Rationality problem for algebraic tori

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The aim of this talk is to introduce some results in

[HY] A. Hoshi, A. Yamasaki, Rationality problem for algebraic tori, arXiv:1210.4525, 125 pages.

§1. Rationality problem for algebraic tori (1/3)

- k: a base field which is NOT algebraically closed! (TODAY)
- T: algebraic torus, i.e. k-form of a split torus;
  T is an algebraic group over k with $T \times_k \overline{k} \simeq (\mathbb{G}_{m,\overline{k}})^n$.

Rationality problem for algebraic tori

Whether T is rational over k?

Let $R^{(1)}_{K/k}(\mathbb{G}_m)$ be the norm one torus of K/k, i.e. the kernel of the norm map $N_{K/k} : R_{K/k}(\mathbb{G}_m) \to \mathbb{G}_m$ where $R_{K/k}$ is the Weil restriction:

$$1 \longrightarrow R^{(1)}_{K/k}(\mathbb{G}_m) \longrightarrow R_{K/k}(\mathbb{G}_m) \xrightarrow{N_{K/k}} \mathbb{G}_m \longrightarrow 1.$$

- $\exists 2$ algebraic tori with $\dim(T) = 1$;
  the trivial torus $\mathbb{G}_m$ and $R^{(1)}_{K/k}(\mathbb{G}_m)$ with $[K : k] = 2$,
  which are rational over k.

Rationality problem for algebraic tori (2/3)

- $\exists 13$ algebraic tori with $\dim(T) = 2$.

Theorem (Voskresenskii, 1967) 2-dim. algebraic tori T

T is rational over k.

- $\exists 73$ algebraic tori with $\dim(T) = 3$.

Theorem (Kunyavskii, 1990) 3-dim. algebraic tori T

(i) $\exists 58$ algebraic tori T which are rational over k;
(ii) $\exists 15$ algebraic tori T which are not rational over k.

- What happens in higher dimensions?

k-tori and G-lattices

- T: k-torus (= algebraic torus over k)
$\Longrightarrow \exists$ finite Galois extension L/k such that $T \times_k L \simeq (\mathbb{G}_{m,L})^n$.
- $G = \mathrm{Gal}(L/k)$ where L is the minimal splitting field.

Category of algebraic k-tori which split$/L \overset{\text{duality}}{\longleftrightarrow}$ Category of G-lattices
(i.e. finitely generated $\mathbb{Z}$-free $\mathbb{Z}[G]$-module)

- $T \mapsto$ the character group $X(T) = \mathrm{Hom}(T, \mathbb{G}_m)$: G-lattice.
- $\exists T$ which splits over L with $X(T) \simeq M \leftarrow\!\shortmid M$: G-lattice
- Tori of dimension $n \overset{1:1}{\longleftrightarrow}$ elements of the set $H^1(\mathcal{G}, \mathrm{GL}(n, \mathbb{Z}))$
where $\mathcal{G} = \mathrm{Gal}(k_{\mathrm{s}}/k)$ since $\mathrm{Aut}(\mathbb{G}_m^n) = \mathrm{GL}(n, \mathbb{Z})$.
- k-torus T of dimension n is determined uniquely by the integral representation $h : \mathcal{G} \to \mathrm{GL}(n, \mathbb{Z})$ up to conjugacy, and the group $h(\mathcal{G})$ is a finite subgroup of $\mathrm{GL}(n, \mathbb{Z})$.
- The function field of $T \overset{\text{identified}}{\longleftrightarrow} L(M)^G$: invariant field.

Rationality problem for algebraic tori (3/3)

- L/k: Galois extension with $G = \mathrm{Gal}(L/k)$.
- $M = \bigoplus_{1 \leq j \leq n} \mathbb{Z} \cdot u_j$: G-lattice with a $\mathbb{Z}$-basis $\{u_1, \ldots, u_n\}$.
- G acts on $L(x_1, \ldots, x_n)$ by

$$\sigma(x_j) = \prod_{i=1}^{n} x_i^{a_{i,j}}, \quad 1 \leq j \leq n$$

  for any $\sigma \in G$, when $\sigma(u_j) = \sum_{i=1}^{n} a_{i,j} u_i$, $a_{i,j} \in \mathbb{Z}$.
- $L(M) := L(x_1, \ldots, x_n)$ with this action of G.
- $\boxed{\text{The function field of algebraic } k\text{-torus } T \overset{\text{identified}}{\longleftrightarrow} L(M)^G}$

Rationality problem for algebraic tori (2nd form)

Whether $L(M)^G$ is rational over k?
(= purely transcendental over k?; $L(M)^G = k(\exists t_1, \ldots, \exists t_n)$?)

§2. Main theorems. Some definitions.

- L/k: a finite generated field extension.

Definition (stably rational)

L is called stably rational over k if $L(y_1, \ldots, y_m)$ is rational over k.

Definition (retract rational)

L is retract rational over k if $\exists k$-algebra $R \subset L$ such that
(i) L is the quotient field of R;
(ii) $\exists f \in k[x_1, \ldots, x_n]$ $\exists k$-algebra hom. $\varphi : R \to k[x_1, \ldots, x_n][1/f]$ and $\psi : k[x_1, \ldots, x_n][1/f] \to R$ satisfying $\psi \circ \varphi = 1_R$.

Definition (unirational)

L is unirational over k if L is a subfield of rational field extension of k.

- "rational" $\Longrightarrow$ "stably rational" $\Longrightarrow$ "retract rational" $\Longrightarrow$ "unirational".
- $L(M)^G$ (resp. T) is always unirational over k.

Rationality of algebraic tori (3-dim.)

- $\exists 73$ $\mathbb{Z}$-coujugacy subgroups $G \leq \mathrm{GL}(3, \mathbb{Z})$
  ($\exists 73$ 3-dim. algebraic tori T).

Theorem (Kunyavskii, 1990) 3-dim. algebraic tori T (precise form)

(i) $\exists 58$ algebraic tori T which are rational over k;
(ii) $\exists 15$ algebraic tori T which are not rational over k;
(iii) T is rational over k
$\iff$ T is stably rational over k
$\iff$ T is retract rational over k.

Main Theorem I. Rationality of algebraic tori (4-dim.)

- $\exists 710$ $\mathbb{Z}$-coujugacy subgroups $G \leq \mathrm{GL}(4, \mathbb{Z})$ ($\exists 710$ 4-dim. algebraic tori T).

Theorem ([HY]) 4-dim. algebraic tori T

(i) $\exists 487$ algebraic tori T which are stably rational over k;
(ii) $\exists \quad 7$ algebraic tori T which are not stably but retract rational over k;
(iii) $\exists 216$ algebraic tori T which are not retract rational over k.

- We do not know "rationality" over k.
- Voskresenskii's conjecture: any stably rational torus over k is rational over k (Zariski problem).
- what happens for dimension 5?

Main Theorem II. Rationality of algebraic tori (5-dim.)

- $\exists 6079$ $\mathbb{Z}$-coujugacy subgroups $G \leq \mathrm{GL}(5,\mathbb{Z})$ ($\exists 6079$ 5-dim. algebraic tori T).

Theorem ([HY]) 5-dim. algebraic tori T

(i) $\exists 3051$ algebraic tori T which are stably rational over k;
(ii) $\exists$ 25 algebraic tori T which are not stably but retract rational over k;
(iii) $\exists 3003$ algebraic tori T which are not retract rational over k.

- what happens for dimension 6?
- BUT we do not know the answer for dimension 6.
- $\exists 85308$ $\mathbb{Z}$-coujugacy subgroups $G \leq \mathrm{GL}(6,\mathbb{Z})$ ($\exists 85308$ 6-dim. algebraic tori T).

§3. Proof: Flabby (Flasque) resolution (1/2)

- $\boxed{\text{The function field of } n\text{-dim. } T \overset{\text{identified}}{\longleftrightarrow} L(M)^G,\ G \leq \mathrm{GL}(n,\mathbb{Z})}$
- M: G-lattice, i.e. f.g. $\mathbb{Z}$-free $\mathbb{Z}[G]$-module.

Definition

(i) M is permutation $\overset{\text{def}}{\iff}$ $M \simeq \oplus_{1\leq i\leq m}\mathbb{Z}[G/H_i]$.

(ii) M is stably permutation $\overset{\text{def}}{\iff}$ $M \oplus \exists P \simeq P'$, P, P': permutation.

(iii) M is invertible $\overset{\text{def}}{\iff}$ $M \oplus \exists M' \simeq P$: permutation.

(iv) M is coflabby $\overset{\text{def}}{\iff}$ $H^1(H,M)=0$ $(\forall H \leq G)$.

(v) M is flabby $\overset{\text{def}}{\iff}$ $\widehat{H}^{-1}(H,M)=0$ $(\forall H \leq G)$. ($\widehat{H}$: Tate cohomology)

- "permutation"
  $\Longrightarrow$ "stably permutation"
  $\Longrightarrow$ "invertible"
  $\Longrightarrow$ "flabby and coflabby".

Proof: Flabby (Flasque) resolution (2/2)

Commutative monoid $\mathcal{M}$

$M_1 \sim M_2 \overset{\text{def}}{\iff} M_1 \oplus P_1 \simeq M_2 \oplus P_2$ ($\exists P_1, \exists P_2$: permutation).
$\Longrightarrow$ commutative monoid $\mathcal{M}$: $[M_1] + [M_2] := [M_1 \oplus M_2]$, $0 = [P]$.

Theorem (Endo-Miyata, 1974, Colliot-Thélène-Sansuc, 1977)

$\exists P$: permutation, $\exists F$: flabby such that

$$0 \to M \to P \to F \to 0\text{: flabby resolution of } M.$$

$[M]^{fl} := [F]$, $[M]^{fl}$ is invertible $\overset{\text{def}}{\iff} [M]^{fl} = [E]$ ($\exists E$: invertible).

Theorem (Endo-Miyata, 1973, Voskresenskii, 1974, Saltman, 1984)

(EM73) $[M]^{fl} = 0 \iff L(M)^G$ is stably rational over k.
(Vos74) $[M]^{fl} = [M']^{fl} \iff L(M)^G(x_1, \ldots, x_m) \simeq L(M')^G(y_1, \ldots, y_n)$.
(Sal84) $[M]^{fl}$ is invertible $\iff L(M)^G$ is retract rational over k.

Our contribution

- We give a procedure to compute a flabby resolution of M, in particular $[M]^{fl} = [F]$, effectively (with smaller rank after base change) by computer software GAP.
- The function `IsFlabby` (resp. `IsCoflabby`) may determine whether M is flabby (resp. coflabby).
- The function `IsInvertibleF` may determine whether $[M]^{fl} = [F]$ is invertible ($\leftrightarrow$ whether $L(M)^G$ (resp. T) is retract rational).
- We provide some functions for checking a possibility of isomorphism

$$\left(\bigoplus_{i=1}^{r} a_i\,\mathbb{Z}[G/H_i]\right) \oplus a_{r+1}F \simeq \bigoplus_{i=1}^{r} b_i'\,\mathbb{Z}[G/H_i] \qquad (*)$$

  by computing some invariants (e.g. trace, $\widehat{Z}^0$, $\widehat{H}^0$) of both sides.
- [HY, Example 10.7]. $G \simeq S_5 \leq \mathrm{GL}(5,\mathbb{Z})$ with number $(5,946,4)$
  $\Longrightarrow \mathrm{rank}(F) = 17$ and $\mathrm{rank}(*) = 88$ holds
  $\Longrightarrow [F] = 0 \Longrightarrow L(M)^G$ (resp. T) is stably rational over k.

Application

Corollary ($[F]=[M]^{fl}$: invertible case, $G\simeq S_5, F_{20}$)

$\exists T$, T'; 4-dim. not stably rational algebraic tori over k such that $T\not\approx T'$ (birational) and $T\times T'$: 8-dim. stably rational over k.
$\because -[M]^{fl}=[M']^{fl}\neq 0$.

Prop. ([HY], Krull-Schmidt fails for permutation D_6-lattices)

$\{1\}$, $C_2^{(1)}$, $C_2^{(2)}$, $C_2^{(3)}$, C_3, C_2^2, C_6, $S_3^{(1)}$, $S_3^{(2)}$, D_6: conj. subgroups of D_6.

$$\mathbb{Z}[D_6]\oplus\mathbb{Z}[D_6/C_2^2]^{\oplus 2}\oplus\mathbb{Z}[D_6/C_6]\oplus\mathbb{Z}[D_6/S_3^{(1)}]\oplus\mathbb{Z}[D_6/S_3^{(2)}]$$
$$\simeq\mathbb{Z}[D_6/C_2^{(1)}]\oplus\mathbb{Z}[D_6/C_2^{(2)}]\oplus\mathbb{Z}[D_6/C_2^{(3)}]\oplus\mathbb{Z}[D_6/C_3]\oplus\mathbb{Z}^{\oplus 2}.$$

- D_6 is the smallest example exhibiting the failure of K-S:

Theorem (Dress, 1973)

Krull-Schmidt holds for permutation G-lattices $\iff$ $G/O_p(G)$ is cyclic where $O_p(G)$ is the maximal normal p-subgroup of G.

Krull-Schmidt and Direct sum cancelation

Theorem (Hindman-Klingler-Odenthal, 1998) Assume $G \neq D_8$

Krull-Schmidt holds for G-lattices $\iff$ (i) $G = C_p$ ($p \leq 19$; prime), (ii) $G = C_n$ ($n = 1, 4, 8, 9$), (iii) $G = V_4$ or (iv) $G = D_4$.

Theorem (Endo-Hironaka, 1979)

Direct sum cancellation holds, i.e. $M_1 \oplus N \simeq M_2 \oplus N \Longrightarrow M_1 \simeq M_2$,
$\Longrightarrow G$ is abelian, dihedral, A_4, S_4 or A_5 (*).

- via projective class group (see Swan (1988) Corollary 1.3, Section 7).
- Except for (*) $\Longrightarrow$ Direct sum cancelation fails $\Longrightarrow$ K-S fails

Theorem ([HY]) $G \leq \mathrm{GL}(n, \mathbb{Z})$ (up to conjugacy)

(i) $n \leq 4 \Longrightarrow$ K-S holds.
(ii) $n = 5$. K-S fails $\iff$ 11 groups (among 6079 groups).
(iii) $n = 6$. K-S fails $\iff$ 131 groups (among 85308 groups).

Special case: $T = R^{(1)}_{K/k}(\mathbb{G}_m)$; norm one tori (1/5)

- Rationality problem for $T = R^{(1)}_{K/k}(\mathbb{G}_m)$ is investigated by S. Endo, Colliot-Thélène and Sansuc, W. Hürlimann, L. Le Bruyn, A. Cortella and B. Kunyavskii, N. Lemire and M. Lorenz, M. Florence, etc.

Theorem (Endo and Miyata, 1974), (Saltman, 1984)

Let K/k be a finite Galois field extension and $G = \mathrm{Gal}(K/k)$.
(i) T is retract k-rational $\iff$ all the Sylow subgroups of G are cyclic;
(ii) T is stably k-rational $\iff$ G is a cyclic group, or a direct product of a cyclic group of order m and a group $\langle \sigma, \tau \,|\, \sigma^n = \tau^{2^d} = 1, \tau\sigma\tau^{-1} = \sigma^{-1} \rangle$, where $d, m \geq 1, n \geq 3, m, n$: odd, and $(m, n) = 1$.

Theorem (Endo, 2011)

Let K/k be a finite non-Galois, separable field extension and L/k be the Galois closure of K/k. Assume that the Galois group of L/k is nilpotent. Then the norm one torus $T = R^{(1)}_{K/k}(\mathbb{G}_m)$ is not retract k-rational.

Special case: $T = R^{(1)}_{K/k}(\mathbb{G}_m)$; norm one tori (2/5)

- Let K/k be a finite non-Galois, separable field extension
- Let L/k be the Galois closure of K/k.
- Let $G = \mathrm{Gal}(L/k)$ and $H = \mathrm{Gal}(L/K) \leq G$.

Theorem (Endo, 2011)

Assume that all the Sylow subgroups of G are cyclic.
Then T is retract k-rational.
$T = R^{(1)}_{K/k}(\mathbb{G}_m)$ is stably k-rational $\iff$ G is the dihedral group D_n of order $2n$ with n odd $(n \geq 3)$ or the direct product of the cyclic group C_m of order m and the dihedral group D_n of order $2n$, where m, n are odd, $m, n \geq 3$, $(m, n) = 1$, and $H \leq D_n$ is of order 2.

Special case: $T = R^{(1)}_{K/k}(\mathbb{G}_m)$; norm one tori (3/5)

Theorem (Endo, 2011)

Assume that $\mathrm{Gal}(L/k) = S_n$, $n \geq 3$, and $\mathrm{Gal}(L/K) = S_{n-1}$ is the stabilizer of one of the letters in S_n.
(i) $R^{(1)}_{K/k}(\mathbb{G}_m)$ is retract k-rational $\iff$ n is a prime;
(ii) $R^{(1)}_{K/k}(\mathbb{G}_m)$ is (stably) k-rational $\iff$ $n = 3$.

Theorem (Endo, 2011)

Assume that $\mathrm{Gal}(L/k) = A_n$, $n \geq 4$, and $\mathrm{Gal}(L/K) = A_{n-1}$ is the stabilizer of one of the letters in A_n.
(i) $R^{(1)}_{K/k}(\mathbb{G}_m)$ is retract k-rational $\iff$ n is a prime;
(ii) $\exists t \in \mathbb{N}$ s.t. $[R^{(1)}_{K/k}(\mathbb{G}_m)]^{(t)}$ is stably k-rational $\iff$ $n = 5$.

- $[R^{(1)}_{K/k}(\mathbb{G}_m)]^{(t)}$: the product of t copies of $R^{(1)}_{K/k}(\mathbb{G}_m)$.

Special case: $T = R^{(1)}_{K/k}(\mathbb{G}_m)$; norm one tori (4/5)

Theorem ([HY], Rationality for $R^{(1)}_{K/k}(\mathbb{G}_m)$ (dim. 4, $[K:k]=5$))

Let K/k be a separable field extension of degree 5 and L/k be the Galois closure of K/k. Assume that $G = \mathrm{Gal}(L/k)$ is a transitive subgroup of S_5 and $H = \mathrm{Gal}(L/K)$ is the stabilizer of one of the letters in G. Then the rationality of $R^{(1)}_{K/k}(\mathbb{G}_m)$ is given by

G		$L(M) = L(x_1,x_2,x_3,x_4)^G$
$5T1$	C_5	stably k-rational
$5T2$	D_5	stably k-rational
$5T3$	F_{20}	not stably but retract k-rational
$5T4$	A_5	stably k-rational
$5T5$	S_5	not stably but retract k-rational

- This theorem is already known except for the case of A_5 (Endo).
- Stably k-rationality for the case A_5 is asked by S. Endo (2011).

Special case: $T = R^{(1)}_{K/k}(\mathbb{G}_m)$; norm one tori (5/5)

By combining this theorem with Endo's theorem, we obtain:

Corollary

Let K/k be a non-Galois separable field extension of degree n and L/k be the Galois closure of K/k. Assume that $\mathrm{Gal}(L/k) = A_n$, $n \geq 4$, and $\mathrm{Gal}(L/K) = A_{n-1}$ is the stabilizer of one of the letters in A_n. Then $R^{(1)}_{K/k}(\mathbb{G}_m)$ is stably k-rational $\iff$ $n = 5$.