

Rationality problem for fields of invariants

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$\mathrm{Br}_{\mathrm{nr}}(X/\mathbb{C}) \simeq H^3(X, \mathbb{Z})_{\mathrm{tors}}$; Artin-Mumford invariant

$H_{\mathrm{nr}}^3(X, \mathbb{Q}/\mathbb{Z}) \simeq \mathrm{Hdg}^4(X, \mathbb{Z})/\mathrm{Hdg}^4(X, \mathbb{Z})_{\mathrm{alg}} \leftrightarrow$ integral Hodge conjecture

cf. Colliot-Thélène and Voisin, *Duke Math. J.* **161** (2012) 735–801.

§0. Introduction

Inverse Galois problem (IGP)

Does every finite group occur as a quotient group of the absolute Galois group $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$?

- ▶ Related to **rationality problem** (**Emmy Noether's strategy**: 1913)

A finite group $G \curvearrowright k(x_g \mid g \in G)$: rational function field over k by permutation

$k(x_g \mid g \in G)^G$ is **rational** over k , i.e. $k(x_g \mid g \in G)^G \simeq k(t_1, \dots, t_n)$
(Noether's problem has an **affirmative** answer)

$\implies k(x_g \mid g \in G)^G$ is **retract rational** over k (weaker concept)

$\iff \exists$ generic extension (polynomial) for (G, k) (Saltman's sense)

$\xrightarrow{k:\text{Hilbertian}}$ IGP for (k, G) has an **affirmative** answer

Rationality problem for quasi-monomial actions

Definition (quasi-monomial action)

Let K/k be a finite field extension and $G \leq \text{Aut}_k(K(x_1, \dots, x_n))$; finite where $K(x_1, \dots, x_n)$ is the rational function field of n variables over K .

The action of G on $K(x_1, \dots, x_n)$ is called **quasi-monomial** if

(i) $\sigma(K) \subset K$ for any $\sigma \in G$;

(ii) $K^G = k$;

(iii) for any $\sigma \in G$,
$$\sigma(x_j) = c_j(\sigma) \prod_{i=1}^n x_i^{a_{ij}}$$

where $c_j(\sigma) \in K^\times$, $1 \leq j \leq n$, $[a_{i,j}]_{1 \leq i,j \leq n} \in GL_n(\mathbb{Z})$.

Rationality problem

Under what situation the fixed field $K(x_1, \dots, x_n)^G$ is **rational** over k , i.e. $K(x_1, \dots, x_n)^G \simeq k(t_1, \dots, t_n)$ (= **purely transcendental** over k), if G acts on $K(x_1, \dots, x_n)$ by **quasi-monomial** k -automorphisms.

Rationality problem for quasi-monomial actions

Definition (quasi-monomial action)

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where $c_j(\sigma) \in K^\times$, $1 \leq j \leq n$, $[a_{i,j}]_{1 \leq i, j \leq n} \in GL_n(\mathbb{Z})$.

- ▶ When $G \curvearrowright K$; trivial (i.e. $K = k$), called (just) **monomial action**.
- ▶ When $G \curvearrowright K$; trivial and permutation \leftrightarrow Noether's problem.
- ▶ When $c_j(\sigma) = 1$ ($\forall \sigma \in G, \forall j$), called **purely (quasi-)monomial**.
- ▶ $G = \text{Gal}(K/k)$ and **purely** \leftrightarrow Rationality problem for algebraic tori.

Exercises (1/2): Noether's problem

- ▶ $S_n \curvearrowright \mathbb{Q}(x_1, \dots, x_n)$; permutation

Q. Is $\mathbb{Q}(x_1, \dots, x_n)^{S_n}$ rational over \mathbb{Q} ? Ans. Yes!

$\mathbb{Q}(x_1, \dots, x_n)^{S_n} = \mathbb{Q}(s_1, \dots, s_n)$; s_i , i th elementary symmetric
 \implies IGP for (\mathbb{Q}, S_n) has affirmative solution.

- ▶ $A_n \curvearrowright \mathbb{Q}(x_1, \dots, x_n)$; permutation

Q. Is $\mathbb{Q}(x_1, \dots, x_n)^{A_n}$ rational over \mathbb{Q} ? Ans. Yes? ?? ??

$\mathbb{Q}(x_1, \dots, x_n)^{A_n} = \mathbb{Q}(s_1, \dots, s_n, \Delta)$; but ...

Open problem Is $\mathbb{Q}(x_1, \dots, x_n)^{A_n}$ rational over \mathbb{Q} ? ($n \geq 6$)

- ▶ $\mathbb{Q}(x_1, \dots, x_5)^{A_5}$ is rational over \mathbb{Q} (Maeda, 1989).

Exercises (2/2): Noether's problem

- ▶ $\mathbb{Q}(x_1, x_2, x_3)^{A_3} = \mathbb{Q}(x_1, x_2, x_3)^{C_3} = \mathbb{Q}(t_1, t_2, t_3)$, $\boxed{\text{Q.}}$ $t_1, t_2, t_3?$
($C_3 : x_1 \mapsto x_2 \mapsto x_3 \mapsto x_1$)

- ▶ $\boxed{\text{Ans.}}$ $\mathbb{Q}(x_1, x_2, x_3)^{C_3} = \mathbb{Q}(t_1, t_2, t_3)$ where

$$t_1 = x_1 + x_2 + x_3,$$

$$t_2 = \frac{x_1x_2^2 + x_2x_3^2 + x_3x_1^2 - 3x_1x_2x_3}{x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_2x_3 - x_3x_1},$$

$$t_3 = \frac{x_1^2x_2 + x_2^2x_3 + x_3^2x_1 - 3x_1x_2x_3}{x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_2x_3 - x_3x_1}.$$

- ▶ $\mathbb{Q}(x_1, x_2, \dots, x_8)^{C_8} = \mathbb{Q}(t_1, t_2, \dots, t_8)$, $\boxed{\text{Q.}}$ $t_1, t_2, \dots, t_8?$
($C_8 : x_1 \mapsto x_2 \mapsto x_3 \mapsto \dots \mapsto x_8 \mapsto x_1$)

- ▶ $\boxed{\text{Ans.}}$ **None:** $\mathbb{Q}(x_1, x_2, \dots, x_8)^{C_8}$ is **not rational** over \mathbb{Q} !

Today's talk (1/2)

Definition (quasi-monomial action)

Let K/k be a finite field extension and $G \leq \text{Aut}_k(K(x_1, \dots, x_n))$; finite where $K(x_1, \dots, x_n)$ is the rational function field of n variables over K .

The action of G on $K(x_1, \dots, x_n)$ is called **quasi-monomial** if

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$$\sigma(x_j) = c_j(\sigma) \prod_{i=1}^n x_i^{a_{ij}}$$

where $c_j(\sigma) \in K^\times$, $1 \leq j \leq n$, $[a_{i,j}]_{1 \leq i,j \leq n} \in GL_n(\mathbb{Z})$.

§1. $G \curvearrowright K$; trivial: monomial action & Noether's problem

§2. $G \curvearrowright K$; trivial and permutation: Noether's problem over \mathbb{C}

§3. (general) quasi-monomial actions (1-dim. and 2-dim. cases)

§4. $G = \text{Gal}(K/k)$ and **purely**: rationality problem for algebraic tori

Today's talk (2/2)

§1. $G \curvearrowright K$; trivial: monomial action & Noether's problem

Hoshi-Kitayama-Yamasaki, *J. Algebra* **341** (2011) 45–108.

§2. $G \curvearrowright K$; trivial and permutation: Noether's problem over \mathbb{C}

Hoshi-Kang-Kunyavskii, *Asian J. Math.* **17** (2013) 689–714.

Chu-Hoshi-Hu-Kang, *J. Algebra* **442** (2015) 233–259.

Hoshi, *J. Algebra* **445** (2016) 394–432.

Hoshi-Kang-Yamasaki, *J. Algebra* **458** (2016) 120–133.

Hoshi-Kang-Yamasaki, [arXiv:1609.04142](https://arxiv.org/abs/1609.04142), 105 pages.

Hoshi-Kang-Yamasaki, [arXiv:1710.01958](https://arxiv.org/abs/1710.01958), 61 pages.

§3. (general) quasi-monomial actions (1-dim. and 2-dim. cases)

Hoshi-Kang-Kitayama, *J. Algebra* **403** (2014) 363–400.

§4. $G = \text{Gal}(K/k)$ and purely: rationality problem for algebraic tori

Hoshi-Yamasaki, *Mem. Amer. Math. Soc.* **248** (2017) no. 1176, 215 pp.

Various rationalities: definitions

$k \subset L$; f.g. field extension, L is **rational** over $k \iff L \simeq k(x_1, \dots, x_n)$.

Definition (stably rational)

L is called **stably rational** over $k \iff L(y_1, \dots, y_m)$ is rational over k .

Definition (retract rational)

L is **retract rational** over $k \iff \exists k$ -algebra $R \subset L$ such that

- (i) L is the quotient field of R ;
- (ii) $\exists f \in k[x_1, \dots, x_n] \exists k$ -algebra hom. $\varphi : R \rightarrow k[x_1, \dots, x_n][1/f]$ and $\psi : k[x_1, \dots, x_n][1/f] \rightarrow R$ satisfying $\psi \circ \varphi = 1_R$.

Definition (unirational)

L is **unirational** over $k \iff L \subset k(t_1, \dots, t_n)$.

- ▶ Assume $L_1(x_1, \dots, x_n) \simeq L_2(y_1, \dots, y_m)$; **stably isomorphic**.
If L_1 is retract rational over k , then so is L_2 over k .
- ▶ “**rational**” \implies “**stably rational**” \implies “**retract rational**” \implies “**unirational**”

“rational” \implies “stably rational” \implies “retract rational” \implies “unirational”

- ▶ The direction of the implication **cannot be reversed**.
- ▶ (Lüroth’s problem) “unirational” \implies “rational” ? YES if $\text{trdeg} = 1$
- ▶ (Castelnuovo, 1894)
 L is unirational over \mathbb{C} and $\text{trdeg}_{\mathbb{C}} L = 2 \implies L$ is rational over \mathbb{C} .
- ▶ (Zariski, 1958) Let k be an alg. closed field and $k \subset L \subset k(x, y)$. If $k(x, y)$ is separable algebraic over L , then L is rational over k .
- ▶ (Zariski cancellation problem) $V_1 \times \mathbb{P}^n \approx V_2 \times \mathbb{P}^n \implies V_1 \approx V_2$?
In particular, “stably rational” \implies “rational”?
- ▶ (Beauville, Colliot-Thélène, Sansuc, Swinnerton-Dyer, 1985, Ann. Math.)
 $L = \mathbb{Q}(x, y, t)$ with $x^2 + 3y^2 = t^3 - 2$ (Châtelet surface)
 $\implies L$ is **not rational** but **stably rational** over \mathbb{Q} .
Indeed, $L(y_1, y_2, y_3)$ is **rational** over \mathbb{Q} .
- ▶ $L(y_1, y_2)$ is **rational** over \mathbb{Q} (Shepherd-Barron, 2002, Fano Conf.).
- ▶ $\mathbb{Q}(x_1, \dots, x_{47})^{C_{47}}$ is **not stably** but **retract rational** over \mathbb{Q} .
- ▶ $\mathbb{Q}(x_1, \dots, x_8)^{C_8}$ is **not retract** but **unirational** over \mathbb{Q} .

Châtelet surface as an invariant field

- ▶ (Beauville, Colliot-Thélène, Sansuc, Swinnerton-Dyer, 1985, Ann. Math.)
 $L = \mathbb{Q}(x, y, t)$ with $x^2 + 3y^2 = t^3 - 2$ (Châtelet surface)
 $\implies L$ is **not rational** but **stably rational** over \mathbb{Q} .
- ▶ $L = \mathbb{Q}(x, y, t) = \mathbb{Q}(\sqrt{-3})(X, Y)^{\langle \sigma \rangle}$ where

$$\sigma : \sqrt{-3} \mapsto -\sqrt{-3}, X \mapsto X, Y \mapsto \frac{X^3 - 2}{Y}.$$

Indeed, we have

$$\begin{aligned}x &= \frac{1}{2} \left(Y + \frac{X^3 - 2}{Y} \right), \\y &= \frac{1}{2\sqrt{-3}} \left(Y - \frac{X^3 - 2}{Y} \right), \\t &= X.\end{aligned}$$

Retract rationality and generic extension

Theorem (Saltman, 1982, DeMeyer)

Let k be an infinite field and G be a finite group.

The following are equivalent:

- (i) $k(x_g \mid g \in G)^G$ is **retract rational** over k .
- (ii) There is a **generic** G -Galois extension over k ;
- (iii) There exists a **generic** G -polynomial over k .

▶ related to Inverse Galois Problem (IGP). (i) \implies IGP(G/k): true

Definition (generic polynomial)

A polynomial $f(t_1, \dots, t_n; X) \in k(t_1, \dots, t_n)[X]$ is **generic** for G over k if

(1) $\text{Gal}(f/k(t_1, \dots, t_n)) \simeq G$;

(2) $\forall L/M \supset k$ with $\text{Gal}(L/M) \simeq G$,

$\exists a_1, \dots, a_n \in M$ such that $L = \text{Spl}(f(a_1, \dots, a_n; X)/M)$.

▶ By Hilbert's irreducibility theorem, $\exists L/\mathbb{Q}$ such that $\text{Gal}(L/\mathbb{Q}) \simeq G$.

§1. Monomial action & Noether's problem

Definition (monomial action) $G \curvearrowright K$; trivial, $k = K^G = K$

An action of G on $k(x_1, \dots, x_n)$ is **monomial** $\stackrel{\text{def}}{\iff}$

$$\sigma(x_j) = c_j(\sigma) \prod_{i=1}^n x_i^{a_{i,j}}, \quad 1 \leq j \leq n, \forall \sigma \in G$$

where $[a_{i,j}]_{1 \leq i, j \leq n} \in \text{GL}_n(\mathbb{Z})$, $c_j(\sigma) \in k^\times := k \setminus \{0\}$.

If $c_j(\sigma) = 1$ for any $1 \leq j \leq n$ then σ is called **purely monomial**.

- ▶ Application to Noether's problem (permutation action)

Noether's problem (1/3) [$G = A$; abelian case]

- ▶ k ; field, G ; finite group
- ▶ $G \curvearrowright k$; trivial, $G \curvearrowright k(x_g \mid g \in G)$; permutation.
- ▶ $k(G) := k(x_g \mid g \in G)^G$; invariant field

Noether's problem (Emmy Noether, 1913)

Is $k(G)$ **rational** over k ?, i.e. $k(G) \simeq k(t_1, \dots, t_n)$?

- ▶ Is the quotient variety \mathbb{P}^n/G **rational** over k ?
- ▶ Assume $G = A$; abelian group.
- ▶ (Fisher, 1915) $\mathbb{C}(A)$ is **rational** over \mathbb{C} .
- ▶ (Masuda, 1955, 1968) $\mathbb{Q}(C_p)$ is **rational** over \mathbb{Q} for $p \leq 11$.
- ▶ (Swan, 1969, Invent. Math.)
 $\mathbb{Q}(C_{47}), \mathbb{Q}(C_{113}), \mathbb{Q}(C_{233})$ are **not rational** over \mathbb{Q} .
- ▶ S. Endo and T. Miyata (1973), V.E. Voskresenskii (1973), ...
e.g. $\mathbb{Q}(C_8)$ is **not rational** over \mathbb{Q} .
- ▶ (Lenstra, 1974, Invent. Math.)
 $k(A)$ is **rational** over $k \iff$ some condition;

Noether's problem (2/3) [$G = A$; abelian case]

- ▶ (Endo-Miyata, 1973) $\mathbb{Q}(C_{p^r})$ is **rational** over \mathbb{Q}
 $\iff \exists \alpha \in \mathbb{Z}[\zeta_{\varphi(p^r)}]$ such that $N_{\mathbb{Q}(\zeta_{\varphi(p^r)})/\mathbb{Q}}(\alpha) = \pm p$
- ▶ $h(\mathbb{Q}(\zeta_m)) = 1$ if $m < 23$
 $\implies \mathbb{Q}(C_p)$ is **rational** over \mathbb{Q} for $p \leq 43$ and $p = 61, 67, 71$.
- ▶ (Endo-Miyata, 1973) For $p = 47, 79, 113, 137, 167, \dots$,
 $\mathbb{Q}(C_p)$ is **not rational** over \mathbb{Q} .
- ▶ However, for $p = 59, 83, 89, 97, 107, 163, \dots$, **unknown**.
Under the GRH, $\mathbb{Q}(C_p)$ is **not rational** for the above primes.
But it is **unknown** for $p = 251, 347, 587, 2459, \dots$
- ▶ For $p \leq 20000$, see speaker's paper (using PARI/GP):
Proc. Japan Acad. Ser. A 91 (2015) 39-44.

Theorem (Plans, 2017, Proc. AMS)

$\mathbb{Q}(C_p)$ is **rational** over $\mathbb{Q} \iff p \leq 43$ or $p = 61, 67, 71$.

- ▶ Using lower bound of height, $\mathbb{Q}(C_p)$ is rational $\implies p < 173$.

Noether's problem (Emmy Noether, 1913)

Is $k(G)$ **rational** over k ?, i.e. $k(G) \simeq k(t_1, \dots, t_n)$?

- ▶ Assume G ; non-abelian group.
- ▶ (Maeda, 1989) $k(A_5)$ is **rational** over k ;
- ▶ (Rikuna, 2003; Plans, 2007)
 $k(GL_2(\mathbb{F}_3))$ and $k(SL_2(\mathbb{F}_3))$ is **rational** over k ;
- ▶ (Serre, 2003)
if 2-Sylow subgroup of $G \simeq C_{8m}$, then $\mathbb{Q}(G)$ is **not rational** over \mathbb{Q} ;
if 2-Sylow subgroup of $G \simeq Q_{16}$, then $\mathbb{Q}(G)$ is **not rational** over \mathbb{Q} ;
e.g. $G = Q_{16}, SL_2(\mathbb{F}_7), SL_2(\mathbb{F}_9),$
 $SL_2(\mathbb{F}_q)$ with $q \equiv 7$ or $9 \pmod{16}$.

From Noether's problem to monomial actions (1/2)

- ▶ $k(G) := k(x_g \mid g \in G)^G$; invariant field

Noether's problem (Emmy Noether, 1913)

Is $k(G)$ rational over k ?, i.e. $k(G) \simeq k(t_1, \dots, t_n)$?

By Hilbert 90, we have:

No-name lemma (e.g. Miyata, 1971, Remark 3)

Let G act faithfully on k -vector space V , $W \subset V$ faithful $k[G]$ -submodule. Then $K(V)^G = K(W)^G(t_1, \dots, t_m)$.

Rationality problem: linear action

Let G act on finite-dimensional k -vector space V and $\rho : G \rightarrow GL(V)$ be a representation. Whether $k(V)^G$ is rational over k ?

- ▶ the quotient variety V/G is rational over k ?

From Noether's problem to monomial actions (2/2)

- ▶ For $\rho : G \rightarrow GL(V)$; **monomial representation**, i.e. matrix rep. has exactly one non-zero entry in each row and each column, G acts on $k(\mathbb{P}(V)) = k(\frac{w_1}{w_n}, \dots, \frac{w_{n-1}}{w_n})$ by **monomial action**.

By Hilbert 90, we have:

Lemma (e.g. Miyata, 1971, Lemma)

$$k(V)^G = k(\mathbb{P}(V))^G(t).$$

- ▶ $V/G \approx \mathbb{P}(V)/G \times \mathbb{P}^1$ (birational)
- ▶ $k(\mathbb{P}(V))^G$ (monomial action) is **rational** over k
 $\implies k(V)^G$ (linear action) is **rational** over k
 $\implies k(G)$ (permutation action) is **rational** over k
(Noether's problem has an **affirmative** answer)

Example: Noether's problem for $GL_2(\mathbb{F}_3)$ and $SL_2(\mathbb{F}_3)$

- ▶ $G = GL_2(\mathbb{F}_3) = \langle A, B, C, D \rangle \subset GL_4(\mathbb{Q})$, $|G| = 48$,
- ▶ $H = SL_2(\mathbb{F}_3) = \langle A, B, C \rangle \subset GL_4(\mathbb{Q})$, $|H| = 24$, where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, C = \begin{bmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

- ▶ G and H act on $k(V) = k(w_1, w_2, w_3, w_4)$ by

$$A : w_1 \mapsto -w_2 \mapsto -w_1 \mapsto w_2 \mapsto w_1, w_3 \mapsto -w_4 \mapsto -w_3 \mapsto w_4 \mapsto w_3,$$

$$B : w_1 \mapsto -w_3 \mapsto -w_1 \mapsto w_3 \mapsto w_1, w_2 \mapsto w_4 \mapsto -w_2 \mapsto -w_4 \mapsto w_2,$$

$$C : w_1 \mapsto -w_2 \mapsto w_3 \mapsto w_1, w_4 \mapsto w_4, \quad D : w_1 \mapsto w_1, w_2 \mapsto -w_2, w_3 \leftrightarrow w_4.$$

- ▶ $k(\mathbb{P}(V)) = k(x, y, z)$, $x = w_1/w_4$, $y = w_2/w_4$, $z = w_3/w_4$.
- ▶ G and H act on $k(x, y, z)$ as $G/Z(G) \simeq S_4$ and $H/Z(H) \simeq A_4$:

$$A : x \mapsto \frac{y}{z}, y \mapsto \frac{-x}{z}, z \mapsto \frac{-1}{z}, \quad B : x \mapsto \frac{-z}{y}, y \mapsto \frac{-1}{y}, z \mapsto \frac{x}{y},$$

$$C : x \mapsto y \mapsto z \mapsto x, \quad D : x \mapsto \frac{x}{z}, y \mapsto \frac{-y}{z}, z \mapsto \frac{1}{z}.$$

- ▶ $k(\mathbb{P}(V))^G$: **rational** $\implies k(V)^G$: **rational** $\implies k(G)$: **rational**.

Monomial action (1/3) [3-dim. case]

Theorem (Hajja,1987) 2-dim. monomial action

$k(x_1, x_2)^G$ is **rational** over k .

Theorem (Hajja-Kang 1994, H-Rikuna 2008) 3-dim. **purely monomial**

$k(x_1, x_2, x_3)^G$ is **rational** over k .

Theorem (Prokhorov, 2010) 3-dim. monomial action over $k = \mathbb{C}$

$\mathbb{C}(x_1, x_2, x_3)^G$ is **rational** over \mathbb{C} .

However,

$\mathbb{Q}(x_1, x_2, x_3)^{\langle \sigma \rangle}$, $\sigma : x_1 \mapsto x_2 \mapsto x_3 \mapsto \frac{-1}{x_1 x_2 x_3}$ is **not rational** over \mathbb{Q}
(Hajja,1983).

Monomial action (2/3) [3-dim. case]

Theorem (Saltman, 2000) char $k \neq 2$

If $[k(\sqrt{a_1}, \sqrt{a_2}, \sqrt{a_3}) : k] = 8$, then $k(x_1, x_2, x_3)^{\langle \sigma \rangle}$,

$$\sigma : x_1 \mapsto \frac{a_1}{x_1}, x_2 \mapsto \frac{a_2}{x_2}, x_3 \mapsto \frac{a_3}{x_3}$$

is **not retract** rational over k (hence **not** rational over k).

Theorem (Kang, 2004)

$k(x_1, x_2, x_3)^{\langle \sigma \rangle}$, $\sigma : x_1 \mapsto x_2 \mapsto x_3 \mapsto \frac{c}{x_1 x_2 x_3} \mapsto x_1$, is **rational** over k

\iff at least one of the following conditions is satisfied:

(i) char $k = 2$; (ii) $c \in k^2$; (iii) $-4c \in k^4$; (iv) $-1 \in k^2$.

If $k(x, y, z)^{\langle \sigma \rangle}$ is **not retract** rational over k , then it is **not retract** rational over k .

Recall that

► “**rational**” \implies “**stably rational**” \implies “**retract rational**” \implies “**unirational**”

Monomial action (3/3) [3-dim. case]

Theorem (Yamasaki, 2012) 3-dim. monomial, char $k \neq 2$

\exists 8 cases $G \leq GL_3(\mathbb{Z})$ s.t $k(x_1, x_2, x_3)^G$ is **not retract rational** over k .
Moreover, the necessary and sufficient conditions are given.

- ▶ Two of 8 cases are Saltman's and Kang's cases.
- ▶ $\exists G \leq GL_3(\mathbb{Z})$; 73 finite subgroups (up to conjugacy)

Theorem (H-Kitayama-Yamasaki, 2011) 3-dim. monomial, char $k \neq 2$

$k(x_1, x_2, x_3)^G$ is **rational** over k except for the 8 cases and $G = A_4$.
For $G = A_4$, if $[k(\sqrt{a}, \sqrt{-1}) : k] \leq 2$, then it is **rational** over k .

Corollary

$\exists L = k(\sqrt{a})$ such that $L(x_1, x_2, x_3)^G$ is **rational** over L .

- ▶ However, \exists 4-dim. $\mathbb{C}(x_1, x_2, x_3, x_4)^{C_2 \times C_2}$ is **not retract rational**.

§2. Noether's problem over \mathbb{C} (1/3)

Let G be a p -group. $\mathbb{C}(G) := \mathbb{C}(x_g \mid g \in G)^G$.

▶ (Fisher, 1915) $\mathbb{C}(A)$ is **rational** over \mathbb{C} if A ; finite abelian group.

▶ (Saltman, 1984, Invent. Math.)

For $\forall p$; prime, \exists meta-abelian p -group G of order p^9
such that $\mathbb{C}(G)$ is **not retract rational** over \mathbb{C} .

▶ (Bogomolov, 1988)

For $\forall p$; prime, \exists p -group G of order p^6
such that $\mathbb{C}(G)$ is **not retract rational** over \mathbb{C} .

Indeed they showed $\text{Br}_{\text{nr}}(\mathbb{C}(G)/\mathbb{C}) \neq 0$; unramified Brauer group

▶ **rational** \implies **stably rational** \implies **retract rational** $\implies \text{Br}_{\text{nr}}(\mathbb{C}(G)) = 0$.

not rational \Leftarrow **not stably rational** \Leftarrow **not retract rational** $\Leftarrow \text{Br}_{\text{nr}}(\mathbb{C}(G)) \neq 0$.

▶ $k(G)$; **retract rational** \implies IGP for (k, G) has an **affirmative** answer.

Unramified Brauer group

Definition (Unramified Brauer group) Saltman (1984)

Let $k \subset K$ be an extension of fields.

$\text{Br}_{\text{nr}}(K/k) = \bigcap_R \text{Image}\{\text{Br}(R) \rightarrow \text{Br}(K)\}$ where $\text{Br}(R) \rightarrow \text{Br}(K)$ is the natural map of Brauer groups and R runs over all the discrete valuation rings R such that $k \subset R \subset K$ and K is the quotient field of R .

- ▶ If K is **retract rational** over k , then $\text{Br}(k) \xrightarrow{\sim} \text{Br}_{\text{nr}}(K/k)$.
In particular, if K is retract rational over \mathbb{C} , then $\text{Br}_{\text{nr}}(K/\mathbb{C}) = 0$.
- ▶ For a smooth projective variety X over \mathbb{C} with function field K , $\text{Br}_{\text{nr}}(K/\mathbb{C}) \simeq H^3(X, \mathbb{Z})_{\text{tors}}$ which is given by Artin-Mumford (1972).

Theorem (Bogomolov 1988, Saltman 1990) $\text{Br}_{\text{nr}}(\mathbb{C}(G)/\mathbb{C}) \simeq B_0(G)$

Let G be a finite group. Then $\text{Br}_{\text{nr}}(\mathbb{C}(G)/\mathbb{C})$ is isomorphic to

$$B_0(G) = \bigcap_A \text{Ker}\{\text{res} : H^2(G, \mathbb{Q}/\mathbb{Z}) \rightarrow H^2(A, \mathbb{Q}/\mathbb{Z})\}$$

where A runs over all the **bicyclic** subgroups of G
(**bicyclic** = cyclic or direct product of two cyclic groups).

- ▶ $\mathbb{C}(G)$: “retract rational” $\implies B_0(G) = 0$.
 $B_0(G) \neq 0 \implies \mathbb{C}(G)$: **not (retract)** rational over k .
- ▶ $B_0(G) \leq H^2(G, \mu) \simeq H_2(G, \mathbb{Z})$; Schur multiplier.
- ▶ $B_0(G)$ is called **Bogomolov multiplier**.

Noether's problem over \mathbb{C} (2/3)

- ▶ (Chu-Kang, 2001) G is p -group ($|G| \leq p^4$) $\implies \mathbb{C}(G)$ is **rational**.

Theorem (Moravec, 2012, Amer. J. Math.)

Assume $|G| = 3^5 = 243$. $B_0(G) \neq 0 \iff G = G(243, i)$, $28 \leq i \leq 30$.
In particular, $\exists 3$ groups G such that $\mathbb{C}(G)$ is **not retract rational** over \mathbb{C} .

- ▶ $\exists G$: 67 groups such that $|G| = 243$.

Theorem (H-Kang-Kunyavskii, 2013, Asian J. Math.)

Assume $|G| = p^5$ where p is odd prime.

$B_0(G) \neq 0 \iff G$ belongs to the isoclinism family Φ_{10} .

In particular, $\exists \gcd(4, p-1) + \gcd(3, p-1) + 1$ (resp. $\exists 3$) groups G of order p^5 ($p \geq 5$) (resp. $p = 3$) s.t. $\mathbb{C}(G)$ is **not retract rational** over \mathbb{C} .

- ▶ $\exists 2p + 61 + \gcd(4, p-1) + 2 \gcd(3, p-1)$ groups such that $|G| = p^5$ ($p \geq 5$). ($\exists \Phi_1, \dots, \Phi_{10}$)

From the proof (1/3)

Definition (isoclinic)

p -groups G_1 and G_2 are **isoclinic** $\stackrel{\text{def}}{\iff}$
isom. $\theta : G_1/Z(G_1) \xrightarrow{\sim} G_2/Z(G_2)$, $\phi : [G_1, G_1] \xrightarrow{\sim} [G_2, G_2]$ such that

$$\begin{array}{ccc} G_1/Z(G_1) \times G_1/Z(G_1) & \xrightarrow[\simeq]{(\theta, \theta)} & G_2/Z(G_2) \times G_2/Z(G_2) \\ \downarrow [\cdot, \cdot] & \circlearrowleft & \downarrow [\cdot, \cdot] \\ [G_1, G_1] & \xrightarrow[\simeq]{\phi} & [G_2, G_2] \end{array}$$

Invariants

- ▶ lower central series
- ▶ # of conj. classes with precisely p^i members
- ▶ # of irr. complex rep. of G of degree p^i

From the proof (2/3)

- ▶ $|G| = p^4 (p > 2)$. $\exists 15$ groups (Φ_1, Φ_2, Φ_3)
- ▶ $|G| = 2^4 = 16$. $\exists 14$ groups (Φ_1, Φ_2, Φ_3)
- ▶ $|G| = p^5 (p > 3)$. $\exists 2p + 61 + (4, p - 1) + 2 \times (3, p - 1)$ groups $(\Phi_1, \dots, \Phi_{10})$

	Φ_1	Φ_2	Φ_3	Φ_4	Φ_5	Φ_6	Φ_7	Φ_8
#	7	15	13	$p + 8$	2	$p + 7$	5	1
$(p = 3)$						7		
	Φ_9			Φ_{10}				
#	$2 + (3, p - 1)$			$1 + (4, p - 1) + (3, p - 1)$				
$(p = 3)$				3				

From the proof (3/3)

[HKK, Question 1.11] (2013) (arXiv:1202.5812)

Let G_1 and G_2 be isoclinic p -groups.

Is it true that the fields $k(G_1)$ and $k(G_2)$ are stably isomorphic, or, at least, that $B_0(G_1)$ is isomorphic to $B_0(G_2)$?

Theorem (Moravec, 2013) (arXiv:1203.2422)

G_1 and G_2 are isoclinic $\implies B_0(G_1) \simeq B_0(G_2)$.

Theorem (Bogomolov-Böhning, 2013) (arXiv: 1204.4747)

G_1 and G_2 are isoclinic $\implies \mathbb{C}(G_1)$ and $\mathbb{C}(G_2)$ are stably isomorphic.

Proof (Φ_{10}): $B_0(G) \neq 0$

Lemma 1. $N \triangleleft G$.

(i) $\text{tr}: H^1(N, \mathbb{Q}/\mathbb{Z})^G \rightarrow H^2(G/N, \mathbb{Q}/\mathbb{Z})$ is **not surjective**

where tr is the transgression map.

(ii) $AN/N \leq G/N$ is **cyclic** ($\forall A \leq G$; bicyclic).

$\implies B_0(G) \neq 0$.

Proof. Consider the Hochschild-Serre 5-term exact sequence

$$\begin{aligned} 0 \rightarrow H^1(G/N, \mathbb{Q}/\mathbb{Z}) \rightarrow H^1(G, \mathbb{Q}/\mathbb{Z}) \rightarrow H^1(N, \mathbb{Q}/\mathbb{Z})^G \\ \xrightarrow{\text{tr}} H^2(G/N, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\psi} H^2(G, \mathbb{Q}/\mathbb{Z}) \end{aligned}$$

where ψ is an inflation map.

(i) $\implies \psi$ is **not** zero-map $\implies \text{Image}(\psi) \neq 0$.

We will show that $\text{Image}(\psi) \subset B_0(G)$ by (ii).

It **suffices** to show that $H^2(G/N, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\psi} H^2(G, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\text{res}} H^2(A, \mathbb{Q}/\mathbb{Z})$ is zero-map ($\forall A \leq G$: bicyclic).

Consider the following commutative diagram:

$$\begin{array}{ccccc}
 H^2(G/N, \mathbb{Q}/\mathbb{Z}) & \xrightarrow{\psi} & H^2(G, \mathbb{Q}/\mathbb{Z}) & \xrightarrow{\text{res}} & H^2(A, \mathbb{Q}/\mathbb{Z}) \\
 \psi_0 \downarrow & & & & \uparrow \psi_1 \\
 H^2(AN/N, \mathbb{Q}/\mathbb{Z}) & \xrightarrow{\tilde{\psi}} & H^2(A/A \cap N, \mathbb{Q}/\mathbb{Z}) & &
 \end{array}$$

where ψ_0 is the restriction map, ψ_1 is the inflation map, $\tilde{\psi}$ is the natural isomorphism.

$$(ii) \implies AN/N \simeq C_m \implies H^2(C_m, \mathbb{Q}/\mathbb{Z}) = 0.$$

$\implies \psi_0$ is zero-map.

$\implies \text{res} \circ \psi: H^2(G/N, \mathbb{Q}/\mathbb{Z}) \rightarrow H^2(A, \mathbb{Q}/\mathbb{Z})$ is zero-map.

$\therefore \text{Image}(\psi) \subset B_0(G)$

$\text{Image}(\psi) \subset B_0(G)$ and $\text{Image}(\psi) \neq 0$ (by (i)) $\implies B_0(G) \neq 0$. □

Proof (Φ_6): $B_0(G) = 0$

- ▶ $G = \Phi_6(211)a = \langle f_1, f_2, f_0, h_1, f_2 \rangle, f_1^p = h_1, f_2^p = h_2,$
 $Z(G) = \langle h_1, h_2 \rangle, f_0^p = h_1^p = h_2^p = 1$
 $[f_1, f_2] = f_0, [f_0, f_1] = h_1, [f_0, f_2] = h_2$

Hochschild-Serre 5-term exact sequence:

$$0 \rightarrow H^1(G/N, \mathbb{Q}/\mathbb{Z}) \rightarrow H^1(G, \mathbb{Q}/\mathbb{Z}) \rightarrow H^1(N, \mathbb{Q}/\mathbb{Z})^G \xrightarrow{\text{tr}} H^2(G/N, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\psi} H^2(G, \mathbb{Q}/\mathbb{Z})$$

Proof (Φ_6) : $B_0(G) = 0$

- ▶ $G = \Phi_6(211)a = \langle f_1, f_2, f_0, h_1, f_2 \rangle, f_1^p = h_1, f_2^p = h_2,$
 $Z(G) = \langle h_1, h_2 \rangle, f_0^p = h_1^p = h_2^p = 1$
 $[f_1, f_2] = f_0, [f_0, f_1] = h_1, [f_0, f_2] = h_2$

Hochschild-Serre 5-term exact sequence:

$$\begin{array}{ccccccc}
 0 \rightarrow H^1(G/N, \mathbb{Q}/\mathbb{Z}) & \rightarrow & H^1(G, \mathbb{Q}/\mathbb{Z}) & \rightarrow & H^1(N, \mathbb{Q}/\mathbb{Z})^G & \xrightarrow{\text{tr}} & H^2(G/N, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\psi} H^2(G, \mathbb{Q}/\mathbb{Z}) \\
 & & & & & & \downarrow \\
 & & & & \text{Ker}\{H^2(G, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\text{res}} H^2(N, \mathbb{Q}/\mathbb{Z})\} & =: & H^2(G, \mathbb{Q}/\mathbb{Z})_1 \\
 & & & & & & \downarrow \\
 & & & & & & H^1(G/N, H^1(N, \mathbb{Q}/\mathbb{Z})) \\
 & & & & & & \lambda \downarrow \\
 & & & & & & H^3(G/N, \mathbb{Q}/\mathbb{Z})
 \end{array}$$

- ▶ Explicit formula for λ is given by Dekimpe-Hartl-Wauters (2012)
- ▶ $N := \langle f_1, f_0, h_1, h_2 \rangle \implies G/N \simeq C_p \implies H^2(G/N, \mathbb{Q}/\mathbb{Z}) = 0$
- ▶ $B_0(G) \subset H^2(G, \mathbb{Q}/\mathbb{Z})_1$
- ▶ We should show $H^2(G, \mathbb{Q}/\mathbb{Z})_1 = 0$ ($\iff \lambda$: injective)

Noether's problem over \mathbb{C} (3/3)

Theorem (H-Kang-Kunyavskii, 2013, Asian J. Math.)

Assume $|G| = p^5$ where p is odd prime.

$B_0(G) \neq 0 \iff G$ belongs to the isoclinism family Φ_{10} .

Theorem (Chu-H-Hu-Kang, 2015, J. Algebra) $|G| = 3^5 = 243$

If $B_0(G) = 0$, then $\mathbb{C}(G)$ is **rational** over \mathbb{C} except for Φ_7 .

- ▶ **Non-rationality** of Φ_7 is **detected by** $H_{\text{nr}}^3(\mathbb{C}(G), \mathbb{Q}/\mathbb{Z})$ (later).
- ▶ Φ_5 and Φ_7 are very similar: $C = 1$ (Φ_5), $C = \omega$ (Φ_7).

$\mathbb{C}(G)$ is stably isomorphic to $\mathbb{C}(z_1, z_2, z_3, z_4, z_5, z_6, z_7, z_8, z_9)^{\langle f_1, f_2 \rangle}$

$$\begin{aligned} f_1 : z_1 &\mapsto z_2, z_2 \mapsto \frac{1}{z_1 z_2}, z_3 \mapsto z_4, z_4 \mapsto \frac{1}{z_3 z_4}, \\ z_5 &\mapsto \frac{z_5}{z_1^2 z_3}, z_6 \mapsto \frac{z_1 z_6}{z_3}, z_7 \mapsto z_8, z_8 \mapsto \frac{1}{z_7 z_8}, z_9 \mapsto \frac{z_4 z_9}{z_1}, \\ f_2 : z_1 &\mapsto z_3, z_2 \mapsto z_4, z_3 \mapsto \frac{1}{z_1 z_3}, z_4 \mapsto \frac{1}{z_2 z_4}, \\ z_5 &\mapsto z_6, z_6 \mapsto \frac{1}{z_5 z_6}, z_7 \mapsto C \frac{z_4 z_7}{z_3}, z_8 \mapsto C \frac{z_8}{z_3 z_4^2}, z_9 \mapsto \frac{z_4 z_9}{z_1}. \end{aligned}$$

Unramified Brauer group: purely monomial case (1/2)

Theorem (H-Kang-Yamasaki, arXiv:1609.04142) purely monomial

Let G be a finite group and M be a faithful G -lattice.

- (1) If $\text{rank}_{\mathbb{Z}} M \leq 3$, then $\text{Br}_{\text{nr}}(\mathbb{C}(M)^G) = 0$.
- (2) When $\text{rank}_{\mathbb{Z}} M = 4$, $\exists 5$ M 's with $\text{Br}_{\text{nr}}(\mathbb{C}(M)^G) \neq 0$.
- (3) When $\text{rank}_{\mathbb{Z}} M = 5$, $\exists 46$ M 's with $\text{Br}_{\text{nr}}(\mathbb{C}(M)^G) \neq 0$.
- (4) When $\text{rank}_{\mathbb{Z}} M = 6$, $\exists 1073$ M 's with $\text{Br}_{\text{nr}}(\mathbb{C}(M)^G) \neq 0$.

rank	# of G -lattices	# of unramified Brauer groups $\neq 0$
1	2	0
2	13	0
3	73	0
4	710	5
5	6079	46
6	85308	1073

- ▶ If M is of rank ≤ 6 and $\text{Br}_{\text{nr}}(\mathbb{C}(M)^G) \neq 0$, then G is **solvable** and **non-abelian**, and $\text{Br}_{\text{nr}}(\mathbb{C}(M)^G) \simeq \mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/3\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

Unramified Brauer group: purely monomial case (2/2)

Theorem (H-Kang-Yamasaki, arXiv:1609.04142) $G = A_6$: simple

Embed $A_6 \simeq PSL_2(\mathbb{F}_9) \hookrightarrow S_{10}$. Let $N = \bigoplus_{1 \leq i \leq 10} \mathbb{Z} \cdot x_i$ be the S_{10} -lattice defined by $\sigma \cdot x_i = x_{\sigma(i)}$ for any $\sigma \in S_{10}$; it becomes an A_6 -lattice by restricting the action of S_{10} to A_6 . Define $M = N / (\mathbb{Z} \cdot \sum_{i=1}^{10} x_i)$ with $\text{rank}_{\mathbb{Z}} M = 9$. $\exists A_6$ -lattices $M = M_1, M_2, \dots, M_6$ which are \mathbb{Q} -conjugate but not \mathbb{Z} -conjugate to each other; in fact, all these M_i form a single \mathbb{Q} -class, but this \mathbb{Q} -class consists of six \mathbb{Z} -classes. Then we have

$$H_{\text{nr}}^2(A_6, M_1) \simeq H_{\text{nr}}^2(A_6, M_3) \simeq \mathbb{Z}/2\mathbb{Z}, \quad H_{\text{nr}}^2(A_6, M_i) = 0 \text{ for } i = 2, 4, 5, 6.$$

In particular, $\mathbb{C}(M_1)^{A_6}$ and $\mathbb{C}(M_3)^{A_6}$ are **not retract \mathbb{C} -rational**.

Furthermore, M_1 and M_3 may be distinguished by Tate cohomologies:

$$\begin{aligned} H^1(A_6, M_1) &= 0, & \widehat{H}^{-1}(A_6, M_1) &= \mathbb{Z}/10\mathbb{Z}, \\ H^1(A_6, M_3) &= \mathbb{Z}/5\mathbb{Z}, & \widehat{H}^{-1}(A_6, M_3) &= \mathbb{Z}/2\mathbb{Z}. \end{aligned}$$

Unramified cohomology (1/4)

Colliot-Thélène and Ojanguren (1989) generalized the notion of the unramified Brauer group $\text{Br}_{\text{nr}}(K/\mathbb{C})$ to the unramified cohomology $H_{\text{nr}}^i(K/\mathbb{C}, \mu_n^{\otimes j})$ of degree $i \geq 1$:

Definition (Colliot-Thélène and Ojanguren, 1989, Invent. Math.)

Let K/\mathbb{C} be a function field, that is finitely generated as a field over \mathbb{C} . The **unramified cohomology group** $H_{\text{nr}}^i(K/\mathbb{C}, \mu_n^{\otimes j})$ of K over \mathbb{C} of degree $i \geq 1$ is defined to be

$$H_{\text{nr}}^i(K/\mathbb{C}, \mu_n^{\otimes j}) = \bigcap_R \text{Image}\{H_{\text{ét}}^i(R, \mu_n^{\otimes j}) \rightarrow H_{\text{ét}}^i(K, \mu_n^{\otimes j})\}$$

where R runs over all the discrete valuation rings R of rank one such that $\mathbb{C} \subset R \subset K$ and K is the quotient field of R .

- ▶ Note that ${}_n\text{Br}_{\text{nr}}(K/\mathbb{C}) \simeq H_{\text{nr}}^2(K/\mathbb{C}, \mu_n)$.

Proposition (Colliot-Thélène and Ojanguren, 1989)

If K and L are stably \mathbb{C} -isomorphic, then

$$H_{\text{nr}}^i(K/\mathbb{C}, \mu_n^{\otimes j}) \xrightarrow{\sim} H_{\text{nr}}^i(L/\mathbb{C}, \mu_n^{\otimes j}).$$

In particular, K is stably \mathbb{C} -rational, then $H_{\text{nr}}^i(K/\mathbb{C}, \mu_n^{\otimes j}) = 0$.

- ▶ Moreover, if K is retract \mathbb{C} -rational, then $H_{\text{nr}}^i(K/\mathbb{C}, \mu_n^{\otimes j}) = 0$.
- ▶ CTO (1989) \exists \mathbb{C} -unirational field K with $\text{trdeg}_{\mathbb{C}} K = 6$ s.t. $H_{\text{nr}}^3(K/\mathbb{C}, \mu_2^{\otimes 3}) \neq 0$ and $\text{Br}_{\text{nr}}(K/\mathbb{C}) = 0$.
- ▶ Peyre (1993) gave a sufficient condition for $H_{\text{nr}}^i(K/\mathbb{C}, \mu_p^{\otimes i}) \neq 0$:
- ▶ $\exists K$ s.t. $H_{\text{nr}}^3(K/\mathbb{C}, \mu_p^{\otimes 3}) \neq 0$ and $\text{Br}_{\text{nr}}(K/\mathbb{C}) = 0$;
- ▶ $\exists K$ s.t. $H_{\text{nr}}^4(K/\mathbb{C}, \mu_2^{\otimes 4}) \neq 0$ and $\text{Br}_{\text{nr}}(K/\mathbb{C}) = 0$.

Unramified cohomology (2/4)

Take the direct limit with respect to n :

$$H^i(K/\mathbb{C}, \mathbb{Q}/\mathbb{Z}(j)) = \varinjlim_n H^i(K/\mathbb{C}, \mu_n^{\otimes j})$$

and we also define the unramified cohomology group

$$H_{\text{nr}}^i(K/\mathbb{C}, \mathbb{Q}/\mathbb{Z}(j)) = \bigcap_R \text{Image}\{H_{\text{ét}}^i(R, \mathbb{Q}/\mathbb{Z}(j)) \rightarrow H_{\text{ét}}^i(K, \mathbb{Q}/\mathbb{Z}(j))\}.$$

Then we have $\text{Br}_{\text{nr}}(K/\mathbb{C}) \simeq H_{\text{nr}}^2(K/\mathbb{C}, \mathbb{Q}/\mathbb{Z}(1))$.

► The case $K = \mathbb{C}(G)$:

Theorem (Peyre, 2008, Invent. Math.) p : odd prime

\exists p -group G of order p^{12} such that $B_0(G) = 0$ and $H_{\text{nr}}^3(\mathbb{C}(G), \mathbb{Q}/\mathbb{Z}) \neq 0$.
In particular, $\mathbb{C}(G)$ is **not (retract, stably) \mathbb{C} -rational**.

- ▶ Asok (2013) generalized Peyre's argument (1993):

Theorem (Asok, 2013, Compos. Math.)

(1) For any $n > 0$, \exists a smooth projective complex variety X that is \mathbb{C} -unirational, for which $H_{\text{nr}}^i(\mathbb{C}(X), \mu_2^{\otimes i}) = 0$ for each $i < n$, yet $H_{\text{nr}}^n(\mathbb{C}(X), \mu_2^{\otimes n}) \neq 0$, and so

X is **not \mathbb{A}^1 -connected, nor (retract, stably) \mathbb{C} -rational**;

(2) For any prime l and any $n \geq 2$, \exists a smooth projective rationally connected complex variety Y such that $H_{\text{nr}}^n(\mathbb{C}(Y), \mu_l^{\otimes n}) \neq 0$.

In particular, Y is **not \mathbb{A}^1 -connected, nor (retract, stably) \mathbb{C} -rational**.

- ▶ Namely, the triviality of the unramified Brauer group or the unramified cohomology of higher degree is just a necessary condition of \mathbb{C} -rationality of fields.
- ▶ It is interesting to consider an analog of above Theorem for quotient varieties V/G , e.g. $\mathbb{C}(V_{\text{reg}}/G) = \mathbb{C}(G)$.

Unramified cohomology (3/4)

Theorem (Peyre, 2008, Invent. Math.) p : odd prime

\exists p -group G of order p^{12} such that $B_0(G) = 0$ and $H_{\text{nr}}^3(\mathbb{C}(G), \mathbb{Q}/\mathbb{Z}) \neq 0$.
In particular, $\mathbb{C}(G)$ is **not (retract, stably) \mathbb{C} -rational**.

Using Peyre's method, we improve this result:

Theorem (H-Kang-Yamasaki, 2016, J. Algebra) p : odd prime

\exists p -group G of order p^9 such that $B_0(G) = 0$ and $H_{\text{nr}}^3(\mathbb{C}(G), \mathbb{Q}/\mathbb{Z}) \neq 0$.
In particular, $\mathbb{C}(G)$ is **not (retract, stably) \mathbb{C} -rational**.

On the other hand, CT and Voisin proved: (\leftrightarrow integral Hodge conjecture)

Theorem (Colliot-Thélène and Voisin, 2012, Duke Math. J.)

Let X be a smooth projective rationally connected complex variety. Then
 $H_{\text{nr}}^3(X, \mathbb{Q}/\mathbb{Z}) \simeq \text{Hdg}^4(X, \mathbb{Z}) / \text{Hdg}^4(X, \mathbb{Z})_{\text{alg}}$.

Unramified cohomology (4/4)

- ▶ Using Peyre's formula [Peyre, 2008, Invent. Math.], we get:

Theorem (H-Kang-Yamasaki, arXiv:1710.01958) $|G| = 3^5$

$H_{\text{nr}}^3(\mathbb{C}(G), \mathbb{Q}/\mathbb{Z}) \neq 0 \iff G$ belongs to the isoclinism family Φ_7 .

In particular, $\mathbb{C}(G)$ is **not rational** over $\mathbb{C} \iff G$ belongs to Φ_7, Φ_{10} .

$ G = 3^5$	Φ_1	Φ_2	Φ_3	Φ_4	Φ_5	Φ_6	Φ_7	Φ_8	Φ_9	Φ_{10}
$H_{\text{nr}}^2(\mathbb{C}(G), \mathbb{Q}/\mathbb{Z})$	0	0	0	0	0	0	0	0	0	$\mathbb{Z}/3\mathbb{Z}$
$H_{\text{nr}}^3(\mathbb{C}(G), \mathbb{Q}/\mathbb{Z})$	0	0	0	0	0	0	$\mathbb{Z}/3\mathbb{Z}$	0	0	0

Theorem (H-Kang-Yamasaki, arXiv:1710.01958) $|G| = 5^5$ or 7^5

$H_{\text{nr}}^3(\mathbb{C}(G), \mathbb{Q}/\mathbb{Z}) \neq 0 \iff G$ belongs to Φ_6, Φ_7 or Φ_{10} .

$ G = p^5$ ($p = 5, 7$)	Φ_1	Φ_2	Φ_3	Φ_4	Φ_5	Φ_6	Φ_7	Φ_8	Φ_9	Φ_{10}
$H_{\text{nr}}^2(\mathbb{C}(G), \mathbb{Q}/\mathbb{Z})$	0	0	0	0	0	0	0	0	0	$\mathbb{Z}/p\mathbb{Z}$
$H_{\text{nr}}^3(\mathbb{C}(G), \mathbb{Q}/\mathbb{Z})$	0	0	0	0	0	$\mathbb{Z}/p\mathbb{Z}$	$\mathbb{Z}/p\mathbb{Z}$	0	0	$\mathbb{Z}/p\mathbb{Z}$

Noether's problem over \mathbb{C} for 2-groups

- ▶ (Chu-Kang, 2001) G is p -group ($|G| \leq p^4$) $\implies \mathbb{C}(G)$ is **rational**.
- ▶ (Chu-Hu-Kang-Prokhorov, 2008)
 $|G| = 32 = 2^5 \implies \mathbb{C}(G)$ is **rational**.
- ▶ $\exists 267$ groups G of order $64 = 2^6$ which are classified into 27 isoclinism families Φ_1, \dots, Φ_{27} .

Theorem (Chu-Hu-Kang-Kunyavskii, 2010) $|G| = 64 = 2^6$

(1) $B_0(G) \neq 0 \iff G$ belongs to Φ_{16} . ($\exists 9$ such G 's)

Moreover, if $B_0(G) \neq 0$, then $B_0(G) \simeq C_2$.

(2) If $B_0(G) = 0$, then $\mathbb{C}(G)$ is **rational** except for Φ_{13} . ($\exists 5$ such G 's)

- ▶ ([CHKK10], [HY14]) ($B_0(G) = 0$, but **rationality unknown**)
If G belongs to Φ_{13} , then $\mathbb{C}(G)$ is stably \mathbb{C} -isomorphic to $L_{\mathbb{C}}^{(0)}$.
- ▶ ([CHKK10], [HKK14]) ($B_0(G) \simeq C_2$, **not retract rational**)
If G belongs to Φ_{16} , then $\mathbb{C}(G)$ is stably \mathbb{C} -isomorphic to $L_{\mathbb{C}}^{(1)}$.

- ▶ ([CHKK10], [HY14]) ($B_0(G) = 0$, but rationality unknown)
If G belongs to Φ_{13} , then $\mathbb{C}(G)$ is stably \mathbb{C} -isomorphic to $L_{\mathbb{C}}^{(0)}$.
- ▶ ([CHKK10], [HKK14]) ($B_0(G) \simeq C_2$, not retract rational)
If G belongs to Φ_{16} , then $\mathbb{C}(G)$ is stably \mathbb{C} -isomorphic to $L_{\mathbb{C}}^{(1)}$.

Definition (The fields $L_{\mathbb{C}}^{(0)}$ and $L_{\mathbb{C}}^{(1)}$)

(i) The field $L_{\mathbb{C}}^{(0)}$ is defined to be $\mathbb{C}(X_1, X_2, X_3, X_4, X_5, X_6)^H$ where $H = \langle \sigma_1, \sigma_2 \rangle \simeq C_2 \times C_2$ act on $\mathbb{C}(X_1, X_2, X_3, X_4, X_5, X_6)$ by

$$\sigma_1 : X_1 \mapsto X_3, X_2 \mapsto \frac{1}{X_1 X_2 X_3}, X_3 \mapsto X_1, X_4 \mapsto X_6, X_5 \mapsto \frac{1}{X_4 X_5 X_6}, X_6 \mapsto X_4,$$

$$\sigma_2 : X_1 \mapsto X_2, X_2 \mapsto X_1, X_3 \mapsto \frac{1}{X_1 X_2 X_3}, X_4 \mapsto X_5, X_5 \mapsto X_4, X_6 \mapsto \frac{1}{X_4 X_5 X_6}.$$

(ii) The field $L_{\mathbb{C}}^{(1)}$ is defined to be $\mathbb{C}(X_1, X_2, X_3, X_4)^{\langle \tau \rangle}$ where $\langle \tau \rangle \simeq C_2$ acts on $\mathbb{C}(X_1, X_2, X_3, X_4)$ by

$$\tau : X_1 \mapsto -X_1, X_2 \mapsto \frac{X_4}{X_2}, X_3 \mapsto \frac{(X_4 - 1)(X_4 - X_1^2)}{X_3}, X_4 \mapsto X_4.$$

- ▶ ([CHKK10], [HY14]) ($B_0(G) = 0$, but rationality unknown)

If G belongs to Φ_{13} , then $\mathbb{C}(G)$ is stably \mathbb{C} -isomorphic to $L_{\mathbb{C}}^{(0)}$.

- ▶ ([CHKK10], [HKK14]) ($B_0(G) \simeq C_2$, not retract rational)

If G belongs to Φ_{16} , then $\mathbb{C}(G)$ is stably \mathbb{C} -isomorphic to $L_{\mathbb{C}}^{(1)}$.

- ▶ $L_{\mathbb{C}}^{(0)} = \mathbb{C}(z_1, z_2, z_3, z_4, u_4, u_5, u_6)$ where

$$(z_1^2 - a)(z_4^2 - d) = (z_2^2 - b)(z_3^2 - c),$$

$$a = u_4(u_4 - 1), b = u_4 - 1, c = u_4(u_4 - u_6^2), d = u_5^2(u_4 - u_6^2).$$

- ▶ $L_{\mathbb{C}}^{(0)} = \mathbb{C}(u, v, t, w_3, w_4, w_5, w_6)$ where

$$u^2 - tv^2 = - (w_4^2(w_5^2 - 1)t^2 + (w_3^2 - w_3^2w_5^2 + 1)t - w_5^2)$$

$$\cdot (w_4^2w_6^2t^2 - (w_4^2 + w_3^2w_6^2)t + w_3^2 - w_6^2 + 1).$$

- ▶ $L_{\mathbb{C}}^{(0)} = \mathbb{C}(m_0, \dots, m_6)$ where

$$m_0^2 = (4m_3 + m_3m_4^2 + m_4^2)(m_3 - m_5^2 + 1)$$

$$\cdot (m_1^2m_3 + m_6^2 - 1)(4m_3 + m_1^2m_2^2m_3 + m_2^2m_6^2).$$

- ▶ $L_{\mathbb{C}}^{(1)} = \mathbb{C}(u, v, t, w_3, w_4)$ where

$$u^2 - tv^2 = (tw_4^2 - w_3^2 + 1)(t + tw_4^2 - w_3^2).$$

- ▶ $\exists 2328$ groups G of order $128 = 2^7$ which are classified into 115 isoclinism families $\Phi_1, \dots, \Phi_{115}$.

Theorem (Moravec, 2012, Amer. J. Math.) $|G| = 128 = 2^7$

$B_0(G) \neq 0$ if and only if G belongs to the isoclinism family $\Phi_{16}, \Phi_{30}, \Phi_{31}, \Phi_{37}, \Phi_{39}, \Phi_{43}, \Phi_{58}, \Phi_{60}, \Phi_{80}, \Phi_{106}$ or Φ_{114} . If $B_0(G) \neq 0$, then

$$B_0(G) \simeq \begin{cases} C_2 & (\Phi_{16}, \Phi_{31}, \Phi_{37}, \Phi_{39}, \Phi_{43}, \Phi_{58}, \Phi_{60}, \Phi_{80}, \Phi_{106}, \Phi_{114}) \\ C_2 \times C_2 & (\Phi_{30}). \end{cases}$$

In particular, $\mathbb{C}(G)$ is **not (retract, stably) \mathbb{C} -rational**.

	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	Total
Family	Φ_{16}	Φ_{31}	Φ_{37}	Φ_{39}	Φ_{43}	Φ_{58}	Φ_{60}	Φ_{80}	Φ_{106}	Φ_{114}	Φ_{30}	
$B_0(G)$	C_2										$C_2 \times C_2$	
# G 's	48	55	18	6	26	20	10	9	2	2	34	220

- ▶ **Q.** Birational classification of $\mathbb{C}(G)$?

In particular, what happens when $B_0(G) \neq 0$?

How many $\mathbb{C}(G)$'s exist up to stably \mathbb{C} -isomorphism?

Theorem (H, 2016, J. Algebra) $|G| = 128 = 2^7$

Assume that $B_0(G) \neq 0$.

Then $\mathbb{C}(G)$ and $L_{\mathbb{C}}^{(m)}$ are stably \mathbb{C} -isomorphic where

$$m = \begin{cases} 1 & \text{if } G \text{ belongs to } \Phi_{16}, \Phi_{31}, \Phi_{37}, \Phi_{39}, \Phi_{43}, \Phi_{58}, \Phi_{60} \text{ or } \Phi_{80}, \\ 2 & \text{if } G \text{ belongs to } \Phi_{106} \text{ or } \Phi_{114}, \\ 3 & \text{if } G \text{ belongs to } \Phi_{30}. \end{cases}$$

In particular, $\text{Br}_{\text{nr}}(L_{\mathbb{C}}^{(1)}) \simeq \text{Br}_{\text{nr}}(L_{\mathbb{C}}^{(2)}) \simeq C_2$ and $\text{Br}_{\text{nr}}(L_{\mathbb{C}}^{(3)}) \simeq C_2 \times C_2$ and hence the fields $L_{\mathbb{C}}^{(1)}$, $L_{\mathbb{C}}^{(2)}$ and $L_{\mathbb{C}}^{(3)}$ are not (retract, stably) \mathbb{C} -rational.

- ▶ $L_{\mathbb{C}}^{(1)} \not\sim L_{\mathbb{C}}^{(3)}$, $L_{\mathbb{C}}^{(2)} \not\sim L_{\mathbb{C}}^{(3)}$ (not stably \mathbb{C} -isomorphic) because their unramified Brauer groups are not isomorphic.
- ▶ However, we do **not** know whether $L_{\mathbb{C}}^{(1)} \sim L_{\mathbb{C}}^{(2)}$.
- ▶ If not, evaluate the higher unramified cohomologies $H_{\text{nr}}^i(i \geq 3)$ (Peyre's formula **can not work** for $|G| = 2^m$)

Definition (The fields $L_{\mathbb{C}}^{(2)}$ and $L_{\mathbb{C}}^{(3)}$)

(i) **The field $L_{\mathbb{C}}^{(2)}$** is defined to be $\mathbb{C}(X_1, X_2, X_3, X_4, X_5, X_6)^{\langle \rho \rangle}$ where $\langle \rho \rangle \simeq C_4$ acts on $\mathbb{C}(X_1, X_2, X_3, X_4, X_5, X_6)$ by

$$\begin{aligned} \rho : X_1 &\mapsto X_2, X_2 \mapsto -X_1, X_3 \mapsto X_4, X_4 \mapsto X_3, \\ X_5 &\mapsto X_6, X_6 \mapsto \frac{(X_1^2 X_2^2 - 1)(X_1^2 X_3^2 + X_2^2 - X_3^2 - 1)}{X_5}. \end{aligned}$$

(ii) **The field $L_{\mathbb{C}}^{(3)}$** is defined to be $\mathbb{C}(X_1, X_2, X_3, X_4, X_5, X_6, X_7)^{\langle \lambda_1, \lambda_2 \rangle}$ where $\langle \lambda_1, \lambda_2 \rangle \simeq C_2 \times C_2$ acts on $\mathbb{C}(X_1, X_2, X_3, X_4, X_5, X_6, X_7)$ by

$$\begin{aligned} \lambda_1 : X_1 &\mapsto X_1, X_2 \mapsto \frac{X_1}{X_2}, X_3 \mapsto \frac{1}{X_1 X_3}, X_4 \mapsto \frac{X_2 X_4}{X_1 X_3}, \\ X_5 &\mapsto -\frac{X_1 X_6^2 - 1}{X_5}, X_6 \mapsto -X_6, X_7 \mapsto X_7, \\ \lambda_2 : X_1 &\mapsto \frac{1}{X_1}, X_2 \mapsto X_3, X_3 \mapsto X_2, X_4 \mapsto \frac{(X_1 X_6^2 - 1)(X_1 X_7^2 - 1)}{X_4}, \\ X_5 &\mapsto -X_5, X_6 \mapsto -X_1 X_6, X_7 \mapsto -X_1 X_7. \end{aligned}$$

§3. (general) quasi-monomial actions

Notion of "quasi-monomial" action is defined in [HKK] J. Algebra (2014).

Theorem (H-Kang-Kitayama) 1-dim. quasi-monomial action

- (1) **purely** quasi-monomial action $\implies K(x)^G$ is **rational** over k .
- (2) $K(x)^G$ is **rational** over k **except for** the case: $\exists N \leq G$ such that
 - (i) $G/N = \langle \sigma \rangle \simeq C_2$;
 - (ii) $K(x)^N = k(\alpha)(y)$, $\alpha^2 = a \in K^\times$, $\sigma(\alpha) = -\alpha$ (if $\text{char } k \neq 2$),
 $\alpha^2 + \alpha = a \in K$, $\sigma(\alpha) = \alpha + 1$ (if $\text{char } k = 2$);
 - (iii) $\sigma \cdot y = b/y$ for some $b \in k^\times$.

For the exceptional case, $K(x)^G = k(\alpha)(y)^{G/N}$ is **rational** over $k \iff$
Hilbert symbol $(a, b)_k = 0$ (if $\text{char } k \neq 2$), $[a, b]_k = 0$ (if $\text{char } k = 2$).
Moreover, $K(x)^G$ is **not rational** over $k \implies$ **not unirational** over k .

Theorem (H-Kang-Kitayama) 2-dim. purely quasi-monomial action

$N = \{\sigma \in G \mid \sigma(x) = x, \sigma(y) = y\}$, $H = \{\sigma \in G \mid \sigma(\alpha) = \alpha(\forall \alpha \in K)\}$.

$K(x, y)^G$ is rational over k except for:

(1) $\text{char } k \neq 2$ and (2) (i) $(G/N, HN/N) \simeq (C_4, C_2)$ or (ii) (D_4, C_2) .

For the exceptional case, we have $k(x, y) = k(u, v)$:

(i) $(G/N, HN/N) \simeq (C_4, C_2)$,

$K^N = k(\sqrt{a})$, $G/N = \langle \sigma \rangle \simeq C_4$, $\sigma : \sqrt{a} \mapsto -\sqrt{a}$, $u \mapsto \frac{1}{u}$, $v \mapsto -\frac{1}{v}$;

(ii) $(G/N, HN/N) \simeq (D_4, C_2)$;

$K^N = k(\sqrt{a}, \sqrt{b})$, $G/N = \langle \sigma, \tau \rangle \simeq D_4$, $\sigma : \sqrt{a} \mapsto -\sqrt{a}$, $\sqrt{b} \mapsto \sqrt{b}$,
 $u \mapsto \frac{1}{u}$, $v \mapsto -\frac{1}{v}$, $\tau : \sqrt{a} \mapsto \sqrt{a}$, $\sqrt{b} \mapsto -\sqrt{b}$, $u \mapsto u$, $v \mapsto -v$.

Case (i), $K(x, y)^G$ is rational over $k \iff$ Hilbert symbol $(a, -1)_k = 0$.

Case (ii), $K(x, y)^G$ is rational over $k \iff$ Hilbert symbol $(a, -b)_k = 0$.

Moreover, $K(x, y)^G$ is not rational over $k \implies$

$\text{Br}(k) \neq 0$ and $K(x, y)^G$ is not unirational over k .

Galois-theoretic interpretation:

(i) rational over $k \iff k(\sqrt{a})$ may be embedded into C_4 -ext. of k .

(ii) rational over $k \iff k(\sqrt{a}, \sqrt{b})$ may be embedded into D_4 -ext. of k .

Application to purely monomial action (1/2)

Theorem (H-Kang-Kitayama), 4-dim. purely monomial

Let M be a G -lattice with $\text{rank}_{\mathbb{Z}} M = 4$ and G act on $k(M)$ by purely monomial k -automorphisms. If M is decomposable, i.e. $M = M_1 \oplus M_2$ as $\mathbb{Z}[G]$ -modules where $1 \leq \text{rank}_{\mathbb{Z}} M_1 \leq 3$, then $k(M)^G$ is **rational** over k .

- ▶ When $\text{rank}_{\mathbb{Z}} M_1 = 1, \text{rank}_{\mathbb{Z}} M_2 = 3$, it is easy to see $k(M)^G$ is **rational**.
- ▶ When $\text{rank}_{\mathbb{Z}} M_1 = \text{rank}_{\mathbb{Z}} M_2 = 2$, we may apply Theorem of 2-dim. to $k(M) = k(x_1, x_2, y_1, y_2) = k(x_1, x_2)(y_1, y_2) = K(y_1, y_2)$.

Theorem (H-Kang-Kitayama) char $k \neq 2$

Let $C_2 = \langle \tau \rangle$ act on the rational function field $k(x_1, x_2, x_3, x_4)$ by k -automorphisms defined as

$$\tau : x_1 \mapsto -x_1, \quad x_2 \mapsto \frac{x_4}{x_2}, \quad x_3 \mapsto \frac{(x_4-1)(x_4-x_1^2)}{x_3}, \quad x_4 \mapsto x_4.$$

Then $k(x_1, x_2, x_3, x_4)^{C_2}$ is **not retract rational** over k .
In particular, it is **not rational** over k .

Theorem A (H-Kang-Kitayama) char $k \neq 2, 5$ -dim. purely monomial

Let $D_4 = \langle \rho, \tau \rangle$ act on the rational function field $k(x_1, x_2, x_3, x_4, x_5)$ by k -automorphisms defined as

$$\begin{aligned} \rho : x_1 &\mapsto x_2, \quad x_2 \mapsto x_1, \quad x_3 \mapsto \frac{1}{x_1 x_2 x_3}, \quad x_4 \mapsto x_5, \quad x_5 \mapsto \frac{1}{x_4}, \\ \tau : x_1 &\mapsto x_3, \quad x_2 \mapsto \frac{1}{x_1 x_2 x_3}, \quad x_3 \mapsto x_1, \quad x_4 \mapsto x_5, \quad x_5 \mapsto x_4. \end{aligned}$$

Then $k(x_1, x_2, x_3, x_4, x_5)^{D_4}$ is **not retract rational** over k .
In particular, it is **not rational** over k .

Application to purely monomial action (2/2)

Theorem (H-Kang-Kitayama), 5-dim. purely monomial

Let M be a G -lattice and G act on $k(M)$ by purely monomial k -automorphisms. Assume that

- (i) $M = M_1 \oplus M_2$ as $\mathbb{Z}[G]$ -modules where $\text{rank}_{\mathbb{Z}} M_1 = 3$ and $\text{rank}_{\mathbb{Z}} M_2 = 2$,
- (ii) either M_1 or M_2 is a faithful G -lattice.

Then $k(M)^G$ is **rational** over k except for the case as in Theorem A.

- ▶ we may apply Theorem of 2-dim. to

$$k(M) = k(x_1, x_2, x_3, y_1, y_2) = k(x_1, x_2, x_3)(y_1, y_2) = K(y_1, y_2).$$

§4. Rationality problem for algebraic tori (2-dim., 3-dim.)

$G \simeq \text{Gal}(K/k) \curvearrowright K(x_1, \dots, x_n)$: purely quasi-monomial,
 $K(x_1, \dots, x_n)^G$ may be regarded as the function field of
algebraic torus T over k which splits over K ($T \otimes_k K \simeq \mathbb{G}_m^n$).

- ▶ T is unirational over k , i.e. $K(x_1, \dots, x_n)^G \subset k(t_1, \dots, t_n)$.
- ▶ $\exists 13$ \mathbb{Z} -conjugacy subgroups $G \leq \text{GL}_2(\mathbb{Z})$.

Theorem (Voskresenskii, 1967) 2-dim. algebraic tori T

T is rational over k .

- ▶ $\exists 73$ \mathbb{Z} -conjugacy subgroups $G \leq \text{GL}_3(\mathbb{Z})$.

Theorem (Kunyavskii, 1990) 3-dim. algebraic tori T

(i) T is rational over $k \iff T$ is stably rational over k

$\iff T$ is retract rational over $k \iff \exists G$: 58 groups;

(ii) T is not rational over $k \iff T$ is not stably rational over k

$\iff T$ is not retract rational over $k \iff \exists G$: 15 groups.

Rationality of algebraic tori (4-dim., 5-dim.)

- ▶ $\exists 710$ \mathbb{Z} -conjugacy subgroups $G \leq \mathrm{GL}_4(\mathbb{Z})$.

Theorem (H-Yamasaki, 2017, Mem. AMS) 4-dim. alg. tori T

- (i) T is **stably rational** over $k \iff \exists G$: 487 groups;
- (ii) T is **not stably** but **retract rational** over $k \iff \exists G$: 7 groups;
- (iii) T is **not retract rational** over $k \iff \exists G$: 216 groups.

- ▶ $\exists 6079$ \mathbb{Z} -conjugacy subgroups $G \leq \mathrm{GL}_5(\mathbb{Z})$.

Theorem (H-Yamasaki, 2017, Mem. AMS) 5-dim. alg. tori T

- (i) T is **stably rational** over $k \iff \exists G$: 3051 groups;
- (ii) T is **not stably** but **retract rational** over $k \iff \exists G$: 25 groups;
- (iii) T is **not retract rational** over $k \iff \exists G$: 3003 groups.

- ▶ (Voskresenskii's conjecture) any **stably rational** torus is **rational**.
- ▶ $\exists 85308$ \mathbb{Z} -conjugacy subgroups $G \leq \mathrm{GL}_6(\mathbb{Z})$!

Proof: Flabby (Flasque) resolution (1/2)

- ▶ The function field of n -dim. $T \xleftrightarrow{\text{identified}} L(M)^G, G \leq \text{GL}(n, \mathbb{Z})$
- ▶ M : G -lattice, i.e. f.g. \mathbb{Z} -free $\mathbb{Z}[G]$ -module.

Definition

- (i) M is **permutation** $\stackrel{\text{def}}{\iff} M \simeq \bigoplus_{1 \leq i \leq m} \mathbb{Z}[G/H_i]$.
- (ii) M is **stably permutation** $\stackrel{\text{def}}{\iff} M \oplus \exists P \simeq P', P, P'$: permutation.
- (iii) M is **invertible** $\stackrel{\text{def}}{\iff} M \oplus \exists M' \simeq P$: permutation.
- (iv) M is **coflabby** $\stackrel{\text{def}}{\iff} H^1(H, M) = 0 (\forall H \leq G)$.
- (v) M is **flabby** $\stackrel{\text{def}}{\iff} \widehat{H}^{-1}(H, M) = 0 (\forall H \leq G)$. (\widehat{H} : Tate cohomology)

- ▶ “permutation”
 - \implies “stably permutation”
 - \implies “invertible”
 - \implies “flabby and coflabby”.

Proof: Flabby (Flasque) resolution (2/2)

Commutative monoid \mathcal{M}

$M_1 \sim M_2 \stackrel{\text{def}}{\iff} M_1 \oplus P_1 \simeq M_2 \oplus P_2$ ($\exists P_1, \exists P_2$: permutation).
 \implies commutative monoid \mathcal{M} : $[M_1] + [M_2] := [M_1 \oplus M_2]$, $0 = [P]$.

Theorem (Endo-Miyata, 1974, Colliot-Thélène-Sansuc, 1977)

$\exists P$: permutation, $\exists F$: flabby such that

$$0 \rightarrow M \rightarrow P \rightarrow F \rightarrow 0: \text{ flabby resolution of } M.$$

$[M]^{fl} := [F]$, $[M]^{fl}$ is invertible $\stackrel{\text{def}}{\iff} [M]^{fl} = [E]$ ($\exists E$: invertible).

Theorem (Endo-Miyata, 1973, Voskresenskii, 1974, Saltman, 1984)

(EM73) $[M]^{fl} = 0 \iff L(M)^G$ is **stably rational** over k .

(Vos74) $[M]^{fl} = [M']^{fl} \iff L(M)^G(x_1, \dots, x_m) \simeq L(M')^G(y_1, \dots, y_n)$.

(Sal84) $[M]^{fl}$ is invertible $\iff L(M)^G$ is **retract rational** over k .

Our contribution

- ▶ We give a procedure to compute a flabby resolution of M , in particular $[M]^{fl} = [F]$, **effectively** (with smaller rank after base change) by computer software GAP.
- ▶ The function `IsFlabby` (resp. `IsCoflabby`) may determine whether M is **flabby** (resp. **coflabby**).
- ▶ The function `IsInvertibleF` may determine whether $[M]^{fl} = [F]$ is **invertible** (\leftrightarrow whether $L(M)^G$ (resp. T) is **retract rational**).
- ▶ We provide some functions for checking **a possibility** of isomorphism

$$\left(\bigoplus_{i=1}^r a_i \mathbb{Z}[G/H_i] \right) \oplus a_{r+1} F \simeq \bigoplus_{i=1}^r b'_i \mathbb{Z}[G/H_i] \quad (*)$$

by computing **some invariants** (e.g. trace, \widehat{Z}^0 , \widehat{H}^0) of both sides.

- ▶ [HY, Example 10.7]. $G \simeq S_5 \leq \mathrm{GL}(5, \mathbb{Z})$ with number $(5, 946, 4)$
 $\implies \mathrm{rank}(F) = 17$ and $\mathrm{rank}(*) = 88$ holds
 $\implies [F] = 0 \implies L(M)^G$ (resp. T) is **stably rational** over k .

Application

Corollary ($[F] = [M]^{fl}$: invertible case, $G \simeq S_5, F_{20}$)

$\exists T, T'$; 4-dim. **not stably rational** algebraic tori over k such that $T \not\sim T'$ (birational) and $T \times T'$: 8-dim. **stably rational** over k .
 $\because -[M]^{fl} = [M']^{fl} \neq 0$.

Prop. ([HY], Krull-Schmidt fails for permutation D_6 -lattices)

$\{1\}, C_2^{(1)}, C_2^{(2)}, C_2^{(3)}, C_3, C_2^2, C_6, S_3^{(1)}, S_3^{(2)}, D_6$: conj. subgroups of D_6 .
$$\mathbb{Z}[D_6] \oplus \mathbb{Z}[D_6/C_2^{(2)}]^{\oplus 2} \oplus \mathbb{Z}[D_6/C_6] \oplus \mathbb{Z}[D_6/S_3^{(1)}] \oplus \mathbb{Z}[D_6/S_3^{(2)}]$$
$$\simeq \mathbb{Z}[D_6/C_2^{(1)}] \oplus \mathbb{Z}[D_6/C_2^{(2)}] \oplus \mathbb{Z}[D_6/C_2^{(3)}] \oplus \mathbb{Z}[D_6/C_3] \oplus \mathbb{Z}^{\oplus 2}.$$

► D_6 is the smallest example exhibiting the failure of K-S:

Theorem (Dress, 1973)

Krull-Schmidt holds for permutation G -lattices $\iff G/O_p(G)$ is cyclic where $O_p(G)$ is the maximal normal p -subgroup of G .

Krull-Schmidt and Direct sum cancelation

Theorem (Hindman-Klingler-Odenthal, 1998) Assume $G \neq D_8$

Krull-Schmidt **holds** for G -lattices \iff (i) $G = C_p$ ($p \leq 19$; prime),
(ii) $G = C_n$ ($n = 1, 4, 8, 9$), (iii) $G = V_4$ or (iv) $G = D_4$.

Theorem (Endo-Hironaka, 1979)

Direct sum cancellation **holds**, i.e. $M_1 \oplus N \simeq M_2 \oplus N \implies M_1 \simeq M_2$,
 $\implies G$ is abelian, dihedral, A_4 , S_4 or A_5 (*).

- ▶ via projective class group (see Swan (1988) Corollary 1.3, Section 7).
- ▶ Except for (*) \implies Direct sum cancelation **fails** \implies K-S **fails**

Theorem ([HY]) $G \leq \text{GL}(n, \mathbb{Z})$ (up to conjugacy)

- (i) $n \leq 4 \implies$ K-S **holds**.
- (ii) $n = 5$. K-S **fails** \iff 11 groups G (among 6079 groups).
- (iii) $n = 6$. K-S **fails** \iff 131 groups G (among 85308 groups).

Special case: $T = R_{K/k}^{(1)}(\mathbb{G}_m)$; norm one tori (1/5)

- ▶ Rationality problem for $T = R_{K/k}^{(1)}(\mathbb{G}_m)$ is investigated by S. Endo, Colliot-Thélène and Sansuc, W. Hürlimann, L. Le Bruyn, A. Cortella and B. Kunyavskii, N. Lemire and M. Lorenz, M. Florence, etc.

Theorem (Endo-Miyata, 1974), (Saltman, 1984)

Let K/k be a finite **Galois** field extension and $G = \text{Gal}(K/k)$.

- (i) T is **retract** k -rational \iff all the Sylow subgroups of G are cyclic;
- (ii) T is **stably** k -rational \iff G is a cyclic group, or a direct product of a cyclic group of order m and a group $\langle \sigma, \tau \mid \sigma^n = \tau^{2^d} = 1, \tau\sigma\tau^{-1} = \sigma^{-1} \rangle$, where $d, m \geq 1, n \geq 3, m, n$: odd, and $(m, n) = 1$.

Theorem (Endo, 2011)

Let K/k be a finite **non-Galois**, separable field extension and L/k be the Galois closure of K/k . Assume that the Galois group of L/k is **nilpotent**. Then the norm one torus $T = R_{K/k}^{(1)}(\mathbb{G}_m)$ is **not retract** k -rational.

Special case: $T = R_{K/k}^{(1)}(\mathbb{G}_m)$; norm one tori (2/5)

- ▶ Let K/k be a finite **non-Galois**, separable field extension
- ▶ Let L/k be the Galois closure of K/k .
- ▶ Let $G = \text{Gal}(L/k)$ and $H = \text{Gal}(L/K) \leq G$.

Theorem (Endo, 2011)

Assume that all the Sylow subgroups of G are cyclic.

Then T is **retract** k -rational.

$T = R_{K/k}^{(1)}(\mathbb{G}_m)$ is **stably** k -rational $\iff G = D_n, n$ odd ($n \geq 3$) or $C_m \times D_n, m, n$ odd ($m, n \geq 3$), $(m, n) = 1, H \leq D_n$ with $|H| = 2$.

Special case: $T = R_{K/k}^{(1)}(\mathbb{G}_m)$; norm one tori (3/5)

Theorem (Endo, 2011) $\dim T = n - 1$

Assume that $\text{Gal}(L/k) = S_n$, $n \geq 3$, and $\text{Gal}(L/K) = S_{n-1}$ is the stabilizer of one of the letters in S_n .

- (i) $R_{K/k}^{(1)}(\mathbb{G}_m)$ is **retract** k -rational $\iff n$ is a prime;
- (ii) $R_{K/k}^{(1)}(\mathbb{G}_m)$ is **(stably)** k -rational $\iff n = 3$.

Theorem (Endo, 2011) $\dim T = n - 1$

Assume that $\text{Gal}(L/k) = A_n$, $n \geq 4$, and $\text{Gal}(L/K) = A_{n-1}$ is the stabilizer of one of the letters in A_n .

- (i) $R_{K/k}^{(1)}(\mathbb{G}_m)$ is **retract** k -rational $\iff n$ is a prime;
- (ii) $\exists t \in \mathbb{N}$ s.t. $[R_{K/k}^{(1)}(\mathbb{G}_m)]^{(t)}$ is **stably** k -rational $\iff n = 5$.

- ▶ $[R_{K/k}^{(1)}(\mathbb{G}_m)]^{(t)}$: the product of t copies of $R_{K/k}^{(1)}(\mathbb{G}_m)$.

Special case: $T = R_{K/k}^{(1)}(\mathbb{G}_m)$; norm one tori (4/5)

Theorem ([HY], Rationality for $R_{K/k}^{(1)}(\mathbb{G}_m)$ (dim. 4, $[K : k] = 5$))

Let K/k be a separable field extension of degree 5 and L/k be the Galois closure of K/k . Assume that $G = \text{Gal}(L/k)$ is a transitive subgroup of S_5 and $H = \text{Gal}(L/K)$ is the stabilizer of one of the letters in G . Then the rationality of $R_{K/k}^{(1)}(\mathbb{G}_m)$ is given by

G	$L(M) = L(x_1, x_2, x_3, x_4)^G$
5T1	C_5 stably k -rational
5T2	D_5 stably k -rational
5T3	F_{20} not stably but retract k -rational
5T4	A_5 stably k -rational
5T5	S_5 not stably but retract k -rational

- ▶ This theorem is already known **except for the case of A_5** (Endo).
- ▶ Stably k -rationality for the case A_5 is asked by S. Endo (2011).

Special case: $T = R_{K/k}^{(1)}(\mathbb{G}_m)$; norm one tori (5/5)

By combining this theorem with Endo's theorem, we obtain:

Corollary

Assume that $\text{Gal}(L/k) = A_n$, $n \geq 4$, and $\text{Gal}(L/K) = A_{n-1}$ is the stabilizer of one of the letters in A_n . Then

$R_{K/k}^{(1)}(\mathbb{G}_m)$ is **stably** k -rational $\iff n = 5$.