# Birational classification for algebraic tori (joint work with Aiichi Yamasaki)

Akinari Hoshi

Niigata University

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   [HY17] A. Hoshi, A. Yamasaki,
   Rationality problem for algebraic tori,
   Mem. Amer. Math. Soc. 248 (2017), no. 1176, v+215 pp.

2. Birational classification for algebraic $k$-tori $T$

   [HY] A. Hoshi, A. Yamasaki,
   Birational classification for algebraic tori, 210 pages,
   arXiv:2112.02280.

# §1. Rationality problem for algebraic tori $T$ (1/3)

- $k$: a base field which is NOT algebraically closed! (TODAY)
- $T$: algebraic $k$-torus, i.e. $k$-form of a split torus;
  an algebraic group over $k$ (group $k$-scheme) with $T \times_k \overline{k} \simeq (\mathbb{G}_{m,\overline{k}})^n$.

**Rationality problem for algebraic tori**

Whether $T$ is $k$-rational?, i.e. $T \approx \mathbb{P}^n$? (birationally $k$-equivalent)

Let $R^{(1)}_{K/k}(\mathbb{G}_m)$ be the norm one torus of $K/k$, i.e. the kernel of the norm map $N_{K/k} : R_{K/k}(\mathbb{G}_m) \to \mathbb{G}_m$ where $R_{K/k}$ is the Weil restriction:

$$1 \longrightarrow R^{(1)}_{K/k}(\mathbb{G}_m) \longrightarrow R_{K/k}(\mathbb{G}_m) \xrightarrow{N_{K/k}} \mathbb{G}_m \longrightarrow 1.$$

$$\dim \qquad\qquad n-1 \qquad\qquad\qquad n \qquad\qquad\qquad 1$$

- $\exists 2$ algebraic $k$-tori $T$ with $\dim(T) = 1$;
  the trivial torus $\mathbb{G}_m$ and $R^{(1)}_{K/k}(\mathbb{G}_m)$ with $[K : k] = 2$, are $k$-rational.

# Rationality problem for algebraic tori $T$ (2/3)

- $\exists 13$ algebraic $k$-tori $T$ with $\dim(T) = 2$.

**Theorem (Voskresenskii 1967) $2$-dim. algebraic tori $T$**

$T$ is $k$-rational.

- $\exists 73$ algebraic $k$-tori $T$ with $\dim(T) = 3$.

**Theorem (Kunyavskii 1990) $3$-dim. algebraic tori $T$**

(i) $\exists 58$ algebraic $k$-tori $T$ which are $k$-rational;
(ii) $\exists 15$ algebraic $k$-tori $T$ which are not $k$-rational.

- What happens in higher dimensions?

# Algebraic $k$-tori $T$ and $G$-lattices

- $T$: algebraic $k$-torus
  $\Longrightarrow \exists$ finite Galois extension $L/k$ such that $T \times_k L \simeq (\mathbb{G}_{m,L})^n$.
- $G = \mathrm{Gal}(L/k)$ where $L$ is the minimal splitting field.

Category of algebraic $k$-tori which split$/L$ $\overset{\text{duality}}{\longleftrightarrow}$ Category of $G$-lattices
(i.e. finitely generated $\mathbb{Z}$-free $\mathbb{Z}[G]$-module)

- $T \mapsto$ the character group $\widehat{T} = \mathrm{Hom}(T, \mathbb{G}_m)$: $G$-lattice.
- $T = \mathrm{Spec}(L[M]^G)$ which splits$/L$ with $\widehat{T} \simeq M \hookleftarrow M$: $G$-lattice
- Tori of dimension $n$ $\overset{1:1}{\longleftrightarrow}$ elements of the set $H^1(\mathcal{G}, \mathrm{GL}(n, \mathbb{Z}))$
  where $\mathcal{G} = \mathrm{Gal}(\bar{k}/k)$ since $\mathrm{Aut}(\mathbb{G}_m^n) = \mathrm{GL}(n, \mathbb{Z})$.
- $k$-torus $T$ of dimension $n$ is determined uniquely by the integral representation $h : \mathcal{G} \to \mathrm{GL}(n, \mathbb{Z})$ up to conjugacy, and the group $h(\mathcal{G})$ is a finite subgroup of $\mathrm{GL}(n, \mathbb{Z})$.
- The function field of $T$ $\overset{\text{identified}}{\longleftrightarrow}$ $L(M)^G$: invariant field.

- $L/k$: Galois extension with $G = \mathrm{Gal}(L/k)$.
- $M = \bigoplus_{1 \leq i \leq n} \mathbb{Z} \cdot u_i$: $G$-lattice with a $\mathbb{Z}$-basis $\{u_1, \ldots, u_n\}$.
- $G$ acts on $L(x_1, \ldots, x_n)$ by

$$x_i^\sigma = \prod_{j=1}^{n} x_j^{a_{i,j}}, \quad 1 \leq i \leq n$$

  for any $\sigma \in G$, when $u_i^\sigma = \sum_{j=1}^{n} a_{i,j} u_j$, $a_{i,j} \in \mathbb{Z}$.
- $L(M) := L(x_1, \ldots, x_n)$ with this action of $G$.
- The function field of algebraic $k$-torus $T \overset{\text{identified}}{\longleftrightarrow} L(M)^G$

**Rationality problem for algebraic tori $T$ (2nd form)**

Whether $L(M)^G$ is $k$-rational?
($=$ purely transcendental over $k$?; $L(M)^G = k(\exists t_1, \ldots, \exists t_n)$?)

# Some definitions.

- $K/k$: a finite generated field extension.

**Definition (stably rational)**

$K$ is called stably $k$-rational if $K(y_1,\ldots,y_m)$ is $k$-rational.

**Definition (retract rational)**

$K$ is retract $k$-rational if $\exists k$-algebra (domain) $R \subset K$ such that
(i) $K$ is the quotient field of $R$;
(ii) $\exists f \in k[x_1,\ldots,x_n]$ $\exists k$-algebra hom. $\varphi : R \to k[x_1,\ldots,x_n][1/f]$ and $\psi : k[x_1,\ldots,x_n][1/f] \to R$ satisfying $\psi \circ \varphi = 1_R$.

**Definition (unirational)**

$K$ is $k$-unirational if $K \subset k(x_1,\ldots,x_n)$.

- $k$-rational $\Rightarrow$ stably $k$-rational $\Rightarrow$ retract $k$-rational $\Rightarrow$ $k$-unirational.
- $L(M)^G$ (resp. $T$) is always $k$-unirational.

# Rationality problem for algebraic tori $T$ (2-dim., 3-dim.)

- The function field of $n$-dim. $T \overset{\text{identified}}{\longleftrightarrow} L(M)^G$, $G \leq \mathrm{GL}(n,\mathbb{Z})$
- $\exists 13$ $\mathbb{Z}$-coujugacy subgroups $G \leq \mathrm{GL}(2,\mathbb{Z})$ ($\exists 13$ $2$-dim. algebraic $k$-tori $T$).

**Theorem (Voskresenskii 1967) $2$-dim. algebraic tori $T$ (restated)**

$T$ is $k$-rational.

- $\exists 73$ $\mathbb{Z}$-coujugacy subgroups $G \leq \mathrm{GL}(3,\mathbb{Z})$ ($\exists 73$ $3$-dim. algebraic $k$-tori $T$).

**Theorem (Kunyavskii 1990) $3$-dim. algebraic tori $T$ (precise form)**

(i) $T$ is $k$-rational $\iff$ $T$ is stably $k$-rational $\iff$ $T$ is retract $k$-rational $\iff$ $\exists G$: $58$ groups;
(ii) $T$ is not $k$-rational $\iff$ $T$ is not stably $k$-rational $\iff$ $T$ is not retract $k$-rational $\iff$ $\exists G$: $15$ groups.

# Rationality problem for algebraic tori $T$ ($4$-dim.)

- The function field of $n$-dim. $T \overset{\text{identified}}{\longleftrightarrow} L(M)^G$, $G \leq \mathrm{GL}(n,\mathbb{Z})$
- $\exists 710$ $\mathbb{Z}$-coujugacy subgroups $G \leq \mathrm{GL}(4,\mathbb{Z})$ ($\exists 710$ $4$-dim. algebraic $k$-tori $T$).

**Theorem ([HY17]) $4$-dim. algebraic tori $T$**

(i) $T$ is stably $k$-rational $\iff$ $\exists G$: $487$ groups;
(ii) $T$ is not stably but retract $k$-rational $\iff$ $\exists G$: $7$ groups;
(iii) $T$ is not retract $k$-rational $\iff$ $\exists G$: $216$ groups.

- We do not know "$k$-rationality".
- Voskresenskii's conjecture: any stably $k$-rational torus is $k$-rational (Zariski problem).
- what happens for dimension $5$?

- The function field of $n$-dim. $T$ $\overset{\text{identified}}{\longleftrightarrow}$ $L(M)^G$, $G \leq \mathrm{GL}(n, \mathbb{Z})$
- $\exists 6079$ $\mathbb{Z}$-coujugacy subgroups $G \leq \mathrm{GL}(5, \mathbb{Z})$ ($\exists 6079$ 5-dim. algebraic $k$-tori $T$).

**Theorem ([HY17]) 5-dim. algebraic tori $T$**

(i) $T$ is stably $k$-rational $\iff$ $\exists G$: $3051$ groups;
(ii) $T$ is not stably but retract $k$-rational $\iff$ $\exists G$: $25$ groups;
(iii) $T$ is not retract $k$-rational $\iff$ $\exists G$: $3003$ groups.

- what happens for dimension $6$?
- BUT we do not know the answer for dimension $6$.
- $\exists 85308$ $\mathbb{Z}$-coujugacy subgroups $G \leq \mathrm{GL}(6, \mathbb{Z})$ ($\exists 85308$ 6-dim. algebraic $k$-tori $T$).

# Strategy: Flabby (Flasque) resolution

- $M$: $G$-lattice, i.e. f.g. $\mathbb{Z}$-free $\mathbb{Z}[G]$-module.

**Definition**

(i) $M$ is permutation $\overset{\text{def}}{\iff}$ $M \simeq \oplus_{1\le i\le m}\mathbb{Z}[G/H_i]$.

(ii) $M$ is stably permutation $\overset{\text{def}}{\iff}$ $M \oplus \exists P \simeq P'$, $P, P'$: permutation.

(iii) $M$ is invertible $\overset{\text{def}}{\iff}$ $M \oplus \exists M' \simeq P$: permutation.

(iv) $M$ is coflabby $\overset{\text{def}}{\iff}$ $H^1(H, M) = 0$ $(\forall H \le G)$.

(v) $M$ is flabby $\overset{\text{def}}{\iff}$ $\widehat{H}^{-1}(H, M) = 0$ $(\forall H \le G)$. ($\widehat{H}$: Tate cohomology)

- "permutation"
  $\Longrightarrow$ "stably permutation"
  $\Longrightarrow$ "invertible"
  $\Longrightarrow$ "flabby and coflabby".

## Commutative monoid $\mathcal{M}$

$M_1 \sim M_2 \overset{\text{def}}{\Longleftrightarrow} M_1 \oplus P_1 \simeq M_2 \oplus P_2$ ($\exists P_1, \exists P_2$: permutation).
$\Longrightarrow$ commutative monoid $\mathcal{M}$: $[M_1] + [M_2] := [M_1 \oplus M_2]$, $0 = [P]$.

## Theorem (Endo-Miyata 1974, Colliot-Thélène-Sansuc 1977)

$\exists P$: permutation, $\exists F$: flabby such that

$$0 \to M \to P \to F \to 0\text{: flabby resolution of } M.$$

- $[M]^{fl} := [F]$; flabby class of $M$.

## Theorem (Endo-Miyata 1973, Voskresenskii 1974, Saltman 1984)

(EM73) $[M]^{fl} = 0 \iff L(M)^G$ is stably $k$-rational.
(Vos74) $[M]^{fl} = [M']^{fl} \iff L(M)^G(x_1, \ldots, x_m) \simeq L(M')^G(y_1, \ldots, y_n)$;
stably $k$-isomorphic.
(Sal84) $[M]^{fl}$ is invertible $\iff L(M)^G$ is retract $k$-rational.

- $M = M_G \simeq \widehat{T} = \mathrm{Hom}(T, \mathbb{G}_m)$, $k(T) \simeq L(M)^G$, $G = \mathrm{Gal}(L/k)$.

# Contributions of [HY17]

- We give a procedure to compute a flabby resolution of $M$, in particular $[M]^{fl} = [F]$, effectively (with smaller rank after base change) by computer software GAP.
- The function `IsFlabby` (resp. `IsCoflabby`) may determine whether $M$ is flabby (resp. coflabby).
- The function `IsInvertibleF` may determine whether $[M]^{fl} = [F]$ is invertible ($\leftrightarrow$ whether $L(M)^G$ (resp. $T$) is retract rational).
- We provide some functions for checking a possibility of isomorphism

$$\left(\bigoplus_{i=1}^{r} a_i\,\mathbb{Z}[G/H_i]\right) \oplus a_{r+1}F \simeq \bigoplus_{i=1}^{r} b_i'\,\mathbb{Z}[G/H_i] \qquad (*)$$

  by computing some invariants (e.g. trace, $\widehat{Z}^0$, $\widehat{H}^0$) of both sides.
- [HY17, Example 10.7]. $G \simeq S_5 \leq \mathrm{GL}(5,\mathbb{Z})$ with number $(5,946,4)$
  $\Longrightarrow \mathrm{rank}(F) = 17$ and $\mathrm{rank}(*) = 88$ holds
  $\Longrightarrow [F] = 0 \Longrightarrow L(M)^G$ (resp. $T$) is stably rational over $k$.

# Application to Krull-Schmidt

**Corollary ($[F]=[M]^{fl}$: invertible case, $G\simeq S_5, F_{20}$)**

$\exists T$, $T'$; $4$-dim. not stably rational algebraic tori over $k$ such that $T\not\sim T'$ (birational) and $T\times T'$: $8$-dim. stably rational over $k$.
$\because -[M]^{fl}=[M']^{fl}\neq 0$.

**Prop. ([HY17], Krull-Schmidt fails for permutation $D_6$-lattices)**

$\{1\}$, $C_2^{(1)}$, $C_2^{(2)}$, $C_2^{(3)}$, $C_3$, $V_4$, $C_6$, $S_3^{(1)}$, $S_3^{(2)}$, $D_6$: conj. subgroups of $D_6$.

$$\mathbb{Z}[D_6]\oplus\mathbb{Z}[D_6/V_4]^{\oplus 2}\oplus\mathbb{Z}[D_6/C_6]\oplus\mathbb{Z}[D_6/S_3^{(1)}]\oplus\mathbb{Z}[D_6/S_3^{(2)}]$$
$$\simeq\mathbb{Z}[D_6/C_2^{(1)}]\oplus\mathbb{Z}[D_6/C_2^{(2)}]\oplus\mathbb{Z}[D_6/C_2^{(3)}]\oplus\mathbb{Z}[D_6/C_3]\oplus\mathbb{Z}^{\oplus 2}.$$

- $D_6$ is the smallest example exhibiting the failure of K-S:

**Theorem (Dress 1973)**

Krull-Schmidt holds for permutation $G$-lattices $\iff$ $G/O_p(G)$ is cyclic where $O_p(G)$ is the maximal normal $p$-subgroup of $G$.

# Krull-Schmidt and Direct sum cancelation

**Theorem (Hindman-Klingler-Odenthal 1998) Assume $G \neq D_8$**

Krull-Schmidt holds for $G$-lattices $\iff$ (i) $G = C_p$ ($p \leq 19$; prime), (ii) $G = C_n$ ($n = 1, 4, 8, 9$), (iii) $G = V_4$ or (iv) $G = D_4$.

**Theorem (Endo-Hironaka 1979)**

Direct sum cancellation holds, i.e. $M_1 \oplus N \simeq M_2 \oplus N \Longrightarrow M_1 \simeq M_2$, $\Longrightarrow G$ is abelian, dihedral, $A_4$, $S_4$ or $A_5$ (*).

- via projective class group (see Swan 1988, Corollary 1.3, Section 7).
- Except for (*) $\Longrightarrow$ Direct sum cancelation fails $\Longrightarrow$ K-S fails

**Theorem ([HY17]) $G \leq \mathrm{GL}(n, \mathbb{Z})$ (up to conjugacy)**

(i) $n \leq 4 \Longrightarrow$ K-S holds.
(ii) $n = 5$. K-S fails $\iff$ $11$ groups $G$ (among $6079$ groups).
(iii) $n = 6$. K-S fails $\iff$ $131$ groups $G$ (among $85308$ groups).

# §2. Birational classification for algebraic tori

- Two algebraic $k$-tori $T$ and $T'$ are stably birationally $k$-equivalent, denoted by $T \overset{\text{s.b.}}{\approx} T'$, if their function fields $k(T)$ and $k(T')$ are stably $k$-isomorphic, i.e. $k(T)(x_1, \ldots, x_m) \simeq k(T')(y_1, \ldots, y_n)$.
- $\mathcal{T}$: the category of algebraic $k$-tori.
- $\mathcal{T}_n$: the category of algebraic $k$-tori of dimension $n$.

**Problem 1: Stably birational classification for algebraic tori**

Determine the structure of $\mathcal{T}/\overset{\text{s.b.}}{\approx}$ (resp. $\mathcal{T}_n/\overset{\text{s.b.}}{\approx}$). In particular, for given two algebraic $k$-tori $T$ and $T'$ (resp. $T$ and $T'$ of dimension $n$) determine whether $T$ and $T'$ are stably birationally $k$-equivalent.

- $V_4 := C_2 \times C_2$.

**Theorem (Colliot-Thélène and Sansuc 1977) $\dim(T) = \dim(T') = 3$**

Let $L/k$ and $L'/k$ be Galois extensions with $\mathrm{Gal}(L/k) \simeq \mathrm{Gal}(L'/k) \simeq V_4$. Let $T = R^{(1)}_{L/k}(\mathbb{G}_m)$ and $T' = R^{(1)}_{L'/k}(\mathbb{G}_m)$ be the corresponding norm one tori. If $T \overset{\text{s.b.}}{\approx} T'$ (stably birationally $k$-equivalent), then $L = L'$.

- In particular, if $k$ is a number field, then there exist infinitely many stably birationally $k$-equivalent classes of (non-rational: $1\mathrm{st}/15$) $k$-tori which correspond to $U_1$ (cf. Main theorem 1, later).
- $T \overset{\text{s.b.}}{\approx} T'$ (stably birationally $k$-equivalent) $\Longleftarrow L = L'$ ???

- $\overline{k}$: a fixed separable closure of $k$ and $\mathcal{G} = \mathrm{Gal}(\overline{k}/k)$.
- $X$: a smooth $k$-compactification of $T$, i.e. smooth projective $k$-variety $X$ containing $T$ as a dense open subvariety.
- $\overline{X} = X \times_k \overline{k}$.

**Theorem (Voskresenskii 1969, 1970)**

There exists an exact sequence of $\mathcal{G}$-lattices

$$0 \to \widehat{T} \to \widehat{Q} \to \mathrm{Pic}\,\overline{X} \to 0$$

where $\widehat{Q}$ is permutation and $\mathrm{Pic}\ \overline{X}$ is flabby.

- $M_G \simeq \widehat{T}$, $[\widehat{T}]^{fl} = [\mathrm{Pic}\,\overline{X}]$ as $\mathcal{G}$-lattices.

**Theorem (Voskresenskii 1970, 1973)**

(i) $T$ is stably $k$-rational if and only if $[\mathrm{Pic}\,\overline{X}] = 0$ as a $\mathcal{G}$-lattice.

(ii) $T \overset{\text{s.b.}}{\approx} T'$ (stably birationally $k$-equivalent) if and only if $[\mathrm{Pic}\,\overline{X}] = [\mathrm{Pic}\,\overline{X'}]$ as $\mathcal{G}$-lattices.

- From $\mathcal{G}$-lattice to $G$-lattice

Let $L$ be the minimal splitting field of $T$ with $G = \mathrm{Gal}(L/k) \simeq \mathcal{G}/\mathcal{H}$.
We obtain a flabby resolution of $\widehat{T}$:

$$0 \to \widehat{T} \to \widehat{Q} \to \mathrm{Pic}\, X_L \to 0$$

with $[\widehat{T}]^{fl} = [\mathrm{Pic}\ X_L]$ as $G$-lattices. We get:

**Theorem (Voskresenskii 1970, 1973)**

(ii)$'$ $T \overset{\text{s.b.}}{\approx} T'$ (stably birationally $k$-equivalent) if and only if $[\mathrm{Pic}\, X_{\widetilde{L}}] = [\mathrm{Pic}\, X'_{\widetilde{L}}]$ as $\widetilde{H}$-lattices where $\widetilde{L} = LL'$ and $\widetilde{H} = \mathrm{Gal}(\widetilde{L}/k)$.

The group $\widetilde{H}$ becomes a *subdirect product* of $G = \mathrm{Gal}(L/k)$ and $G' = \mathrm{Gal}(L'/k)$, i.e. a subgroup $\widetilde{H}$ of $G \times G'$ with surjections $\varphi_1 : \widetilde{H} \twoheadrightarrow G$ and $\varphi_2 : \widetilde{H} \twoheadrightarrow G'$.

- This observation yields a concept of "*weak stably $k$-equivalence*".

## Definition

(i) $[M]^{fl}$ and $[M']^{fl}$ are *weak stably $k$-equivalent*, if there exists a subdirect product $\widetilde{H} \leq G \times G'$ of $G$ and $G'$ with surjections $\varphi_1 : \widetilde{H} \twoheadrightarrow G$ and $\varphi_2 : \widetilde{H} \twoheadrightarrow G'$ such that $[M]^{fl} = [M']^{fl}$ as $\widetilde{H}$-lattices where $\widetilde{H}$ acts on $M$ (resp. $M'$) through the surjection $\varphi_1$ (resp. $\varphi_2$).
(ii) Algebraic $k$-tori $T$ and $T'$ are *weak stably birationally $k$-equivalent*, denoted by $T \overset{\mathrm{s.b.}}{\sim} T'$, if $[\widehat{T}]^{fl}$ and $[\widehat{T'}]^{fl}$ are weak stably $k$-equivalent.

## Remark

(1) $T \overset{\mathrm{s.b.}}{\approx} T'$ (stably bir. $k$-equiv.) $\Rightarrow T \overset{\mathrm{s.b.}}{\sim} T'$ (weak stably bir. $k$-equiv.).
(2) $\overset{\mathrm{s.b.}}{\sim}$ becomes an equivalence relation and we call this equivalent class *the weak stably $k$-equivalent class* of $[\widehat{T}]^{fl}$ (or $T$) denoted by $\mathrm{WSEC}_r$ $(r \geq 0)$ with the stably $k$-rational class $\mathrm{WSEC}_0$.

- Let $L$ be the minimal splitting field of $T$ with $G = \mathrm{Gal}(L/k) \simeq \mathcal{G}/\mathcal{H}$.
- By the inflation-restriction exact sequence $0 \to H^1(G, \mathrm{Pic}\, X_L) \xrightarrow{\mathrm{inf}} H^1(k, \mathrm{Pic}\, \overline{X}) \xrightarrow{\mathrm{res}} H^1(L, \mathrm{Pic}\, \overline{X})$, we get $\mathrm{inf} : H^1(G, \mathrm{Pic}\, X_L) \xrightarrow{\sim} H^1(k, \mathrm{Pic}\, \overline{X})$ because $H^1(L, \mathrm{Pic}\, \overline{X}) = 0$.
- $H^1(k, \mathrm{Pic}\, \overline{X}) \simeq \mathrm{Br}(X)/\mathrm{Br}(k) \simeq \mathrm{Br}_{\mathrm{nr}}(k(X)/k)/\mathrm{Br}(k)$
  by Colliot-Thélène-Sansuc 1987
  where $\mathrm{Br}(X)$ is the étale cohomological/Azumaya Brauer group of $X$ and $\mathrm{Br}_{\mathrm{nr}}(k(X)/k)$ is the unramified Brauer group of $k(X)$ over $k$.

- Let $G$, $G' \leq \mathrm{GL}(n,\mathbb{Z})$ be $\mathrm{GL}(n,\mathbb{Z})$-conjugate. Then $\exists\psi : G \xrightarrow{\sim} G'$, $g \mapsto u^{-1}gu$ $(u \in \mathrm{GL}(n,\mathbb{Z}))$.
- Let $T$, $T'$ be algebraic $k$-tori of dimension $n$ with the minimal splitting fields $L$ and $L'$ and the character modules $\widehat{T} = M_G$ and $\widehat{T}' = M_{G'}$.
- Assume that $L = L'$. Then $G \simeq G' \simeq \mathrm{Gal}(L/k)$ and $\exists\varphi_1 : \mathrm{Gal}(L/k) \xrightarrow{\sim} G \leq \mathrm{GL}(n,\mathbb{Z})$, $f \mapsto \varphi_1(f)$, $\exists\varphi_2 : \mathrm{Gal}(L'/k) \xrightarrow{\sim} G' \leq \mathrm{GL}(n,\mathbb{Z})$, $f \mapsto \varphi_2(f)$, $\exists$ a subdirect product $\widetilde{H} = \{(\varphi_1(f), \varphi_2(f)) \mid f \in \mathrm{Gal}(L/k)\} \leq G \times G'$ $(\widetilde{H} \simeq G)$.
- Hence we can obtain $\sigma \in \mathrm{Aut}(G)$ such that $(\psi^{-1})(\varphi_2\varphi_1^{-1})(g) = g^\sigma$ $(\forall g \in G)$ and hence we can identify $\psi^{-1} : G' \simeq G^\sigma$ $(\sigma \in \mathrm{Aut}(G))$.
- Let $T^\sigma$ be an algebraic $k$-torus of dimension $n$ with the minimal splitting field $L$ and the character module $\widehat{T}^\sigma = M_{G^\sigma}$ $(\sigma \in \mathrm{Aut}(G))$ with $G \simeq G^\sigma \simeq \mathrm{Gal}(L/k)$.
- Then the set

$$\{T^\sigma \mid \sigma \in \mathrm{Aut}(G)\}$$

gives all algebraic $k$-tori of dimension $n$ with the minimal splitting field $L$ and the character module $\widehat{T}^\sigma \simeq M_G$.

## Definition (the groups $X, Y, Z$)

We define the following subgroups of $\mathrm{Aut}(G)$ for $G \leq \mathrm{GL}(n\,\mathbb{Z})$:

$$\mathrm{Inn}(G) \leq X \leq Y \leq Z \leq \mathrm{Aut}(G),$$

$$\begin{aligned} X &= \mathrm{Aut}_{\mathrm{GL}(n,\mathbb{Z})}(G) \\ &= \{\sigma \in \mathrm{Aut}(G) \mid \exists u \in \mathrm{GL}(n,\mathbb{Z}) \text{ s.t. } u^{-1}gu = g^{\sigma} (\forall g \in G)\} \\ &\simeq N_{\mathrm{GL}(n,\mathbb{Z})}(G)/Z_{\mathrm{GL}(n,\mathbb{Z})}(G), \\ Y &= \{\sigma \in \mathrm{Aut}(G) \mid [M_G]^{fl} = [M_{G^\sigma}]^{fl} \text{ as } \widetilde{H}\text{-lattices} \\ &\quad \text{where } \widetilde{H} = \{(g, g^\sigma) \mid g \in G\} \simeq G\}, \\ Z &= \{\sigma \in \mathrm{Aut}(G) \mid [M_H]^{fl} \sim [M_{H^\sigma}]^{fl} \text{ for any } H \leq G\}. \end{aligned}$$

## Corollary

The set $\{T^\sigma \mid \sigma \in \mathrm{Aut}(G)\}$ gives all algebraic $k$-tori of dimension $n$ with the minimal splitting field $L$ and $\widehat{T}^\sigma \simeq M_G$ and splits into $\lambda$ stably birationally $k$-equivalent classes consist of $\mu$ birationally $k$-equivalent classes where $\lambda = |Y\backslash\mathrm{Aut}(G)| \leq \mu \leq |X\backslash\mathrm{Aut}(G)|$.

## Theorem ([HY, Theorem 1.26])

(1) For $\sigma \in \mathrm{Aut}(G)$, $T$ and $T^\sigma$ are weak stably birationally $k$-equivalent;
(2) For $\sigma \in \mathrm{Aut}(G)$, $\sigma \in X$ if and only if $M_G \simeq M_{G^\sigma}$ as $\mathrm{Gal}(L/k)$-lattices. In particular, $T$ and $T^\sigma$ are birationally $k$-equivalent, i.e. $k(T) \simeq L(M)^G \simeq L(M^\sigma)^{G^\sigma} \simeq k(T^\sigma)$;
(3) For $\sigma \in \mathrm{Aut}(G)$, $\sigma \in Y$ if and only if $T$ and $T^\sigma$ are stably birationally $k$-equivalent. In particular, $\{T^\sigma \mid \sigma \in \mathrm{Aut}(G)\}$ splits into $\lambda$ stably birationally $k$-equivalent classes where $\lambda = |Y\backslash\mathrm{Aut}(G)|$;
(4) For $\sigma \in \mathrm{Aut}(G)$, $\sigma \in Z$ if and only if $T \times_k K$ and $T^\sigma \times_k K$ are weak stably birationally $K$-equivalent for any $k \subset K \subset L$.
In particular, (i) if $Y = Z$, then $\sigma \in Z$ if and only if $T$ and $T^\sigma$ are stably birationally $k$-equivalent; and (ii) if $X = Y = Z$ (resp. $X = Y$), then $\sigma \in Z$ (resp. $\sigma \in Y$) if and only if $T$ and $T^\sigma$ are birationally $k$-equivalent.

Rationality problem for 3-dimensional algebraic $k$-tori $T$ was solved by Kunyavskii (1990). Stably/retract rationality for algebraic $k$-tori $T$ of dimensions $4$ and $5$ are given in Hoshi and Yamasaki [HY17, Chapter 1].

## Definition $(N_{3,i}, N_{31,i}, N_{4,i}, I_{4,i})$

(1) The $15$ groups $G = N_{3,i} \leq \mathrm{GL}(3,\mathbb{Z})$ $(1 \leq i \leq 15)$ for which $k(T) \simeq L(M)^G$ is not retract $k$-rational are as in [HY, Table 6].
(2) The $64$ groups $G = N_{31,i} \leq \mathrm{GL}(4,\mathbb{Z})$ $(1 \leq i \leq 64)$ for which $k(T) \simeq L(M)^G$ is not retract $k$-rational where $M \simeq M_1 \oplus M_2$ with $\mathrm{rank}$ $M = 3+1$ are as in [HY, Table 7].
(3) The $152$ groups $G = N_{4,i} \leq \mathrm{GL}(4,\mathbb{Z})$ $(1 \leq i \leq 152)$ for which $k(T) \simeq L(M)^G$ is not retract $k$-rational with $\mathrm{rank}$ $M = 4$ are as in [HY, Table 8].
(4) The $7$ groups $G = I_{4,i} \leq \mathrm{GL}(4,\mathbb{Z})$ $(1 \leq i \leq 7)$ for which $k(T) \simeq L(M)^G$ is not stably but retract $k$-rational with $\mathrm{rank}$ $M = 4$ are as in [HY, Table 9].

# Main Theorems 1, 2, 3, 4, 5, 6, 7

- Main theorem 1 $\dim(T) = 3$: up to $\overset{\text{s.b.}}{\sim}$ (weak)
- Main theorem 2 $\dim(T) = 3$: up to $\overset{\text{s.b.}}{\approx}$
- Main theorem 3 $\dim(T) = 4$: up to $\overset{\text{s.b.}}{\sim}$ (weak)
- Main theorem 4 $\dim(T) = 4$ ($N_{4,i}$): up to $\overset{\text{s.b.}}{\approx}$
- Main theorem 5 $\dim(T) = 4$ ($I_{4,i}$): up to $\overset{\text{s.b.}}{\approx}$
- Main theorem 6 $\dim(T) = 4$: $I_{4,i}$ cases $(1 \leq i \leq 7)$
- Main theorem 7 higher dimensional cases: $\dim(T) \geq 3$

**Definition**

*The $G$-lattice $M_G$ of rank $n$* is defined to be the $G$-lattice with a $\mathbb{Z}$-basis $\{u_1, \ldots, u_n\}$ on which $G$ acts by $u_i^\sigma = \sum_{j=1}^n a_{i,j} u_j$ for any $\sigma = [a_{i,j}] \in G \leq \mathrm{GL}(n, \mathbb{Z})$.

## Main theorem 1 ([HY, Theorem 1.28]) $\dim(T)=3$: up to $\overset{\text{s.b.}}{\sim}$

There exist exactly $14 = 1 + 13$ weak stably birationally $k$-equivalent classes of algebraic $k$-tori $T$ of dimension $3$ which consist of the stably rational class $\mathrm{WSEC}_0$ and $13$ classes $\mathrm{WSEC}_r$ $(1 \le r \le 13)$ for $[\widehat{T}]^{fl}$ with $\widehat{T} = M_G$ and $G = N_{3,i}$ $(1 \le i \le 15)$: (red $\leftrightarrow$ norm one tori)

| $r$ | $G = N_{3,i} : [\widehat{T}]^{fl} = [M_G]^{fl} \in \mathrm{WSEC}_r$ | $\mathrm{G}_r \simeq G$ | $\lambda_r = \lvert\mathrm{WSEC}_{r,L}\rvert$ |
|---|---|---|---|
| 1 | $N_{3,1} \simeq U_1$ ([CTS 1977]) | $V_4$ | 1 |
| **2** | $N_{3,2} \simeq U_2$ | $C_2^3$ | **7** |
| 3 | $N_{3,3} \simeq W_2$ | $C_2^3$ | 1 |
| 4 | $N_{3,4} \simeq W_1$ | $C_4 \times C_2$ | 1 |
| **5** | $N_{3,5} \simeq U_3$, $N_{3,6} \simeq U_4$ | $D_4$ | **2** |
| 6 | $N_{3,7} \simeq U_6$ | $D_4 \times C_2$ | 4 |
| 7 | $N_{3,8} \simeq U_5$ | $A_4$ | 1 |
| 8 | $N_{3,9} \simeq U_7$ | $A_4 \times C_2$ | 1 |
| 9 | $N_{3,10} \simeq W_3$ | $A_4 \times C_2$ | 1 |
| 10 | $N_{3,11} \simeq U_9$, $N_{3,13} \simeq U_{10}$ | $S_4$ | 1 |
| 11 | $N_{3,12} \simeq U_8$ | $S_4$ | 1 |
| 12 | $N_{3,14} \simeq U_{12}$ | $S_4 \times C_2$ | 1 |
| 13 | $N_{3,15} \simeq U_{11}$ | $S_4 \times C_2$ | 2 |

## Main theorem 2 ([HY, Theorem 1.29]) $\dim(T)=3$: up to $\overset{\text{s.b.}}{\approx}$

Let $T_i$ and $T'_j$ $(1\leq i,j\leq 15)$ be algebraic $k$-tori of dimension $3$ with the minimal splitting fields $L_i$ and $L'_j$, and $\widehat{T}_i=M_G$ and $\widehat{T}'_j=M_{G'}$ which satisfy that $G$ and $G'$ are $\mathrm{GL}(3,\mathbb{Z})$-conjugate to $N_{3,i}$ and $N_{3,j}$ respectively. For $1\leq i,j\leq 15$, the following conditions are equivalent:

(1) $T_i\overset{\text{s.b.}}{\approx}T'_j$ (stably birationally $k$-equivalent);

(2) $G\simeq G'$, $L_i=L'_j$, $T_i\times_k K$ and $T'_j\times_k K$ are weak stably birationally $K$-equivalent for any $k\subset K\subset L_i$;

(3) $G\simeq G'$, $L_i=L'_j$, $T_i\times_k K$ and $T'_j\times_k K$ are weak stably birationally $K$-equivalent for any $k\subset K\subset L_i$ corresponding to $\mathrm{WSEC}_r$ $(r\geq 1)$;

(4) $G\simeq G'$, $L_i=L'_j$, $T_i\times_k K$ and $T'_j\times_k K$ are weak stably birationally $K$-equivalent for any $k\subset K\subset L_i$ corresponding to $\mathrm{WSEC}_r$ $(r\geq 1)$ with $[K:k]=d$ where

$$d=\begin{cases}1 & (i=1,3,4,8,9,10,11,12,13,14),\\ 1,2 & (i=2,5,6,7,15).\end{cases}$$

In particular, for $d=1$, (4) $\Leftrightarrow$ $G\simeq G'$, $L_i=L'_j$, i.e. $\widetilde{H}\simeq G\simeq G'$.

## Main theorem 2 ([HY, Theorem 1.29]) $\dim(T)=3$: up to $\overset{\text{s.b.}}{\approx}$

Moreover, if $i=j$ with $G\simeq G'$, $L_i=L'_j$, i.e. $\widetilde{H}\simeq G\simeq G'$, then $Y=Z$ (which is equivalent to $(1)\Leftrightarrow(2)$)
and for $1\leq r\leq 13$, we get the following disjoint union decompositions

$$\mathrm{WSEC}_r=\coprod_{\substack{L/k\\ \mathrm{Gal}(L/k)\simeq \mathrm{G}_r}}\mathrm{WSEC}_{r,L},\ \ \mathrm{WSEC}_{r,L}=\coprod_{t=1}^{\lambda_r}\mathrm{SEC}_{r,L,t}$$

modulo $\overset{\text{s.b.}}{\approx}$ where $\mathrm{Gal}(L/k)\simeq \mathrm{G}_r\simeq N_{3,i}$ $(1\leq r\leq 13)$ and
$\lambda_r=|\mathrm{WSEC}_{r,L}|=|Y\backslash \mathrm{Aut}(G)|$ is given as in Main theorem 1.
Furthermore, for the cases $G=N_{3,i}$ $(i=1,5,6,8,9,10,11,12,13,15)$ with $X=Y$, the following conditions are also equivalent:
(0) $T_i$ and $T'_i$ are birationally $k$-equivalent;
(1) $T_i$ and $T'_i$ are stably birationally $k$-equivalent.

## Corollary (Stably birational classification for $T$ with $\dim(T) = 3$)

Let $\mathcal{T}_3$ be the category of algebraic $k$-tori of dimension $3$. We get a classification (disjoint union decomposition) of $\mathcal{T}_3$ with respect to the stably birationally $k$-equivalence $\overset{\text{s.b.}}{\approx}$:

$$\mathcal{T}_3 = \coprod_{r=0}^{13} \mathrm{WSEC}_r = \mathrm{SEC}_0 \coprod \left( \coprod_{r=1}^{13} \coprod_{\substack{L/k \\ \mathrm{Gal}(L/k) \simeq \mathrm{G}_r}} \coprod_{t=1}^{\lambda_r} \mathrm{SEC}_{r,L,t} \right)$$

modulo $\overset{\text{s.b.}}{\approx}$ where $\mathrm{SEC}_0$ is the stably $k$-equivalent class consists of stably $k$-rational tori $T \in \mathcal{T}_3$, $\mathrm{Gal}(L/k) \simeq \mathrm{G}_r \simeq N_{3,i}$ $(1 \le r \le 13)$ and $\lambda_r = |\mathrm{WSEC}_{r,L}| = |Y \backslash \mathrm{Aut}(G)|$ is given as in Main theorem 1.

# Example of Main theorem 2: $\mathrm{WSEC}_5$; $G = N_{3,6} \simeq U_4 \simeq D_4$, $G' = N_{3,6} \simeq U_4 \simeq D_4$ with $\lambda_5 = 2$

- $k = \mathbb{Q}$, $K_4 = \mathbb{Q}(\sqrt[4]{2})$, $K_4' = \mathbb{Q}(\sqrt[4]{2}\zeta_8)$ with $[K_4 : \mathbb{Q}] = [K_4' : \mathbb{Q}] = 4$ and the same Galois closure $L = \mathbb{Q}(\sqrt[4]{2}, \sqrt{-1})$, $\mathrm{Gal}(L/\mathbb{Q}) \simeq D_4$.
- $T = R^{(1)}_{K_4/\mathbb{Q}}(\mathbb{G}_m)$, $T' = R^{(1)}_{K_4'/\mathbb{Q}}(\mathbb{G}_m)$: the corresponding norm one tori with $\widehat{T} = M_G$, $\widehat{T}' = M_{G'}$, $G = N_{3,6} \simeq U_4 \simeq D_4$, $G' = N_{3,6} \simeq U_4 \simeq D_4$, $[\widehat{T}]^{fl}, [\widehat{T}']^{fl} \in \mathrm{WSEC}_2$ (not retract $\mathbb{Q}$-rational).
- $\exists$ 2 subgroups $H_1, H_2 \leq G$ with $H_1 \simeq H_2 \simeq V_4$, $H_1 \simeq U_1$ (not retract $\mathbb{Q}$-rational) and $[M_G\mid_{H_2}]^{fl} = [M_{H_2}]^{fl} = 0$ (stably $\mathbb{Q}$-rational).
- By Main theorem 2, $\lambda_5 = |\mathrm{WSEC}_{5,L}| = |Y \backslash \mathrm{Aut}(G)| = 2$ with $\mathrm{Aut}(G) \simeq D_4$, $T$ and $T'$ are not stably birationally $\mathbb{Q}$-equivalent.
- $\mathrm{WSEC}_{5,L} = \coprod_{t=1}^{2} \mathrm{SEC}_{5,L,t} = \{[\widehat{T}]^{fl}, [\widehat{T}']^{fl}\}$.
- Because $\lambda_5 = 2$, if we take an algebraic $\mathbb{Q}$-torus $T''$ of dimension $3$ with the minimal splitting field $L = \mathbb{Q}(\sqrt[4]{2}, \sqrt{-1})$ and $[T'']^{fl} \in N_{3,5} \simeq U_3$, then $T''$ is stably birationally $\mathbb{Q}$-equivalent to either $T = R^{(1)}_{K_4/\mathbb{Q}}(\mathbb{G}_m)$ or $T' = R^{(1)}_{K_4'/\mathbb{Q}}(\mathbb{G}_m)$.

# Example of Main theorem 2: $\mathrm{WSEC}_2$; $G = N_{3,2} \simeq U_2 \simeq C_2^3$, $G' = N_{3,2} \simeq U_2 \simeq C_2^3$ with $\lambda_2 = 7$

- $k = \mathbb{Q}$ and $T$ is an algebraic $\mathbb{Q}$-torus with the minimal splitting field $L = \mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5})$, $[L : \mathbb{Q}] = 8$, $\widehat{T} = M_G$, $G = N_{3,2} \simeq U_2 \simeq C_2^3$, $[\widehat{T}]^{fl} \in \mathrm{WSEC}_2$ (not retract $\mathbb{Q}$-rational).
- $\exists$ exactly 7 subgroups $H_i \leq G$ $(1 \leq i \leq 7)$ with $[G : H_i] = 2$ and $\exists$ exactly 7 subgroups $H'_i \leq G$ $(1 \leq i \leq 7)$ with $[G : H'_i] = 4$.
- If $H_1 \simeq U_1 \simeq C_2^2$, then $[M_G\,|_{H_i}]^{fl} = [M_{H_i}]^{fl} = 0$ $(2 \leq i \leq 7)$ and $[M_G\,|_{H'_i}]^{fl} = [M_{H'_i}]^{fl} = 0$ $(1 \leq i \leq 7)$ (stably $\mathbb{Q}$-rational).
- $G' = G^\sigma \leq \mathrm{GL}(3, \mathbb{Z})$ with $G' \simeq G \simeq C_2^3 \simeq (\mathbb{F}_2)^3$ $(\sigma \in \mathrm{Aut}(G))$ where $\mathrm{Aut}(G) \simeq \mathrm{GL}(3, \mathbb{F}_2) \simeq \mathrm{PGL}(3, \mathbb{F}_2) \simeq \mathrm{SL}(3, \mathbb{F}_2) \simeq \mathrm{PSL}(3, \mathbb{F}_2) \simeq \mathrm{PSL}(2, \mathbb{F}_7)$ with $|\mathrm{Aut}(G)| = 168$.
- By Main theorem 2, we have $\lambda_2 = |\mathrm{WSEC}_{2,L}| = |Y \backslash \mathrm{Aut}(G)| = 7$ where $Y = \mathrm{Stab}_{H_1}(\mathrm{Aut}(G)) \simeq S_4$ with $|Y| = 24$.
- $\mathrm{WSEC}_{2,L} = \coprod_{t=1}^{7} \mathrm{SEC}_{2,L,t} = \{[\widehat{T}^\sigma]^{fl} \mid \sigma \in Y \backslash \mathrm{Aut}(G)\}$.
- $Y \backslash \mathrm{Aut}(G) \simeq \mathrm{Gr}_{\mathbb{F}_2}(2,3) \simeq \mathrm{Gr}_{\mathbb{F}_2}(1,3) \simeq \mathbb{P}^2_{\mathbb{F}_2}$ with $|\mathbb{P}^2_{\mathbb{F}_2}| = 7$ where $\mathrm{Gr}_{\mathbb{F}_2}(d,3)$ is the Grassmannian of $d$-dimensional subspaces of $(\mathbb{F}_2)^3$.

# Main theorems 3, 4, 5: $\dim(T) = 4$

- $\exists G = N_{31,i} \leq \mathrm{GL}(4,\mathbb{Z})$ $(1 \leq i \leq 64)$ for which $k(T) \simeq L(M)^G$ is not retract $k$-rational where $M \simeq M_1 \oplus M_2$ with $\mathrm{rank}\ M = 3 + 1$.
- $\exists G = N_{4,i} \leq \mathrm{GL}(4,\mathbb{Z})$ $(1 \leq i \leq 152)$ for which $k(T) \simeq L(M)^G$ is not retract $k$-rational with $\mathrm{rank}\ M = 4$.
- $\exists G = I_{4,i} \leq \mathrm{GL}(4,\mathbb{Z})$ $(1 \leq i \leq 7)$ for which $k(T) \simeq L(M)^G$ is not stably but retract $k$-rational with $\mathrm{rank}\ M = 4$.

## Main theorem 3 ([HY, Theorem 1.34]) $\dim(T)=4$: up to $\overset{\rm s.b.}{\sim}$

There exist exactly $129 = 1 + 121 + 7$ weak stably birationally $k$-equivalent classes of algebraic $k$-tori $T$ of dimension $4$ which consist of the stably rational class $\mathrm{WSEC}_0$, $121$ classes $\mathrm{WSEC}_r$ $(1 \leq r \leq 121)$ for $[\widehat{T}]^{fl}$ with $\widehat{T} = M_G$ and $G = N_{31,i}$ $(1 \leq i \leq 64)$ as in [HY, Table 3] and for $[\widehat{T}]^{fl}$ with $\widehat{T} = M_G$ and $G = N_{4,i}$ $(1 \leq i \leq 152)$ as in [HY, Table 4], and $7$ classes $\mathrm{WSEC}_r$ $(122 \leq r \leq 128)$ for $[\widehat{T}]^{fl}$ with $\widehat{T} = M_G$ and $G = I_{4,i}$ $(1 \leq i \leq 7)$: (red $\leftrightarrow$ norm one tori)

| $r$ | $G = I_{4,i} : [\widehat{T}]^{fl} = [M_G]^{fl} \in \mathrm{WSEC}_r$ | $\mathrm{G}_r \simeq G$ | $\lambda_r = \lvert\mathrm{WSEC}_{r,L}\rvert$ |
|---|---|---|---|
| 122 | $I_{4,1}$ | $F_{20}$ | 1 |
| 123 | $I_{4,2}$ | $F_{20}$ | 1 |
| 124 | $I_{4,3}$ | $F_{20} \times C_2$ | 2 |
| 125 | $I_{4,4}$ | $S_5$ | 1 |
| 126 | $I_{4,5}$ | $S_5$ | 1 |
| 127 | $I_{4,6}$ | $S_5 \times C_2$ | 2 |
| 128 | $I_{4,7}$ | $C_3 \rtimes C_8$ | 2 |

## Main theorem 4 ([HY, Theorem 1.36]) $\dim(T)=4$ $(N_{4,i})$: up to $\overset{\text{s.b.}}{\approx}$

Let $T_i$ and $T_j'$ $(1\leq i,j\leq 152)$ be algebraic $k$-tori of dimension $4$ with the minimal splitting fields $L_i$ and $L_j'$ and the character modules $\widehat{T}_i=M_G$ and $\widehat{T}_j'=M_{G'}$ which satisfy that $G$ and $G'$ are $\mathrm{GL}(4,\mathbb{Z})$-conjugate to $N_{4,i}$ and $N_{4,j}$ respectively. For $1\leq i,j\leq 152$ except for the cases $i=j=137,139,145,147$, the following conditions are equivalent:

(1) $T_i \overset{\text{s.b.}}{\approx} T_j'$ (stably birationally $k$-equivalent);

(2) $G\simeq G'$, $L_i=L_j'$, $T_i\times_k K$ and $T_j'\times_k K$ are weak stably birationally $K$-equivalent for any $k\subset K\subset L_i$;

(3) $G\simeq G'$, $L_i=L_j'$, $T_i\times_k K$ and $T_j'\times_k K$ are weak stably birationally $K$-equivalent for any $k\subset K\subset L_i$ corresponding to $\mathrm{WSEC}_r$ $(r\geq 1)$;

(4) $G\simeq G'$, $L_i=L_j'$, $T_i\times_k K$ and $T_j'\times_k K$ are weak stably birationally $K$-equivalent for any $k\subset K\subset L_i$ corresponding to $\mathrm{WSEC}_r$ $(r\geq 1)$ with $[K:k]=d$ where $d$ is given as in [HY, Theorem 1.26].

In particular, for $d=1$, $(4)\Leftrightarrow G\simeq G'$, $L_i=L_j'$, i.e. $\widetilde{H}\simeq G\simeq G'$.

## Main theorem 4 ([HY, Theorem 1.36]) $\dim(T) = 4$ $(N_{4,i})$: up to $\overset{\text{s.b.}}{\approx}$

For the exceptional cases $i = j = 137, 139, 145, 147$ ($G \simeq Q_8 \times C_3$, $(Q_8 \times C_3) \rtimes C_2$, $\mathrm{SL}(2, \mathbb{F}_3) \rtimes C_4$, $(\mathrm{GL}(2, \mathbb{F}_3) \rtimes C_2) \rtimes C_2 \simeq (\mathrm{SL}(2, \mathbb{F}_3) \rtimes C_4) \rtimes C_2$), we have the implications $(1) \Rightarrow (2) \Leftrightarrow (3) \Leftrightarrow (4)$, there exists $\sigma \in \mathrm{Aut}(G)$ such that $G' = G^\sigma$ and $X = Y \lhd Z$ with $Z/Y \simeq C_2, C_2^2, C_2, C_2$ respectively ($Y = Z$ is equivalent to $(1) \Leftrightarrow (2)$) and we have $(1) \Leftrightarrow M_G \simeq M_{G^\sigma}$ as $\widetilde{H}$-lattices (this is equivalent to $X = Y$) $\Leftrightarrow M_G \otimes_{\mathbb{Z}} \mathbb{F}_p \simeq M_{G^\sigma} \otimes_{\mathbb{Z}} \mathbb{F}_p$ as $\mathbb{F}_p[\widetilde{H}]$-lattices for $p = 2$ $(i = j = 137)$, for $p = 2$ and $3$ $(i = j = 139)$, for $p = 3$ $(i = j = 145, 147)$.

Furthermore, for the cases $G = N_{4,i}$ with $X = Y$ (82 cases of 152), the following conditions are also equivalent:

$(0)$ $T_i$ and $T_i'$ are birationally $k$-equivalent;

$(1)$ $T_i$ and $T_i'$ are stably birationally $k$-equivalent.

## Main theorem 5 ([HY, Theorem 1.39]) $\dim(T)=4$ $(I_{4,i})$: up to $\overset{\text{s.b.}}{\approx}$

Let $T_i$ and $T_j'$ $(1\leq i,j\leq 7)$ be algebraic $k$-tori of dimension $4$ with the minimal splitting fields $L_i$ and $L_j'$ and the character modules $\widehat{T}_i=M_G$ and $\widehat{T}_j'=M_{G'}$ which satisfy that $G$ and $G'$ are $\mathrm{GL}(4,\mathbb{Z})$-conjugate to $I_{4,i}$ and $I_{4,j}$ respectively. For $1\leq i,j\leq 7$ except for the case $i=j=7$, the following conditions are equivalent:

(1) $T_i \overset{\text{s.b.}}{\approx} T_j'$ (stably birationally $k$-equivalent);
(2) $G\simeq G'$, $L_i=L_j'$, $T_i\times_k K$ and $T_j'\times_k K$ are weak stably birationally $K$-equivalent for any $k\subset K\subset L_i$;
(3) $G\simeq G'$, $L_i=L_j'$, $T_i\times_k K$ and $T_j'\times_k K$ are weak stably birationally $K$-equivalent for any $k\subset K\subset L_i$ corresponding to $\mathrm{WSEC}_r$ $(r\geq 1)$;
(4) $G\simeq G'$, $L_i=L_j'$, $T_i\times_k K$ and $T_j'\times_k K$ are weak stably birationally $K$-equivalent for any $k\subset K\subset L_i$ corresponding to $\mathrm{WSEC}_r$ $(r\geq 1)$ with $[K:k]=d$ where $d=1$ $(i=1,2,4,5,7)$, $d=1,2$ $(i=3,6)$.
In particular, for $d=1$, $(4)\Leftrightarrow G\simeq G'$, $L_i=L_j'$, i.e. $\widetilde{H}\simeq G\simeq G'$.

## Main theorem 5 ([HY, Theorem 1.39]) $\dim(T)=4$ $(I_{4,i})$: up to $\overset{\text{s.b.}}{\approx}$

For the exceptional case $i=j=7$ $(G\simeq C_3\rtimes C_8)$, we have the implications $(1)\Rightarrow(2)\Leftrightarrow(3)\Leftrightarrow(4)$, there exists $\sigma\in\mathrm{Aut}(G)$ such that $G'=G^\sigma$ and $D_6\simeq X=Y\lhd Z$ with $Z/Y\simeq C_2$ ($Y=Z$ is equivalent to $(1)\Leftrightarrow(2)$) and we have $(1)\Leftrightarrow M_G\simeq M_{G^\sigma}$ as $\widetilde{H}$-lattices (this is equivalent to $X=Y$) $\Leftrightarrow M_G\otimes_{\mathbb{Z}}\mathbb{F}_3\simeq M_{G^\sigma}\otimes_{\mathbb{Z}}\mathbb{F}_3$ as $\mathbb{F}_3[\widetilde{H}]$-lattices. Furthermore, we have $X=Y$ for all the cases $G=I_{4,i}$ $(1\leq i\leq 7)$, and hence the following conditions are also equivalent:

(0) $T_i$ and $T_i'$ are birationally $k$-equivalent;

(1) $T_i$ and $T_i'$ are stably birationally $k$-equivalent.

## Corollary (Stably birational classification for $T$ with $\dim(T) = 4$)

Let $\mathcal{T}_4$ be the category of algebraic $k$-tori of dimension $4$. We get a classification (disjoint union decomposition) of $\mathcal{T}_4$ with respect to the stably birationally $k$-equivalence $\overset{\text{s.b.}}{\approx}$:

$$\mathcal{T}_4 = \coprod_{r=0}^{128} \mathrm{WSEC}_r = \mathrm{SEC}_0 \coprod \left( \coprod_{r=1}^{128} \coprod_{\substack{L/k \\ \mathrm{Gal}(L/k) \simeq \mathrm{G}_r}} \coprod_{t=1}^{\lambda_r} \mathrm{SEC}_{r,L,t} \right)$$

modulo $\overset{\text{s.b.}}{\approx}$ where $\mathrm{SEC}_0$ is the stably $k$-equivalent class consists of stably $k$-rational tori $T \in \mathcal{T}_4$, $\mathrm{Gal}(L/k) \simeq \mathrm{G}_r \simeq N_{4,i}$ $(1 \leq r \leq 128)$ and $\lambda_r = |\mathrm{WSEC}_{r,L}| = |Y \backslash \mathrm{Aut}(G)|$ is given as in [HY, Table 4] and Main theorem $3$.

## Main theorem 6 ([HY, Theorem 1.42]) $\dim(T) = 4$: seven $I_{4,i}$ cases

Let $T_i$ $(1 \leq i \leq 7)$ be an algebraic $k$-torus of dimension $4$ with the minimal splitting field $L_i$ and the character module $\widehat{T}_i = M_{G_i}$ which satisfies that $G_i$ is $\mathrm{GL}(4,\mathbb{Z})$-conjugate to $I_{4,i}$. Let $T_i^\sigma$ be the algebraic $k$-torus with $\widehat{T}_i^\sigma = M_{G^\sigma}$ $(\sigma \in \mathrm{Aut}(G_i))$. Then $T_i$ and $T_i^\sigma$ are not stably but retract $k$-rational, i.e. $[M_{G_i}]^{fl} \neq 0$, $[M_{G_i^\sigma}]^{fl} \neq 0$ but invertible of infinite order. We have:

(1) [HY17, Theorem 1.27] If $L_1 = L_2$, then $T_1 \times_k T_2$ is stably $k$-rational;
(2) $T_3 \times_k T_3^\sigma$ is stably $k$-rational for $\sigma \in \mathrm{Aut}(G_3)$ with $1 \neq \overline{\sigma} \in \mathrm{Aut}(G_3)/\mathrm{Inn}(G_3) \simeq C_2$;
(3) [HY17, Theorem 1.27] If $L_4 = L_5$, then $T_4 \times_k T_5$ is stably $k$-rational;
(4) $T_6 \times_k T_6^\sigma$ is stably $k$-rational for $\sigma \in \mathrm{Aut}(G_6)$ with $1 \neq \overline{\sigma} \in \mathrm{Aut}(G_6)/\mathrm{Inn}(G_6) \simeq C_2$;
(5) $T_7 \times_k T_7^\sigma$ is stably $k$-rational for $\sigma \in \mathrm{Aut}(G_7)$ with $1 \neq \overline{\sigma} \in \mathrm{Aut}(G_7)/X \simeq C_2$ where

$$X = \mathrm{Aut}_{\mathrm{GL}(4,\mathbb{Z})}(G_7) = \{\sigma \in \mathrm{Aut}(G_7) \mid G_7 \text{ and } G_7^\sigma \text{ are conjugate in } \mathrm{GL}(4,\mathbb{Z})\} \simeq D_6.$$

The following theorem can answer Problem 1 for algebraic $k$-tori $T$ and $T'$ of dimensions $m \geq 3$ and $n \geq 3$ respectively with $[\widehat{T}]^{fl}, [\widehat{T}']^{fl} \in \mathrm{WSEC}_r$ $(1 \leq r \leq 128)$ via Main theorem 2, Main theorem 4, and Main theorem 5.

**Main theorem 7 ([HY, Theorem 1.44]) higher dimensional cases**

Let $T$ be an algebraic $k$-torus of dimension $m \geq 3$ with the minimal splitting field $L$, $\widehat{T} = M_G$, $G \leq \mathrm{GL}(m, \mathbb{Z})$ and $[\widehat{T}]^{fl} \in \mathrm{WSEC}_r$ $(1 \leq r \leq 128)$. Then there exists an algebraic $k$-torus $T''$ of dimension 3 or 4 with the minimal splitting field $L''$, $\widehat{T}'' = M_{G''}$, and $G'' = N_{3,i}$ $(1 \leq i \leq 15)$, $G'' = N_{4,i}$ $(1 \leq i \leq 152)$ or $G'' = I_{4,i}$ $(1 \leq i \leq 7)$ such that $T''$ and $T$ are stably birationally $k$-equivalent and $L'' \subset L$, i.e. $[M_{G''}]^{fl} = [M_G]^{fl}$ as $G$-lattices and $G$ acts on $[M_{G''}]^{fl}$ through $G'' \simeq G/N$ for the corresponding normal subgroup $N \lhd G$.

## Corollary (Stably birational classification for $T$ with $[\widehat{T}]^{fl} \in \mathrm{WSEC}_r$ $(0 \leq r \leq 128)$)

Let $\mathcal{T}'$ be the category of algebraic $k$-tori $T$ with $[\widehat{T}]^{fl} \in \mathrm{WSEC}_r$ $(0 \leq r \leq 128)$. We get a classification (disjoint union decomposition) of $\mathcal{T}'$ with respect to the stably birationally $k$-equivalence $\overset{\text{s.b.}}{\approx}$:

$$\mathcal{T}' = \coprod_{r=0}^{128} \mathrm{WSEC}_r = \mathrm{SEC}_0 \coprod \left( \coprod_{r=1}^{128} \coprod_{\substack{L/k \\ \mathrm{Gal}(L/k) \simeq \mathrm{G}_r}} \coprod_{t=1}^{\lambda_r} \mathrm{SEC}_{r,L,t} \right)$$

modulo $\overset{\text{s.b.}}{\approx}$ where $\mathrm{SEC}_0$ is the stably $k$-equivalent class consists of stably $k$-rational tori $T \in \mathcal{T}'$, $\mathrm{Gal}(L/k) \simeq \mathrm{G}_r \simeq N_{3,i} \in \mathrm{WSEC}_r$ $(1 \leq r \leq 13)$, $N_{4,i} \in \mathrm{WSEC}_r$ $(14 \leq r \leq 121)$, $I_{4,i} \in \mathrm{WSEC}_r$ $(122 \leq r \leq 128)$ and $\lambda_r = |\mathrm{WSEC}_{r,L}| = |Y \backslash \mathrm{Aut}(G)|$ is given as in Main theorem 1, [HY, Table 4] and Main theorem 3.

# Sketch: Proof of Main theorems 1, 2, 3, 4, 5, 6, 7

- For Main theorems 1 (and 3), we get $\mathrm{WSEC}_r$ using torus invariants and establish $[\widehat{T}_i]^{fl} = [\widehat{T}_j]^{fl}$ as $\widetilde{H}$-lattices for some $i < j$ for $\widehat{T}_i = M_{G_i}$, $G_i = N_{3,i}$ with $G_i \simeq G'_j$, $L_i = L'_j$, i.e. $\widetilde{H} \simeq G_i \simeq G'_j$.
- For Main theorems 2 (and 4, 5), $(1) \Rightarrow (2)$, we should show that $[\widehat{T}_i]^{fl} = [\widehat{T}_j]^{fl}$ as $\widetilde{H}$-lattices for $i = j$ with $G \simeq G'$, $L_i = L'_j$, i.e. $\widetilde{H} \simeq G \simeq G'$ via the $p$-part of the flabby class $[\widehat{T}]^{fl}$ $(p = 2, 3)$ as a (faithful and indecomposable) $\mathbb{Z}_p[\mathrm{Syl}_p(G)]$-lattice via $p$-adic analysis.
- For Main theorems 2 (and 4, 5), $(2) \Rightarrow (1)$, we should show that if $i = j$ with $G \simeq G'$, $L_i = L'_j$, i.e. $\widetilde{H} \simeq G \simeq G'$, then $Y = Z$ (which is equivalent to $(1) \Leftrightarrow (2)$) with some exceptional cases.
- For Main theorems 2 (and 4, 5), $(2) \Leftrightarrow (3) \Leftrightarrow (4)$, we should show that $(4) \Rightarrow (2)$ because we already have $(2) \Rightarrow (3) \Rightarrow (4)$.
- For Main theorem 6, we should show that $[M_{G_i}]^{fl} + [M_{G_i^\sigma}]^{fl} = 0$ $(i = 3, 6, 7)$.
- For Main theorem 7, we should show that $L'' \subset L$.

# Torus invariants (1/2)

Define

$$Ш^i_\omega(G,M) := \mathrm{Ker}\left\{ H^i(G,M) \xrightarrow{\mathrm{res}} \prod_{g\in G} H^i(\langle g\rangle, M)\right\} \quad (i \geq 1).$$

**Theorem (Kunyavskii, Skorobogatov and Tsfasman, 1989)**

Let $M$ be a $G$-lattice. Then three groups

$$\begin{aligned}
&Ш^1_\omega(G,[M]^{fl}) \simeq Ш^2_\omega(G,M) \simeq H^1(G,[M]^{fl}),\\
&Ш^2_\omega(G,([M]^{fl})^\circ) \simeq Ш^1_\omega(G,M^\circ) \simeq H^1(G,(([M]^{fl})^\circ)^{fl}),\\
&Ш^2_\omega(G,[M]^{fl}) \simeq Ш^1_\omega(G,([M]^{fl})^{fl}) \simeq H^1(G,([M]^{fl})^{fl})
\end{aligned}$$

are invariants of the flabby class $[M]^{fl}$ of $M$.

# Torus invariants (2/2)

## Definition (Torus invariants)

Let $G \leq \mathrm{GL}(n,\mathbb{Z})$ and $M_G$ be the corresponding $G$-lattice of $\mathbb{Z}$-rank $n$. The torus invariants $TI_G = [l_1, l_2, l_3, l_4]$ of $[M_G]^{fl}$ are defined to be

$$l_1 = \begin{cases} 0 & \text{if } \ [M_G]^{fl} = 0, \\ 1 & \text{if } \ [M_G]^{fl} \neq 0 \text{ but is invertible,} \\ 2 & \text{if } \ [M_G]^{fl} \text{ is not invertible,} \end{cases}$$

$l_2$ (resp. $l_3$, $l_4$) is the abelian invariants of $Ш^1_\omega(G, [M_G]^{fl})$ (resp. $Ш^2_\omega(G, ([M_G]^{fl})^\circ)$, $Ш^2_\omega(G, [M_G]^{fl})$).

## Definition (The $p$-part $\widetilde{N}_p$ of $[M]^{fl}$ as a $\mathbb{Z}_p[\mathrm{Syl}_p(G)]$-lattice)

Let $G$ be a finite group, $M$ be a $G$-lattice and

$$0 \to M \to P \to F \to 0$$

be a flabby resolution of $M$ where $P$ is permutation and $F$ is flabby.
(1) (Kunyavskii, 1990) By tensering with $\mathbb{Z}_2$, we also get a flabby resolution of $\widetilde{M} = M \otimes_{\mathbb{Z}} \mathbb{Z}_2$ as $\mathbb{Z}_2[G]$-lattices:

$$0 \to \widetilde{M} \to \widetilde{P} \to \widetilde{F} \to 0$$

where $\widetilde{P} = P \otimes_{\mathbb{Z}} \mathbb{Z}_2$ is permutation and $\widetilde{F} = F \otimes_{\mathbb{Z}} \mathbb{Z}_2$ is flabby. Take a direct sum decomposition $\widetilde{F} \simeq \widetilde{N} \oplus \widetilde{Q}$ where $\widetilde{N}$ does not contain a permutation direct summand and $\widetilde{Q}$ is permutation.
(2) Take a flabby resolution of $M_p = M \mid_{\mathrm{Syl}_p(G)}$ as $\mathbb{Z}[\mathrm{Syl}_p(G)]$-lattices:

$$0 \to M_p \to P_p \to F_p \to 0$$

where $P_p$ is permutation and $F_p$ is flabby. Then $[F_p] = [F \mid_{\mathrm{Syl}_p(G)}]$ and take the direct sum decomposition $\widetilde{F}_p = F_p \otimes_{\mathbb{Z}} \mathbb{Z}_p \simeq \widetilde{N}_p \oplus \widetilde{Q}_p$ as $\mathbb{Z}_p[\mathrm{Syl}_p(G)]$-lattices where $\widetilde{N}_p$ does not contain a permutation direct summand and $\widetilde{Q}_p$ is permutation.

# Example ($\widetilde{N}$ is not uniquely determined)

- $G = I_{4,1} \leq \mathrm{GL}(4,\mathbb{Z})$ with $G \simeq F_{20}$.
- $M_G$ is the corresponding $G$-lattice with rank $M_G = 4$.
- Take $\widetilde{M} = M_G \otimes \mathbb{Z}_2$.
- Then we see that $\widetilde{M}$ is an indecomposable $\mathbb{Z}_2[G]$-lattice and $\widetilde{M} \oplus \mathbb{Z}_2 \simeq \mathbb{Z}_2[G/C_4]$ as $\mathbb{Z}_2[G]$-lattices.
- In particular, $\widetilde{M}^{\oplus r}$ does not contain a permutatin direct summand for any $r \geq 1$.
- We get that $F = [M_G]^{fl}$ with rank $F = 16$ and $\widetilde{F} = F \otimes_{\mathbb{Z}} \mathbb{Z}_2 \simeq \widetilde{M} \oplus \widetilde{M} \oplus \widetilde{M} \oplus \mathbb{Z}_2[G/C_5]$ as $\mathbb{Z}_2[G]$-lattices with $\widetilde{N} \simeq \widetilde{M} \oplus \widetilde{M} \oplus \widetilde{M}$.
- On the other hand, we also see that $\widetilde{F} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \simeq \mathbb{Z}_2[G/C_4] \oplus \mathbb{Z}_2[G/C_4] \oplus \mathbb{Z}_2[G/C_4] \oplus \mathbb{Z}_2[G/C_5]$ and hence $\widetilde{N} = 0$.
- $\widetilde{N}$ is not uniquely determined in general when $G$ is not a $2$-group.

- Retract non-rationality of an algebraic $k$-torus $T$ can be detected by the non-vanishing of $\widetilde{N}_p$.

**Lemma ([HY, Lemma 7.5])**

Let $M$ be a $G$-lattice and $\widetilde{N}_p$ be the $p$-part of $[M]^{fl}$ as a $\mathbb{Z}_p[\mathrm{Syl}_p(G)]$-lattice. If $\widetilde{N}_p \neq 0$, then $[M]^{fl}$ is not invertible. In particular, the corresponding torus $T$ with $\widehat{T} = M$ is not retract $k$-rational.

- The following theorem is given by Kunyavskii (1990) except for the indecomposability of $\widetilde{N}_2$.

**Theorem ([HY, Theorem 7.14], see also Kunyavskii (1990))**

Let $G = N_{3,i}$ $(1 \leq i \leq 15)$ and $M_G$ be the corresponding $G$-lattice. The 2-part $\widetilde{N}_2$ of $[M_G]^{fl}$ as a $\mathbb{Z}_2[\mathrm{Syl}_2(G)]$-lattice is a faithful and indecomposable $\mathbb{Z}_2[\mathrm{Syl}_2(G)]$-lattice and the $\mathbb{Z}_2$-rank of $\widetilde{N}_2$ is given as in [HY, Table 14].

**Theorem ([HY, Theorem 7.15])**

Let $G = N_{4,i}$ $(1 \leq i \leq 152)$ and $M_G$ be the corresponding $G$-lattice. The 2-part $\widetilde{N}_2$ (resp. 3-part $\widetilde{N}_3$) of $[M_G]^{fl}$ as a $\mathbb{Z}_2[\mathrm{Syl}_2(G)]$-lattice (resp. $\mathbb{Z}_3[\mathrm{Syl}_3(G)]$-lattice) is a faithful and indecomposable $\mathbb{Z}_2[\mathrm{Syl}_2(G)]$-lattice (resp. $\mathbb{Z}_3[\mathrm{Syl}_3(G)]$-lattice) unless it vanishes and the $\mathbb{Z}_2$-rank of $\widetilde{N}_2$ (resp. the $\mathbb{Z}_3$-rank of $\widetilde{N}_3$) is given as in [HY, Table 15].