On the simplest number fields and related Thue equations

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§1 Introduction: known results of degree 3 case

We consider Thomas' family of cubic Thue equations

$$\boxed{F_m^{(3)}(X,Y) := X^3 - mX^2Y - (m+3)XY^2 - Y^3 = \lambda}$$

for $m \in \mathbb{Z}$ and $\lambda \in \mathbb{Z}$ $(\lambda \neq 0)$.

- For fixed $m, \lambda \in \mathbb{Z}$, $\exists^{<\infty}$ $(x,y) \in \mathbb{Z}^2$ s.t. $F_m^{(3)}(x,y) = \lambda$ (Thue's theorem, 1909)
- The splitting fields $L_m^{(3)} := \mathrm{Spl}_{\mathbb{Q}} F_m^{(3)}(X,1)$ are totally real cyclic cubic fields called Shanks' simplest cubic.
- We may assume that $-1 \leq m$ and $0 < \lambda$ because

$$F_{-m-3}^{(3)}(X,Y) = F_m^{(3)}(-Y,-X),$$
$$-F_m^{(3)}(X,Y) = F_m^{(3)}(-X,-Y).$$

- $L_m^{(3)} = L_{-m-3}^{(3)}$ $(m \in \mathbb{Z})$.

$$F_m^{(3)}(X,Y) := X^3 - mX^2Y - (m+3)XY^2 - Y^3 = \lambda$$

for $m \in \mathbb{Z}$ and $\lambda \in \mathbb{Z}$ $(\lambda \neq 0)$.

- $\lambda = a^3$ for some $a \in \mathbb{Z}$, $F_m^{(3)}(x,y) = a^3$ has three trivial solutions $(a,0),(0,-a),(-a,a)$, i.e. $xy(x+y)=0$.
- If $(x,y) \in \mathbb{Z}^2$ is solution, then $(y,-x-y)$, $(-x-y,x)$ are also solutions because $F_m^{(3)}(x,y)$ is invariant under the action $x \longmapsto y \longmapsto -x-y \longmapsto x$ of order three.
- $3 \mid \#\{(x,y) \mid F_m^{(3)}(x,y) = \lambda\}$.
- $\mathrm{disc}_X F_m^{(3)}(X,1) = (m^2+3m+9)^2$.
- For $\lambda = 1$, Thomas and Mignotte solved completely a family of the equations $(\forall m)$ as follows:

Thomas' theorem for a family of Thue equations

$$F_m^{(3)}(X,Y) = X^3 - mX^2Y - (m+3)XY^2 - Y^3 = 1$$

By using Baker's theory, Thomas proved:

Theorem (Thomas 1990)

If $-1 \le m \le 10^3$ or $1.365 \times 10^7 \le m$, then all solutions of $F_m^{(3)}(x,y) = 1$ are given by trivial solutions $(x,y) = (0,-1), (-1,1), (1,0)$ for $\forall m$ and additionally

$$\begin{aligned}
(x,y) &= (-1,-1), (-1,2), (2,-1) && \text{for} \quad m = -1,\\
(x,y) &= (5,4), (4,-9), (-9,5) && \text{for} \quad m = -1,\\
(x,y) &= (2,1), (1,-3), (-3,2) && \text{for} \quad m = 0,\\
(x,y) &= (-7,-2), (-2,9), (9,-7) && \text{for} \quad m = 2.
\end{aligned}$$

Theorem (Mignotte 1993)

For the remaining case, $\exists$ only trivial solutions.

Mignotte-Pethö-Lemmermeyer (1996)

$$F_m^{(3)}(X,Y) = X^3 - mX^2Y - (m+3)XY^2 - Y^3 = \lambda$$

By using Baker's theory, they proved:

Theorem Mignotte-Pethö-Lemmermeyer (1996)

Let $m \geq 1649$ and $\lambda > 1$. If $F_m^{(3)}(x,y) = \lambda$, then

$$\log|y| < c_1 \log^2(m+3) + c_2 \log(m+1) \log\lambda$$

where

$$c_1 = 700 + 476.4\left(1 - \frac{1432.1}{m+1}\right)^{-1}\left(1.501 - \frac{1902}{m+1}\right) < 1956.4,$$

$$c_2 = 29.82 + \left(1 - \frac{1432.1}{m+1}\right)^{-1}\frac{1432}{(m+1)\log(m+1)} < 30.71.$$

Example (much smaller than previous bounds)

- If $m = 1649$ and $\lambda = 10^9$, then $|y| < 10^{48698}$.

Mignotte-Pethö-Lemmermeyer (1996)

$$\boxed{F_m^{(3)}(X,Y) = X^3 - mX^2Y - (m+3)XY^2 - Y^3 = \lambda}$$

Theorem Mignotte-Pethö-Lemmermeyer (1996)

For $-1 \le m$ and $1 < \lambda \le 2m+3$, all solutions to $F_m^{(3)}(x,y) = \lambda$ are given by trivial solutions for $\lambda = a^3$ and

$$(x,y) \in \{(-1,2),(2,-1),(-1,-1),\\ (-1,m+2),(m+2,-m-1),(-m-1,-1)\}$$

for $\lambda = 2m+3$,
except for $m=1$ in which case ∃extra solutions:

$$(x,y) \in \{(1,-4),(-4,3),(3,1),(3,-11),(-11,8),(8,3)\}$$

for $\lambda = 5$ $(= 2m+3)$.

Lettl-Pethö-Voutier (1999)

Let θ_2 be a root of $f_m(X) := F_m(X,1)$ with $-\frac{1}{2} < \theta_2 < 0$.
By using hypergeometric method, they proved:

Theorem Lettl-Pethö-Voutier (1999)

Let $m \geq 1$ and assume that $(x,y) \in \mathbb{Z}^2$ is a primitive solution to $|F_m^{(3)}(x,y)| \leq \lambda(m)$ with $-\frac{y}{2} < x \leq y$ and $\frac{8\lambda(m)}{2m+3} \leq y$ where $\lambda(m) : \mathbb{Z} \to \mathbb{N}$. Then
(i) x/y is a convergent to θ_2, and we have either $y = 1$ or

$$\left|\frac{x}{y} - \theta_2\right| < \frac{\lambda(m)}{y^3(m+1)} \quad \text{and} \quad y \geq m+2.$$

(ii)Define

$$\kappa = \frac{\log(\sqrt{m^2+3m+9}) + 0.83}{\log(m+\frac{3}{2}) - 1.3}.$$

If $m \geq 30$, then $y^{2-\kappa} < 17.78 \cdot 2.59^{\kappa}\lambda(m)$.

Example (comparing with MPL (1996))

- For $m = 1649$, $|y| < 635\lambda(m)^{1.54}$ instead of $|y| < 10^{46649}\lambda(m)^{288}$.

§2 Main thms: Thm C and Thm S

$$f_m^{(3)}(X) := F_m^{(3)}(X,1), \quad L_m^{(3)} := \mathsf{Spl}_{\mathbb{Q}} f_m^{(3)}(X)$$

Go back to

Theorem (Thomas 1990, Mignotte 1993)

All solutions of $F_m^{(3)}(x,y) = 1$ are given by trivial solutions $(x,y) = (0,-1), (-1,1), (1,0)$ for $\forall m$ and additionally

$$\begin{array}{lll} (x,y) = (-1,-1), (-1,2), (2,-1) & \text{for} & m = -1, \\ (x,y) = (5,4), (4,-9), (-9,5) & \text{for} & m = -1, \\ (x,y) = (2,1), (1,-3), (-3,2) & \text{for} & m = 0, \\ (x,y) = (-7,-2), (-2,9), (9,-7) & \text{for} & m = 2. \end{array}$$

$\boxed{\text{Q.}}$ Why $\exists$ 12 (non-trivial) solutions? meaning?

- $L_{-1}^{(3)} = L_{12}^{(3)}$, $L_{-1}^{(3)} = L_{1259}^{(3)}$, $L_0^{(3)} = L_{54}^{(3)}$, $L_2^{(3)} = L_{2389}^{(3)}$.

Splitting fields $L_m^{(3)}$ know solutions!

$$f_m^{(3)}(X) := F_m^{(3)}(X,1), \quad L_m^{(3)} := \mathrm{Spl}_{\mathbb{Q}} f_m^{(3)}(X)$$

- $L_m^{(3)} = L_{-m-3}^{(3)}$ for $m \in \mathbb{Z}$. $\mathrm{disc}_X f_m^{(3)} = (m^2+3m+9)^2$.

Theorem C (Correspondence)

For a given $m \in \mathbb{Z}$,
$\exists (x,y) \in \mathbb{Z}^2$ with $xy(x+y) \neq 0$ s.t. $F_m^{(3)}(x,y) = \lambda$
for some $\lambda \in \mathbb{N}$ with $\lambda \mid m^2+3m+9$
$\iff \exists n \in \mathbb{Z} \setminus \{m, -m-3\}$ s.t. $L_m^{(3)} = L_n^{(3)}$.
Moreover integers n, m and $(x,y) \in \mathbb{Z}^2$ satisfy

$$N = m + \frac{(m^2+3m+9)xy(x+y)}{F_m^{(3)}(x,y)}$$

where N is either n or $-n-3$.

- ($\Rightarrow$) Using Theorem (Morton 1994, Chapman 1996, Hoshi-Miyake 2009) ($\Leftarrow$) Using resultant method.

Theorem C

For a fixed $m \in \mathbb{Z}$, we obtain the correspondence

$$\boxed{\exists n \in \mathbb{Z} \setminus \{m, -m-3\} \text{ s.t. } L_m^{(3)} = L_n^{(3)}} \quad \text{(I)}$$

$$1:3 \ \Updownarrow \ \text{Theorem C}$$

$$\boxed{\begin{array}{l}\exists (x,y) \in \mathbb{Z}^2 \text{ with } xy(x+y) \neq 0 \\ \quad \text{s.t. } F_m^{(3)}(x,y) = \lambda \,|\, m^2+3m+9\end{array}} \quad \text{(II)}$$

- $\mathrm{disc}(F_m^{(3)}(X,Y)) = (m^2+3m+9)^2$.

R. Okazaki's theorems O_1, O_2

Okazaki announced the following theorems in 2002.
He use his result on gaps between sol's (2002) which is based on Baker's theory: Laurent-Mignotte-Nesterenko (1995).

R. Okazaki, Geometry of a cubic Thue equation, Publ. Math. Debrecen 61 (2002) 267–314.

Theorem O_1 (Okazaki $2002+\alpha$)
For $-1 \leq m < n \in \mathbb{Z}$, if $L_m^{(3)} = L_n^{(3)}$ then $m \leq 35731$.

Theorem O_2 (Okazaki unpublished)
For $-1 \leq m < n \in \mathbb{Z}$, if $L_m^{(3)} = L_n^{(3)}$ then
$m, n \in \{-1, 0, 1, 2, 3, 5, 12, 54, 66, 1259, 2389\}$.

In particular, we get

$$L_{-1}^{(3)} = L_5^{(3)} = L_{12}^{(3)} = L_{1259}^{(3)},$$
$$L_0^{(3)} = L_3^{(3)} = L_{54}^{(3)}, \qquad L_1^{(3)} = L_{66}^{(3)}, \qquad L_2^{(3)} = L_{2389}^{(3)}.$$

Thomas' $4 \times 3 = 12$ non-trivial solutions for $\lambda = 1$

$$\begin{aligned}
(x,y) &= (-1,-1),(-1,2),(2,-1) && \text{for} \quad m=-1,\\
(x,y) &= (5,4),(4,-9),(-9,5) && \text{for} \quad m=-1,\\
(x,y) &= (2,1),(1,-3),(-3,2) && \text{for} \quad m=0,\\
(x,y) &= (-7,-2),(-2,9),(9,-7) && \text{for} \quad m=2
\end{aligned}$$

correspond to

$$\boxed{L_{-1}^{(3)} = L_{12}^{(3)}, \quad L_{-1}^{(3)} = L_{1259}^{(3)}, \quad L_{0}^{(3)} = L_{54}^{(3)}, \quad L_{2}^{(3)} = L_{2389}^{(3)}.}$$

$$\boxed{\begin{array}{l} L_{-1}^{(3)} = L_{5}^{(3)}, \quad L_{0}^{(3)} = L_{3}^{(3)}, \quad L_{1}^{(3)} = L_{66}^{(3)}, \quad L_{3}^{(3)} = L_{54}^{(3)}, \\ L_{5}^{(3)} = L_{12}^{(3)}, \quad L_{5}^{(3)} = L_{1259}^{(3)}, \quad L_{12}^{(3)} = L_{1259}^{(3)} \end{array}}$$

correspond to $7 \times 3 = \exists 21$ (non-trivial) solutions for $\lambda > 1$.

$L_m^{(3)} = L_n^{(3)}$ (33 solutions), $L_n^{(3)} = L_m^{(3)}$ (33 solutions)

Conclusion: in total $\exists 66$ solutions.

Theorem S: Solutions

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$$F_m^{(3)}(X,Y) = X^3 - mX^2Y - (m+3)XY^2 - Y^3 = \lambda$$

By Theorem C and Theorem O_2, we get:

Theorem S (Solutions)

For $m \geq -1$,
all integer solutions $(x,y) \in \mathbb{Z}^2$ with $xy(x+y) \neq 0$
to $F_m^{(3)}(x,y) = \lambda$ with $\lambda \in \mathbb{N}$ and $\lambda \mid m^2+3m+9$
are given in Table 1. (66 solutions)

Table 1

m	n	$-n-3$	$2m+3$	λ	m^2+3m+9	(x,y)
-1	-15	12	1	1	7	$(-1,2)$, $(2,-1)$, $(-1,-1)$
-1	1259	-1262	1	1	7	$(4,-9)$, $(-9,5)$, $(5,4)$
-1	5	-8	1	7	7	$(1,-3)$, $(-3,2)$, $(2,1)$
0	54	-57	3	1	9	$(1,-3)$, $(-3,2)$, $(2,1)$
0	-6	3	3	3	9	$(-1,2)$, $(2,-1)$, $(-1,-1)$
1	-69	66	5	13	13	$(-2,7)$, $(7,-5)$, $(-5,-2)$
2	-2392	2389	7	1	19	$(-2,9)$, $(9,-7)$, $(-7,-2)$
3	-3	0	9	9	27	$(-1,2)$, $(2,-1)$, $(-1,-1)$
3	-57	54	9	9	27	$(-1,5)$, $(5,-4)$, $(-4,-1)$
5	1259	-1262	13	49	49	$(3,-22)$, $(-22,19)$, $(19,3)$
5	-15	12	13	49	49	$(-1,5)$, $(5,-4)$, $(-4,-1)$
5	-1	-2	13	49	49	$(-1,-2)$, $(-2,3)$, $(3,-1)$
12	-2	-1	27	27	$3^3\cdot 7$	$(-1,2)$, $(2,-1)$, $(-1,-1)$
12	-1262	1259	27	27	$3^3\cdot 7$	$(-1,14)$, $(14,-13)$, $(-13,-1)$
12	-8	5	27	$3^3\cdot 7$	$3^3\cdot 7$	$(-1,5)$, $(5,-4)$, $(-4,-1)$
54	0	-3	111	7^3	$3^2\cdot 7^3$	$(-1,-2)$, $(-2,3)$, $(3,-1)$
54	-6	3	111	$3\cdot 7^3$	$3^2\cdot 7^3$	$(-1,5)$, $(5,-4)$, $(-4,-1)$
66	-4	1	135	$3^3\cdot 13^2$	$3^3\cdot 13^2$	$(-2,7)$, $(7,-5)$, $(-5,-2)$
1259	-1	-2	2521	61^3	$7\cdot 61^3$	$(-4,-5)$, $(-5,9)$, $(9,-4)$
1259	-15	12	2521	61^3	$7\cdot 61^3$	$(-1,14)$, $(14,-13)$, $(-13,-1)$
1259	5	-8	2521	$7\cdot 61^3$	$7\cdot 61^3$	$(-3,-19)$, $(-19,22)$, $(22,-3)$
2389	-5	2	4781	67^3	$19\cdot 67^3$	$(-2,9)$, $(9,-7)$, $(-7,-2)$

§3 Theorem O_1: Okazaki's Theorem

For $m \in \mathbb{Z}$, we take

$$F_m^{(3)}(X,Y) = (X - \theta_1^{(m)}Y)(X - \theta_2^{(m)}Y)(X - \theta_3^{(m)}Y),$$

and $L_m = \mathbb{Q}(\theta_1^{(m)})$. We see

$$-2 < \theta_3^{(m)} < -1, \quad -\tfrac{1}{2} < \theta_2^{(m)} < 0, \quad 1 < \theta_1^{(m)}.$$

Take the exterior product

$$\boldsymbol{\delta} = {}^t(\delta_1, \delta_2, \delta_3) := \mathbf{1} \times \boldsymbol{\theta} = {}^t(\theta_2 - \theta_3, \theta_3 - \theta_1, \theta_1 - \theta_2)$$

where $\mathbf{1} = {}^t(1,1,1)$, $\boldsymbol{\theta} = {}^t(\theta_1, \theta_2, \theta_3) \in \mathbb{R}^3$.

The norm $\mathrm{N}(\boldsymbol{\delta}) = \delta_1\delta_2\delta_3 = -\sqrt{D}$ where $D = ((\theta_1 - \theta_2)(\theta_1 - \theta_3)(\theta_2 - \theta_3))^2$.

The canonical lattice

$$\mathcal{L}^\natural = \boldsymbol{\delta}(\mathbb{Z}\mathbf{1} + \mathbb{Z}\boldsymbol{\theta})$$

of F is orthogonal to $\mathbf{1}$, where the product of vectors is the component-wise product. We consider the curve $\mathcal{H}$

$$\mathcal{H} : z_1 + z_2 + z_3 = 0, \quad z_1 z_2 z_3 = \sqrt{D}.$$

on the plane $\mathit{\Pi} = \{{}^t(z_1, z_2, z_3) \in \mathbb{R}^3 \mid z_1 + z_2 + z_3 = 0\}$. For (x, y) with $F_m^{(3)}(x, y) = 1$, we see $x\mathbf{1} - y\boldsymbol{\theta} \in (\mathcal{O}_{L_m}^\times)^3$ because $\mathrm{N}(x\mathbf{1} - y\boldsymbol{\theta}) = 1$. Then we get a bijection

$$(x, y) \longleftrightarrow \boldsymbol{z} = \boldsymbol{\delta}(-x\mathbf{1} + y\boldsymbol{\theta}) \in \mathcal{L}^\natural \cap \mathcal{H}$$

via $\mathrm{N}(\boldsymbol{z}) = \mathrm{N}(\boldsymbol{\delta})\mathrm{N}(-x\mathbf{1} + y\boldsymbol{\theta}) = (-\sqrt{D})(-1) = \sqrt{D}$. Let

$$\log : (\mathbb{R}^\times)^3 \ni {}^t(z_1, z_2, z_3) \mapsto {}^t(\log|z_1|, \log|z_2|, \log|z_3|) \in \mathbb{R}^3$$

be the logarithmic map. By Dirichlet's unit theorem, the set

$$\mathcal{E}(L_m) := \{\log \boldsymbol{\varepsilon} \mid \boldsymbol{\varepsilon} = {}^t(\varepsilon, \varepsilon^\sigma, \varepsilon^{\sigma^2}), \varepsilon \in \mathcal{O}_{L_m}^\times\}$$

is a lattice of rank 2 on the plane
$\mathit{\Pi}_{\log} := \{{}^t(u_1, u_2, u_3) \in \mathbb{R}^3 \mid u_1 + u_2 + u_3 = 0\}$.

We use the modified logarithmic map

$$\phi : (\mathbb{R}^\times)^3 \ni \boldsymbol{z} \mapsto \boldsymbol{u} = {}^t(u_1, u_2, u_3) = \log(D^{-1/6}\boldsymbol{z}) \in \mathbb{R}^3.$$

For (x, y) with $F_m^{(3)}(x, y) = 1$ and
$\boldsymbol{z} = \boldsymbol{\delta}(-x\mathbf{1} + y\boldsymbol{\theta}) \in \mathcal{L}^\natural \cap \mathcal{H}$,
$\boldsymbol{u} = \phi(\boldsymbol{z}) = \phi(\boldsymbol{\delta}(-x\mathbf{1} + y\boldsymbol{\theta})) \in \phi(\boldsymbol{\delta}) + \mathcal{E}(L_m) \subset \mathit{\Pi}_{\log}$; the displaced lattice, since $-x\mathbf{1} + y\boldsymbol{\theta} \in (\mathcal{O}_{L_m}^\times)^3$. We can show

- $3\,\phi(\boldsymbol{\delta}) \in \mathcal{E}(L_m)$.

We now assume that $L_m = L_n$ for $-1 \leq m < n$ and take a common trivial solution $(x, y) = (1, 0)$. Then

$$\boldsymbol{u}^{(m)}, \boldsymbol{u}^{(n)} \in \mathcal{M} = \mathbb{Z}\,\phi(\boldsymbol{\delta}^{(m)}) + \mathbb{Z}\,\phi(\boldsymbol{\delta}^{(n)}) + \mathcal{E}(L_m) \subset \mathit{\Pi}_{\log}$$

where $\mathcal{M}$ is a lattice with discriminant
$d(\mathcal{M}) = d(\mathcal{E}(L_m))$, $\frac{1}{3}d(\mathcal{E}(L_m))$ or $\frac{1}{9}d(\mathcal{E}(L_m))$. We may get:

- $d(\mathcal{M}) = d(\mathcal{E}(L_m))$ or $\frac{1}{3}d(\mathcal{E}(L_m))$.

We adopt local coordinates for $\mathcal{C} := \phi(\mathcal{H}) \subset \Pi_{\log}$ by

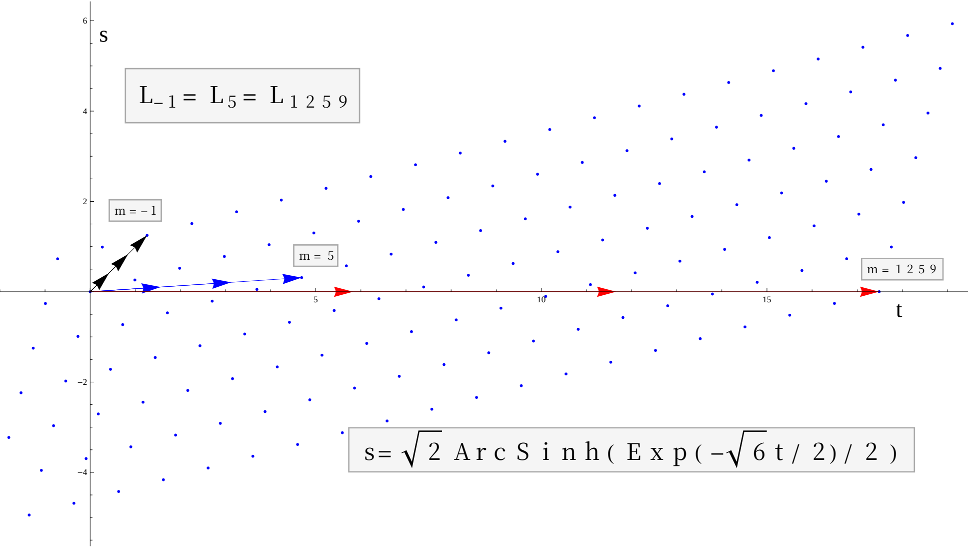
$$s = s(\boldsymbol{u}) := \frac{u_2 - u_3}{\sqrt{2}}, \quad t = t(\boldsymbol{u}) := -\frac{\sqrt{6}u_1}{2}.$$

Then

$$s = \sqrt{2}\ \mathrm{arcsinh}\left(\exp\left(-\sqrt{6}t/2\right)/2\right), \qquad 0 \le s \le \sqrt{3}t.$$

Example

m	-1	0	1	2	3	4	5
s	0.4163	0.3016	0.2263	0.1773	0.1444	0.1212	0.1042
t	0.4206	0.6893	0.9267	1.1269	1.2952	1.4385	1.5624



s
$L_{-1} = L_5 = L_{1259}$
m = −1
m = 5
m = 1259
t
$s = \sqrt{2}\,\mathrm{ArcSinh}(\mathrm{Exp}(-\sqrt{6}\,t/2)/2)$

Using a result of Laurent-Mignotte-Nesterenko (1995) in Baker's theory, Okazaki proved:

Theorem (Okazaki 2002)

Assume distinct points $\boldsymbol{u} = \boldsymbol{u}^{(m)}$ and $\boldsymbol{u}' = \boldsymbol{u}^{(n)}$ of $\mathcal{M}$ on $\mathcal{C}$. Assume $t = t(\boldsymbol{u}) \le t' = t(\boldsymbol{u}')$. Then

$$\frac{\sqrt{2}\, d(\mathcal{M}) \exp(\sqrt{6}t/2)}{1 + \exp(-2(t'-t)/\sqrt{6}\log 2)} \le t'.$$

Theorem (Okazaki 2002)

For $\boldsymbol{z}' \in \mathcal{L}^{\natural} \cap \mathcal{H}$ and $t' = t(\boldsymbol{z}')$, we have

$$\frac{t'}{d(\mathbb{Z}\phi(\boldsymbol{\delta}) + \mathcal{E}(L_m))} \le 5.04 \times 10^4.$$

Combining these two theorems, we have: (Theorem O_1)
$L_m^{(3)} = L_n^{(3)}$ $(-1 \le m < n) \Rightarrow t \le 8.56$ and $m \le 35731$.

§4 Theorem C+Theorem $O_1 \Rightarrow$ Theorem S

It is enough to find all non-trivial solutions $(x, y) \in \mathbb{Z}^2$ to $F_m^{(3)}(x, y) = \lambda \mid m^2 + 3m + 9$ for $-1 \leq m \leq 35731$.
Indeed if there exists a non-trivial solution $(x, y) \in \mathbb{Z}^2$ to $F_n^{(3)}(x, y) = \lambda \mid n^2 + 3n + 9$ for $n \geq 35732$ then there exists $-1 \leq m \leq 35731$ such that $L_m = L_n$ (by Thms C and O_1).
(i) $-1 \leq m \leq 2407$. For small m, we can use MAGMA (Bilu-Hanrot).
(ii) $2408 \leq m \leq 35731$ and $2(2m + 3 + \frac{27}{2m+3}) \leq y$. We consider $|F_m^{(3)}(x, y)| \leq m^2 + 3m + 9$. Applying Lettel-Pethö-Voutier Theorem $\lambda(m) = m^2 + 3m + 9$, $\frac{8\lambda(m)}{2m+3} = 2\left(2m + 3 + \frac{27}{2m+3}\right)$, x/y is a convergent to θ_2.
But we see that this case has no solution.
(iii) $2408 \leq m \leq 35731$ and $y < 2(2m + 3 + \frac{27}{2m+3})$. The bound is small enough to reach using a computer.

- This gives another proof of Thm O_2 because Thm C+Thm S$\Rightarrow$ Thm O_2. Theorem O_2

Degree 6 case

$$\begin{aligned} F_m^{(6)}(x,y) = x^6 - 2mx^5y - (5m+15)x^4y^2 \\ -20x^3y^3 + 5mx^2y^4 + (2m+6)xy^5 + y^6 = \lambda \end{aligned}$$

- $f_m^{(6)}(X) := F_m^{(6)}(X,1)$.
- $f_m^{(6)}(X)$ is irreducible/$\mathbb{Q}$ for $m \in \mathbb{Z} \setminus \{-8,-3,0,5\}$.
- $L_m^{(6)} := \mathrm{Spl}_{\mathbb{Q}} f_m^{(6)}(X)$, then $L_m^{(6)} = L_{-m-3}^{(6)}$; the simplest sextic fields.
- $L_m^{(3)} \subset L_m^{(6)}$ for $\forall m \in \mathbb{Z}$.

Theorem (Theorem C)

For a given $m \in \mathbb{Z}$, $\exists n \in \mathbb{Z} \setminus \{m, -m-3\}$ s.t. $L_m^{(6)} = L_n^{(6)}$
$\iff$ $\exists (x,y) \in \mathbb{Z}^2$ with
$xy(x+y)(x-y)(x+2y)(2x+y) \neq 0$ s.t $F_m^{(6)}(x,y) = \lambda$
for some $\lambda \in \mathbb{N}$ with $\lambda \,|\, 27(m^2+3m+9)$.

Moreover integers n, m and $(x, y) \in \mathbb{Z}^2$ satisfy

$$N = m + \frac{(m^2 + 3m + 9)xy(x+y)(x-y)(x+2y)(2x+y)}{F_m^{(6)}(x, y)}$$

where N is either n or $-n-3$.

By Theorem O_2 and the fact $L_m^{(3)} \subset L_m^{(6)}$, we get:

Theorem

For $m, n \in \mathbb{Z}$, $L_m^{(6)} = L_n^{(6)} \iff m = n$ or $m = -n-3$.

Theorem (Theorem S)

For $m \in \mathbb{Z}$, $F_m^{(6)}(x, y) = \lambda$ with $\lambda \,|\, 27(m^2 + 3m + 9)$ has only trivial solutions, i.e. $xy(x+y)(x-y)(x+2y)(2x+y) = 0$.

- (Compare) $F_m^{(6)}(x, y) = \pm 1, \pm 27$ is solved by Lettl-Pethö-Voutier (1998). $|F_m^{(6)}(x, y)| \leq 120m + 323$ is solved by Lettl-Pethö-Voutier (1999).

Degree 4 case

$$F_m^{(4)}(x,y) = x^4 - mx^3y - 6x^2y^2 + mxy^3 + y^4 = \lambda$$

- $f_m^{(4)}(X) := F_m^{(4)}(X,1)$.
- $f_m^{(4)}(X)$ is irreducible/$\mathbb{Q}$ for $m \in \mathbb{Z} \setminus \{0, \pm 3\}$.
- $L_m^{(4)} := \mathrm{Spl}_{\mathbb{Q}} f_m^{(4)}(X)$, then $L_m^{(4)} = L_{-m}^{(4)}$; the simplest quartic fields.

Theorem (Theorem C)

For a given $m \in \mathbb{Z}$, $\exists n \in \mathbb{Z} \setminus \{m, -m\}$ s.t. $L_m^{(4)} = L_n^{(4)}$
$\iff$ $\exists (x,y) \in \mathbb{Z}^2$ with $xy(x+y)(x-y) \neq 0$ s.t
$F_m^{(4)}(x,y) = \lambda$ for some $\lambda \in \mathbb{N}$ with $\lambda \mid 4(m^2+16)$.
Moreover integers n, m and $(x,y) \in \mathbb{Z}^2$ satisfy

$$N = m + \frac{(m^2+16)xy(x+y)(x-y)}{F_m^{(4)}(x,y)}$$

where N is either n or $-n$.

BUT we do not know

- For $m, n \in \mathbb{Z}$, $L_m^{(4)} = L_n^{(4)} \iff$???

- $L_1^{(4)} = L_{103}^{(4)}$, $L_2^{(4)} = L_{22}^{(4)}$, $L_4^{(4)} = L_{956}^{(4)}$.
- For $0 \le m < n \le 100000$, $L_m^{(4)} = L_n^{(4)} \iff (m,n) \in \{(1,103),(2,22),(4,956)\}$.

By using PARI/GP or Magma, we may check:

Theorem

For $0 \le m \le 1000$, all solutions with $xy(x+y)(x-y) \neq 0$ and $\gcd(x,y) = 1$ to $F_m^{(4)}(x,y) = \lambda$ where $\lambda \mid 4(m^2+16)$ are given as in Table 2.
In particular, for $0 \le m \le 1000$, $m \notin \{1,2,4,22,103,956\}$ and $n \in \mathbb{Z}$, $L_m^{(4)} = L_n^{(4)} \Rightarrow m = \pm n$.

- (Compare) $F_m^{(4)}(x,y) = \pm 1, \pm 4$ is solved by Lettl-Pethö (1995) and Chen-Voutier (1997). $|F_m^{(4)}(x,y)| \le 6m+7$ is solved by Lettl-Pethö-Voutier (1999).

Table 2

m	n	$6m+7$	$F_m^{(4)}(x,y)=\lambda$	m^2+16	(x,y)
1	103	13	-1	17	$(\pm1,\pm2),\ (\pm2,\mp1)$
1	103	13	4	17	$(\mp1,\pm3),\ (\pm3,\pm1)$
2	-22	19	5	20	$(\pm1,\pm2),\ (\pm2,\mp1)$
2	-22	19	-20	20	$(\mp1,\pm3),\ (\pm3,\pm1)$
4	-956	31	1	32	$(\pm2,\pm3),\ (\pm3,\mp2)$
4	-956	31	-4	32	$(\mp1,\pm5),\ (\pm5,\pm1)$
22	-2	139	125	500	$(\pm1,\pm2),\ (\pm2,\mp1)$
22	-2	139	-500	500	$(\mp1,\pm3),\ (\pm3,\pm1)$
103	1	5^4	-5^4	$5^4\cdot17$	$(\mp1,\pm2),\ (\pm2,\pm1)$
103	1	5^4	$2^2\cdot5^4$	$5^4\cdot17$	$(\pm1,\pm3),\ (\pm3,\mp1)$
956	-4	5743	13^4	$2^5\cdot13^4$	$(\pm2,\pm3),\ (\pm3,\mp2)$
956	-4	5743	$-2^2\cdot13^4$	$2^5\cdot13^4$	$(\mp1,\pm5),\ (\pm5,\pm1)$