(1)}_{K/k}(\mathbb{G}_m)$ with $[K : k] = 2$, are k-rational.

Rationality problem for algebraic tori T (2/3)

- $\exists 13$ algebraic k-tori T with $\dim(T) = 2$.

Theorem (Voskresenskii, 1967) 2-dim. algebraic tori T

T is k-rational.

- $\exists 73$ algebraic k-tori T with $\dim(T) = 3$.

Theorem (Kunyavskii, 1990) 3-dim. algebraic tori T

(i) $\exists 58$ algebraic k-tori T which are k-rational;
(ii) $\exists 15$ algebraic k-tori T which are not k-rational.

- What happens in higher dimensions?

k-tori T and G-lattices

- T: algebraic k-torus
  $\Longrightarrow \exists$ finite Galois extension L/k such that $T \times_k L \simeq (\mathbb{G}_{m,L})^n$.
- $G = \mathrm{Gal}(L/k)$ where L is the minimal splitting field.

Category of algebraic k-tori which split$/L$ $\overset{\text{duality}}{\longleftrightarrow}$ Category of G-lattices
(i.e. finitely generated $\mathbb{Z}$-free $\mathbb{Z}[G]$-module)

- $T \mapsto$ the character group $X(T) = \mathrm{Hom}(T, \mathbb{G}_m)$: G-lattice.
- $T = \mathrm{Spec}(L[M]^G)$ which splits$/L$ with $X(T) \simeq M \leftarrow\!\shortmid M$: G-lattice
- Tori of dimension $n \overset{1:1}{\longleftrightarrow}$ elements of the set $H^1(\mathcal{G}, \mathrm{GL}(n,\mathbb{Z}))$
  where $\mathcal{G} = \mathrm{Gal}(\overline{k}/k)$ since $\mathrm{Aut}(\mathbb{G}_m^n) = \mathrm{GL}(n,\mathbb{Z})$.
- k-torus T of dimension n is determined uniquely by the integral representation $h : \mathcal{G} \to \mathrm{GL}(n,\mathbb{Z})$ up to conjugacy, and the group $h(\mathcal{G})$ is a finite subgroup of $\mathrm{GL}(n,\mathbb{Z})$.
- The function field of T $\overset{\text{identified}}{\longleftrightarrow}$ $L(M)^G$: invariant field.

Rationality problem for algebraic tori T (3/3)

- L/k: Galois extension with $G = \mathrm{Gal}(L/k)$.
- $M = \bigoplus_{1 \le j \le n} \mathbb{Z} \cdot u_j$: G-lattice with a $\mathbb{Z}$-basis $\{u_1, \ldots, u_n\}$.
- G acts on $L(x_1, \ldots, x_n)$ by

$$\sigma(x_j) = \prod_{i=1}^{n} x_i^{a_{i,j}}, \quad 1 \le j \le n$$

for any $\sigma \in G$, when $\sigma(u_j) = \sum_{i=1}^{n} a_{i,j} u_i$, $a_{i,j} \in \mathbb{Z}$.
- $L(M) := L(x_1, \ldots, x_n)$ with this action of G.
- $\boxed{\text{The function field of algebraic } k\text{-torus } T \overset{\text{identified}}{\longleftrightarrow} L(M)^G}$

Rationality problem for algebraic tori T (2nd form)

Whether $L(M)^G$ is k-rational?
($=$ purely transcendental over k?; $L(M)^G = k(\exists t_1, \ldots, \exists t_n)$?)

Some definitions.

- K/k: a finite generated field extension.

Definition (stably rational)

K is called stably k-rational if $K(y_1, \ldots, y_m)$ is k-rational.

Definition (retract rational)

K is retract k-rational if $\exists k$-algebra (domain) $R \subset K$ such that
(i) K is the quotient field of R;
(ii) $\exists f \in k[x_1, \ldots, x_n]$ $\exists k$-algebra hom. $\varphi : R \to k[x_1, \ldots, x_n][1/f]$ and $\psi : k[x_1, \ldots, x_n][1/f] \to R$ satisfying $\psi \circ \varphi = 1_R$.

Definition (unirational)

K is k-unirational if $K \subset k(x_1, \ldots, x_n)$.

- k-rational $\Rightarrow$ stably k-rational $\Rightarrow$ retract k-rational $\Rightarrow$ k-unirational.
- $L(M)^G$ (resp. T) is always k-unirational.

Rationality problem for algebraic tori T (2-dim., 3-dim.)

- $\boxed{\text{The function field of } n\text{-dim. } T \overset{\text{identified}}{\longleftrightarrow} L(M)^G,\ G \leq \mathrm{GL}(n,\mathbb{Z})}$

- $\exists 13$ algebraic k-tori T with $\dim(T) = 2$.

Theorem (Voskresenskii, 1967) 2-dim. algebraic tori T (restated)

T is k-rational.

- $\exists 73$ $\mathbb{Z}$-coujugacy subgroups $G \leq \mathrm{GL}(3,\mathbb{Z})$ ($\exists 73$ 3-dim. algebraic k-tori T).

Theorem (Kunyavskii, 1990) 3-dim. algebraic tori T (precise form)

(i) T is k-rational $\iff$ T is stably k-rational $\iff$ T is retract k-rational $\iff$ $\exists G$: 58 groups;

(ii) T is not k-rational $\iff$ T is not stably k-rational $\iff$ T is not retract k-rational $\iff$ $\exists G$: 15 groups.

Rationality problem for algebraic tori T (4-dim.)

- The function field of n-dim. $T \overset{\text{identified}}{\longleftrightarrow} L(M)^G$, $G \leq \mathrm{GL}(n, \mathbb{Z})$
- $\exists 710$ $\mathbb{Z}$-coujugacy subgroups $G \leq \mathrm{GL}(4, \mathbb{Z})$
  ($\exists 710$ 4-dim. algebraic k-tori T).

Theorem ([HY17]) 4-dim. algebraic tori T

(i) T is stably k-rational $\iff$ $\exists G$: 487 groups;
(ii) T is not stably but retract k-rational $\iff$ $\exists G$: 7 groups;
(iii) T is not retract k-rational $\iff$ $\exists G$: 216 groups.

- We do not know "k-rationality".
- Voskresenskii's conjecture:
  any stably k-rational torus is k-rational (Zariski problem).
- what happens for dimension 5?

Rationality problem for algebraic tori T (5-dim.)

- $\boxed{\text{The function field of } n\text{-dim. } T \overset{\text{identified}}{\longleftrightarrow} L(M)^G,\ G \leq \mathrm{GL}(n, \mathbb{Z})}$
- $\exists 6079$ $\mathbb{Z}$-coujugacy subgroups $G \leq \mathrm{GL}(5, \mathbb{Z})$ ($\exists 6079$ 5-dim. algebraic k-tori T).

Theorem ([HY17]) 5-dim. algebraic tori T

(i) T is stably k-rational $\iff$ $\exists G$: 3051 groups;
(ii) T is not stably but retract k-rational $\iff$ $\exists G$: 25 groups;
(iii) T is not retract k-rational $\iff$ $\exists G$: 3003 groups.

- what happens for dimension 6?
- BUT we do not know the answer for dimension 6.
- $\exists 85308$ $\mathbb{Z}$-coujugacy subgroups $G \leq \mathrm{GL}(6, \mathbb{Z})$ ($\exists 85308$ 6-dim. algebraic k-tori T).

Flabby (Flasque) resolution

- M: G-lattice, i.e. f.g. $\mathbb{Z}$-free $\mathbb{Z}[G]$-module.

Definition

(i) M is permutation $\overset{\text{def}}{\iff}$ $M \simeq \oplus_{1\leq i\leq m}\mathbb{Z}[G/H_i]$.

(ii) M is stably permutation $\overset{\text{def}}{\iff}$ $M \oplus \exists P \simeq P'$, P, P': permutation.

(iii) M is invertible $\overset{\text{def}}{\iff}$ $M \oplus \exists M' \simeq P$: permutation.

(iv) M is coflabby $\overset{\text{def}}{\iff}$ $H^1(H, M) = 0$ $(\forall H \leq G)$.

(v) M is flabby $\overset{\text{def}}{\iff}$ $\widehat{H}^{-1}(H, M) = 0$ $(\forall H \leq G)$. ($\widehat{H}$: Tate cohomology)

- "permutation"
  $\Longrightarrow$ "stably permutation"
  $\Longrightarrow$ "invertible"
  $\Longrightarrow$ "flabby and coflabby".

Commutative monoid $\mathcal{M}$

$M_1 \sim M_2 \overset{\text{def}}{\iff} M_1 \oplus P_1 \simeq M_2 \oplus P_2$ ($\exists P_1, \exists P_2$: permutation).
$\Longrightarrow$ commutative monoid $\mathcal{M}$: $[M_1] + [M_2] := [M_1 \oplus M_2]$, $0 = [P]$.

Theorem (Endo-Miyata, 1974, Colliot-Thélène-Sansuc, 1977)

$\exists P$: permutation, $\exists F$: flabby such that

$$0 \to M \to P \to F \to 0\text{: flabby resolution of } M.$$

- $[M]^{fl} := [F]$; flabby class of M

Theorem (Endo-Miyata, 1973, Voskresenskii, 1974, Saltman, 1984)

(EM73) $[M]^{fl} = 0 \iff L(M)^G$ is stably k-rational.
(Vos74) $[M]^{fl} = [M']^{fl} \iff L(M)^G(x_1, \ldots, x_m) \simeq L(M')^G(y_1, \ldots, y_n)$;
stably k-equivalent.
(Sal84) $[M]^{fl}$ is invertible $\iff L(M)^G$ is retract k-rational.

- $M = M_G \simeq \widehat{T}$, $k(T) \simeq L(M)^G$

§2. Birational classification for algebraic tori

(Stably) birational classification for algebraic tori

For given two algebraic k-tori T and T',

whether T and T' are stably birationally k-equivalent?, i.e. $T \stackrel{\text{s.b.}}{\approx} T'$?

Theorem (Colliot-Thélène and Sansuc, 1977) $\dim(T) = \dim(T') = 3$

Let L/k and L'/k be Galois extensions with $\mathrm{Gal}(L/k) \simeq \mathrm{Gal}(L'/k) \simeq C_2^2$. Let $T = R^{(1)}_{L/k}(\mathbb{G}_m)$ and $T' = R^{(1)}_{L'/k}(\mathbb{G}_m)$ be the corresponding norm one tori. Then $T \stackrel{\text{s.b.}}{\approx} T'$ (stably birationally k-equivalent) if and only if $L = L'$.

- In particular, if k is a number field, then there exist infinitely many stably birationally k-equivalent classes of (non-rational: $1\text{st}/15$) k-tori which correspond to U_1 (cf. Main theorem 1, later).

- $\overline{k}$: a fixed separable closure of k and $\mathcal{G} = \mathrm{Gal}(\overline{k}/k)$
- X: a smooth k-compactification of T, i.e. smooth projective k-variety X containing T as a dense open subvariety
- $\overline{X} = X \times_k \overline{k}$

Theorem (Voskresenskii, 1969, 1970)

There exists an exact sequence of $\mathcal{G}$-lattices

$$0 \to \widehat{T} \to \widehat{Q} \to \mathrm{Pic}\,\overline{X} \to 0$$

where $\widehat{Q}$ is permutation and $\mathrm{Pic}\,\overline{X}$ is flabby.

- $M_G \simeq \widehat{T}$, $[\widehat{T}]^{fl} = [\mathrm{Pic}\,\overline{X}]$ as $\mathcal{G}$-lattices

Theorem (Voskresenskii, 1970, 1973)

(i) T is stably k-rational if and only if $[\mathrm{Pic}\,\overline{X}] = 0$ as a $\mathcal{G}$-lattice.

(ii) $T \overset{\text{s.b.}}{\approx} T'$ (stably birationally k-equivalent) if and only if $[\mathrm{Pic}\,\overline{X}] = [\mathrm{Pic}\,\overline{X'}]$ as $\mathcal{G}$-lattices.

- From $\mathcal{G}$-lattice to G-lattice

Let L be the minimal splitting field of T with $G = \mathrm{Gal}(L/k) \simeq \mathcal{G}/\mathcal{H}$. We obtain a flabby resolution of $\widehat{T}$:

$$0 \to \widehat{T} \to \widehat{Q} \to \mathrm{Pic}\, X_L \to 0$$

with $[\widehat{T}]^{fl} = [\mathrm{Pic}\ X_L]$ as G-lattices.

By the inflation-restriction exact sequence $0 \to H^1(G, \mathrm{Pic}\, X_L) \xrightarrow{\mathrm{inf}} H^1(k, \mathrm{Pic}\, \overline{X}) \xrightarrow{\mathrm{res}} H^1(L, \mathrm{Pic}\, \overline{X})$, we get $\mathrm{inf} : H^1(G, \mathrm{Pic}\, X_L) \xrightarrow{\sim} H^1(k, \mathrm{Pic}\, \overline{X})$ because $H^1(L, \mathrm{Pic}\, \overline{X}) = 0$. We get:

Theorem (Voskresenskii, 1970, 1973)

(ii)$'$ $T \overset{\text{s.b.}}{\approx} T'$ (stably birationally k-equivalent) if and only if $[\mathrm{Pic}\, X_{\widetilde{L}}] = [\mathrm{Pic}\, X'_{\widetilde{L}}]$ as $\widetilde{H}$-lattices where $\widetilde{L} = LL'$ and $\widetilde{H} = \mathrm{Gal}(\widetilde{L}/k)$.

The group $\widetilde{H}$ becomes a *subdirect product* of $G = \mathrm{Gal}(L/k)$ and $G' = \mathrm{Gal}(L'/k)$, i.e. a subgroup $\widetilde{H}$ of $G \times G'$ with surjections $\varphi_1 : \widetilde{H} \twoheadrightarrow G$ and $\varphi_2 : \widetilde{H} \twoheadrightarrow G'$.

- This observation yields a concept of "*weak stably k-equivalence*".

Definition

(i) $[M]^{fl}$ and $[M']^{fl}$ are *weak stably k-equivalent*, if there exists a subdirect product $\widetilde{H} \leq G \times G'$ of G and G' with surjections $\varphi_1 : \widetilde{H} \twoheadrightarrow G$ and $\varphi_2 : \widetilde{H} \twoheadrightarrow G'$ such that $[M]^{fl} = [M']^{fl}$ as $\widetilde{H}$-lattices where $\widetilde{H}$ acts on M (resp. M') through the surjection φ_1 (resp. φ_2).
(ii) Algebraic k-tori T and T' are *weak stably birationally k-equivalent*, denoted by $T \overset{\text{s.b.}}{\sim} T'$, if $[\widehat{T}]^{fl}$ and $[\widehat{T}']^{fl}$ are weak stably k-equivalent.

Remark

(1) $T \overset{\text{s.b.}}{\approx} T'$ (birational k-equiv.) $\Rightarrow T \overset{\text{s.b.}}{\sim} T'$ (weak birational k-equiv.).
(2) $\overset{\text{s.b.}}{\sim}$ becomes an equivalence relation and we call this equivalent class *the weak stably k-equivalent class* of $[\widehat{T}]^{fl}$ (or T) denoted by WSEC_r $(r \geq 0)$ with the stably k-rational class WSEC_0.

Main theorem 1 ([HY, Theorem 1.22]) $\dim(T) = 3$: up to $\overset{\text{s.b.}}{\sim}$

There exist exactly 14 weak stably birationally k-equivalent classes of algebraic k-tori T of dimension 3 which consist of the stably rational class WSEC_0 and 13 classes WSEC_r $(1 \le r \le 13)$ for $[\widehat{T}]^{fl}$ with $\widehat{T} = M_G$ and $G = N_{3,i}$ $(1 \le i \le 15)$ as in the following:

r	$G = N_{3,i} : [\widehat{T}]^{fl} = [M_G]^{fl} \in \mathrm{WSEC}_r$	G
1	$N_{3,1} = U_1$ ([CTS 1977])	C_2^2
2	$N_{3,2} = U_2$	C_2^3
3	$N_{3,3} = W_2$	C_2^3
4	$N_{3,4} = W_1$	$C_4 \times C_2$
5	$N_{3,5} = U_3$, $N_{3,6} = U_4$	D_4
6	$N_{3,7} = U_6$	$D_4 \times C_2$
7	$N_{3,8} = U_5$	A_4
8	$N_{3,9} = U_7$	$A_4 \times C_2$
9	$N_{3,10} = W_3$	$A_4 \times C_2$
10	$N_{3,11} = U_9$, $N_{3,13} = U_{10}$	S_4
11	$N_{3,12} = U_8$	S_4
12	$N_{3,14} = U_{12}$	$S_4 \times C_2$
13	$N_{3,15} = U_{11}$	$S_4 \times C_2$

Main theorem 2 ([HY, Theorem 1.23]) $\dim(T) = 3$: up to $\overset{\text{s.b.}}{\approx}$

Let T_i and T'_j $(1 \leq i, j \leq 15)$ be algebraic k-tori of dimension 3 with the minimal splitting fields L_i and L'_j, and $\widehat{T}_i = M_G$ and $\widehat{T}'_j = M_{G'}$ which satisfy that G and G' are $\mathrm{GL}(3, \mathbb{Z})$-conjugate to $N_{3,i}$ and $N_{3,j}$ respectively. For $1 \leq i, j \leq 15$, the following conditions are equivalent:

(1) $T_i \overset{\text{s.b.}}{\approx} T'_j$ (stably birationally k-equivalent);
(2) $L_i = L'_j$, $T_i \times_k K$ and $T'_j \times_k K$ are weak stably birationally K-equivalent for any $k \subset K \subset L_i$;
(3) $L_i = L'_j$, $T_i \times_k K$ and $T'_j \times_k K$ are weak stably birationally K-equivalent for any $k \subset K \subset L_i$ corresponding to WSEC_r $(r \geq 1)$;
(4) $L_i = L'_j$, $T_i \times_k K$ and $T'_j \times_k K$ are weak stably birationally K-equivalent for any $k \subset K \subset L_i$ corresponding to WSEC_r $(r \geq 1)$ with $[K : k] = d$ where

$$d = \begin{cases} 1 & (i = 1, 3, 4, 8, 9, 10, 11, 12, 13, 14), \\ 1, 2 & (i = 2, 5, 6, 7, 15). \end{cases}$$