

Norm one tori and Hasse norm principle

Akinari Hoshi (joint work with Kazuki Kanai and Aiichi Yamasaki)

Niigata University, Japan

November 6, 2024

Table of contents

- 1 Introduction & Main theorems 1,2,3,4
- 2 Rationality problem for algebraic tori
- 3 Proof of Theorem 3
- 4 Applications: R -equivalence & Tamagawa number

A. Hoshi, K. Kanai, A. Yamasaki,

[HKY22] Norm one tori and Hasse norm principle, Math. Comp. (2022).

[HKY23] Norm one tori and Hasse norm principle, II: Degree 12 case, JNT (2023).

[HKY1] Norm one tori and Hasse norm principle, III: Degree 16 case,
arXiv:2404.01362.

[HKY2] Hasse norm principle for M_{11} and J_1 extensions, arXiv:2210.09119.

We use GAP. The related algorithms/functions are available from

<https://doi.org/10.57723/289563> (KURENAI: repository of Kyoto University),

<http://mathweb.sc.niigata-u.ac.jp/~hoshi/Algorithm/Norm1ToriHNP/>,

<https://www.math.kyoto-u.ac.jp/~yamasaki/Algorithm/Norm1ToriHNP/>.

§1 Introduction & Main theorems 1,2,3,4

- ▶ k : a global field, i.e. a number field or a finite extension of $\mathbb{F}_q(t)$.

Definition (Hasse norm principle)

Let k be a global field. K/k be a finite extension and \mathbb{A}_K^\times be the idele group of K . We say that **the Hasse norm principle holds for K/k** if

$$\text{Obs}(K/k) := (N_{K/k}(\mathbb{A}_K^\times) \cap k^\times) / N_{K/k}(K^\times) = 1$$

where $N_{K/k}$ is the norm map.

Theorem (Hasse's norm theorem 1931)

If K/k is a cyclic extension of a number field, then

$$\text{Obs}(K/k) = 1.$$

Example (Hasse [Has31]): $\text{Obs}(\mathbb{Q}(\sqrt{-39}, \sqrt{-3})/\mathbb{Q}) \simeq \mathbb{Z}/2\mathbb{Z}$.

$$\text{Obs}(\mathbb{Q}(\sqrt{2}, \sqrt{-1})/\mathbb{Q}) = 1.$$

In both cases, Galois group $G \simeq V_4$ (Klein four-group).

Tate's theorem (1967)

For any Galois extension K/k , Tate gave:

Theorem (Tate 1967, in *Alg. Num. Th. ed. by Cassels and Fröhlich*)

Let K/k be a finite Galois extension with Galois group $\text{Gal}(K/k) \simeq G$. Let V_k be the set of all places of k and G_v be the decomposition group of G at $v \in V_k$. Then

$$\text{Obs}(K/k) \simeq \text{Coker} \left\{ \bigoplus_{v \in V_k} \widehat{H}^{-3}(G_v, \mathbb{Z}) \xrightarrow{\text{cores}} \widehat{H}^{-3}(G, \mathbb{Z}) \right\}$$

where \widehat{H} is the Tate cohomology. In particular, the Hasse norm principle holds for K/k if and only if the restriction map $H^3(G, \mathbb{Z}) \xrightarrow{\text{res}} \bigoplus_{v \in V_k} H^3(G_v, \mathbb{Z})$ is injective.

- ▶ If $G \simeq C_n$ is cyclic, then $H^3(C_n, \mathbb{Z}) \simeq H^1(C_n, \mathbb{Z}) = 0$ and hence the Hasse's original theorem follows.
- ▶ If $G \simeq V_4$, then $\text{Obs}(K/k) = 0 \iff \exists v \in V_k$ such that $G_v = V_4$ ($H^3(V_4, \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$) (v : should be ramified).

Known results for HNP (1/2)

The HNP for Galois extensions K/k was investigated by Gerth [Ger77], [Ger78], Gurak [Gur78a], [Gur78b], [Gur80], Morishita [Mor90], Horie [Hor93], Takeuchi [Tak94], Kagawa [Kag95], etc.

- ▶ (Gurak 1978; Endo-Miyata 1975 + Ono 1963)
If all the Sylow subgroups of $\text{Gal}(K/k)$ is cyclic, then $\text{Obs}(K/k) = 0$.

However, for non-Galois extensions K/k , very little is known whether the Hasse norm principle holds:

- ▶ (Bartels 1981) $[K : k] = p$; prime \Rightarrow HNP for K/k holds.
- ▶ (Bartels 1981) $[K : k] = n$ and Galois closure $\text{Gal}(L/k) \simeq D_n$
 \Rightarrow HNP for K/k holds.
- ▶ (Voskresenskii-Kunyavskii 1984) $[K : k] = n$ and $\text{Gal}(L/k) \simeq S_n$
 \Rightarrow HNP for K/k holds.
- ▶ (Macedo 2020) $[K : k] = n$ and $\text{Gal}(L/k) \simeq A_n$
 \Rightarrow HNP for K/k holds if $n \geq 5$; $n = 6$ using Hoshi-Yamasaki [HY17].

Ono's theorem (1963)

- ▶ T : algebraic k -torus, i.e. $T \times_k \bar{k} \simeq (\mathbb{G}_{m, \bar{k}})^n$.
- ▶ $\text{III}(T) := \text{Ker}\{H^1(k, T) \xrightarrow{\text{res}} \bigoplus_{v \in V_k} H^1(k_v, T)\}$: Shafarevich-Tate gp.
- ▶ The norm one torus $R_{K/k}^{(1)}(\mathbb{G}_m)$ of K/k :

$$1 \longrightarrow R_{K/k}^{(1)}(\mathbb{G}_m) \longrightarrow R_{K/k}(\mathbb{G}_{m,K}) \xrightarrow{N_{K/k}} \mathbb{G}_{m,k} \longrightarrow 1$$

where $R_{K/k}$ is the Weil restriction.

- ▶ $R_{K/k}^{(1)}(\mathbb{G}_m)$ is biregularly isomorphic to the norm hyper surface $f(x_1, \dots, x_n) = 1$ where $f \in k[x_1, \dots, x_n]$ is the norm form of K/k .

Theorem (Ono 1963, Ann. of Math.)

Let K/k be a finite extension and $T = R_{K/k}^{(1)}(\mathbb{G}_m)$. Then

$$\text{III}(T) \simeq \text{Obs}(K/k).$$

Known results for HNP (2/2)

- ▶ $T = R_{K/k}^{(1)}(\mathbb{G}_m)$.
- ▶ $\text{III}(T) \simeq \text{Obs}(K/k)$.

Theorem (Kunyavskii 1984)

Let $[K : k] = 4$, $G = \text{Gal}(L/k) \simeq 4Tm$ ($1 \leq m \leq 5$).

Then $\text{III}(T) = 0$ except for $4T2$ and $4T4$. For $4T2 \simeq V_4$, $4T4 \simeq A_4$,

(i) $\text{III}(T) \leq \mathbb{Z}/2\mathbb{Z}$;

(ii) $\text{III}(T) = 0 \Leftrightarrow \exists v \in V_k$ such that $V_4 \leq G_v$.

Theorem (Drakokhrust-Platonov 1987)

Let $[K : k] = 6$, $G = \text{Gal}(L/k) \simeq 6Tm$ ($1 \leq m \leq 16$).

Then $\text{III}(T) = 0$ except for $6T4$ and $6T12$. For $6T4 \simeq A_4$, $6T12 \simeq A_5$,

(i) $\text{III}(T) \leq \mathbb{Z}/2\mathbb{Z}$;

(ii) $\text{III}(T) = 0 \Leftrightarrow \exists v \in V_k$ such that $V_4 \leq G_v$.

Main theorems 1,2,3,4 (1/3)

- ▶ \exists 2, 13, 73, 710, 6079 cases of alg. k -tori T of $\dim(T) = 1, 2, 3, 4, 5$.
- ▶ X : a smooth k -compactification of T , $\bar{X} = X \times_k \bar{k}$.

Theorem 1 ([HKY22, Theorem 1.5 and Theorem 1.6])

(i) $\dim(T) = 4$. Among the 216 cases (of 710) of **not retract rational** T ,

$$H^1(k, \text{Pic } \bar{X}) \simeq \begin{cases} 0 & (194 \text{ of } 216), \\ \mathbb{Z}/2\mathbb{Z} & (20 \text{ of } 216), \\ (\mathbb{Z}/2\mathbb{Z})^{\oplus 2} & (2 \text{ of } 216). \end{cases}$$

(ii) $\dim(T) = 5$. Among 3003 cases (of 6079) of **not retract rational** T ,

$$H^1(k, \text{Pic } \bar{X}) \simeq \begin{cases} 0 & (2729 \text{ of } 3003), \\ \mathbb{Z}/2\mathbb{Z} & (263 \text{ of } 3003), \\ (\mathbb{Z}/2\mathbb{Z})^{\oplus 2} & (11 \text{ of } 3003). \end{cases}$$

- ▶ Kunyavskii (1984) showed that among the 15 cases (of 73) of **not retract rational** T of $\dim(T) = 3$, $H^1(k, \text{Pic } \bar{X}) = 0$ (13 of 15), $H^1(k, \text{Pic } \bar{X}) \simeq \mathbb{Z}/2\mathbb{Z}$ (2 of 15).

Main theorems 1,2,3,4 (2/3)

- ▶ k : a field, K/k : a separable field extension of $[K : k] = n$.
- ▶ $T = R_{K/k}^{(1)}(\mathbb{G}_m)$ with $\dim(T) = n - 1$.
- ▶ X : a smooth k -compactification of T .
- ▶ L/k : Galois closure of K/k , $G := \text{Gal}(L/k)$ and $H = \text{Gal}(L/K)$ with $[G : H] = n \implies G = nTm \leq S_n$: transitive.
- ▶ The number of transitive subgroups nTm of S_n ($2 \leq n \leq 15$) up to conjugacy is given as follows:

n	2	3	4	5	6	7	8	9	10	11	12	13	14	15
# of nTm	1	2	5	5	16	7	50	34	45	8	301	9	63	104

Theorem 2 ([HKY22, Theorem 1.5], [HKY23, Theorem 1.1])

Let $2 \leq n \leq 15$ be an integer. Then $H^1(k, \text{Pic } \overline{X}) \neq 0 \iff G = nTm$ is given as in [HKY22, Table 1] ($n \neq 12$) or [HKY23, Table 1] ($n = 12$).

[HKY22, Table 1]: $H^1(k, \text{Pic } \bar{X}) \simeq H^1(G, [J_{G/H}]^{fl}) \neq 0$
 where $G = nTm$ with $2 \leq n \leq 15$ and $n \neq 12$

G	$H^1(k, \text{Pic } \bar{X}) \simeq H^1(G, [J_{G/H}]^{fl})$
$4T2 \simeq V_4$	$\mathbb{Z}/2\mathbb{Z}$
$4T4 \simeq A_4$	$\mathbb{Z}/2\mathbb{Z}$
$6T4 \simeq A_4$	$\mathbb{Z}/2\mathbb{Z}$
$6T12 \simeq A_5$	$\mathbb{Z}/2\mathbb{Z}$
$8T2 \simeq C_4 \times C_2$	$\mathbb{Z}/2\mathbb{Z}$
$8T3 \simeq (C_2)^3$	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 3}$
$8T4 \simeq D_4$	$\mathbb{Z}/2\mathbb{Z}$
$8T9 \simeq D_4 \times C_2$	$\mathbb{Z}/2\mathbb{Z}$
$8T11 \simeq (C_4 \times C_2) \rtimes C_2$	$\mathbb{Z}/2\mathbb{Z}$
$8T13 \simeq A_4 \times C_2$	$\mathbb{Z}/2\mathbb{Z}$
$8T14 \simeq S_4$	$\mathbb{Z}/2\mathbb{Z}$
$8T15 \simeq C_8 \rtimes V_4$	$\mathbb{Z}/2\mathbb{Z}$
$8T19 \simeq (C_2)^3 \rtimes C_4$	$\mathbb{Z}/2\mathbb{Z}$
$8T21 \simeq (C_2)^3 \rtimes C_4$	$\mathbb{Z}/2\mathbb{Z}$
$8T22 \simeq (C_2)^3 \rtimes V_4$	$\mathbb{Z}/2\mathbb{Z}$
$8T31 \simeq ((C_2)^4 \rtimes C_2) \rtimes C_2$	$\mathbb{Z}/2\mathbb{Z}$
$8T32 \simeq ((C_2)^3 \rtimes V_4) \rtimes C_3$	$\mathbb{Z}/2\mathbb{Z}$
$8T37 \simeq \text{PSL}_3(\mathbb{F}_2) \simeq \text{PSL}_2(\mathbb{F}_7)$	$\mathbb{Z}/2\mathbb{Z}$
$8T38 \simeq (((C_2)^4 \rtimes C_2) \rtimes C_2) \rtimes C_3$	$\mathbb{Z}/2\mathbb{Z}$

[HKY22, Table 1]: $H^1(k, \text{Pic } \overline{X}) \simeq H^1(G, [J_{G/H}]^{fl}) \neq 0$
 where $G = nTm$ with $2 \leq n \leq 15$ and $n \neq 12$

G	$H^1(k, \text{Pic } \overline{X}) \simeq H^1(G, [J_{G/H}]^{fl})$
$9T2 \simeq (C_3)^2$	$\mathbb{Z}/3\mathbb{Z}$
$9T5 \simeq (C_3)^2 \rtimes C_2$	$\mathbb{Z}/3\mathbb{Z}$
$9T7 \simeq (C_3)^2 \rtimes C_3$	$\mathbb{Z}/3\mathbb{Z}$
$9T9 \simeq (C_3)^2 \rtimes C_4$	$\mathbb{Z}/3\mathbb{Z}$
$9T11 \simeq (C_3)^2 \rtimes C_6$	$\mathbb{Z}/3\mathbb{Z}$
$9T14 \simeq (C_3)^2 \rtimes Q_8$	$\mathbb{Z}/3\mathbb{Z}$
$9T23 \simeq ((C_3)^2 \rtimes Q_8) \rtimes C_3$	$\mathbb{Z}/3\mathbb{Z}$
$10T7 \simeq A_5$	$\mathbb{Z}/2\mathbb{Z}$
$10T26 \simeq \text{PSL}_2(\mathbb{F}_9) \simeq A_6$	$\mathbb{Z}/2\mathbb{Z}$
$10T32 \simeq S_6$	$\mathbb{Z}/2\mathbb{Z}$
$14T30 \simeq \text{PSL}_2(\mathbb{F}_{13})$	$\mathbb{Z}/2\mathbb{Z}$
$15T9 \simeq (C_5)^2 \rtimes C_3$	$\mathbb{Z}/5\mathbb{Z}$
$15T14 \simeq (C_5)^2 \rtimes S_3$	$\mathbb{Z}/5\mathbb{Z}$

Main theorems 1,2,3,4 (3/3)

- ▶ k : a number field, K/k : a separable field extension of $[K : k] = n$.
- ▶ $T = R_{K/k}^{(1)}(\mathbb{G}_m)$, X : a smooth k -compactification of T .

Theorem 3 ([HKY22, Theorem 1.18], [HKY23, Theorem 1.3])

Let $2 \leq n \leq 15$ be an integer. For the cases in [HKY22, Table 1] ($n \neq 12$) or [HKY23, Table 1] ($n = 12$),

$$\text{III}(T) = 0 \iff G = nTm \text{ satisfies } \boxed{\text{some conditions}} \text{ of } G_v$$

where G_v is the decomposition group of G at v .

- ▶ By Ono's theorem $\text{III}(T) \simeq \text{Obs}(K/k)$, **Theorem 3 gives a necessary and sufficient condition for HNP for K/k with $[K : k] \leq 15$.**

Theorem 4 ([HKY22, Theorem 1.17])

Assume that $G = M_n \leq S_n$ ($n = 11, 12, 22, 23, 24$) is the Mathieu group of degree n . Then $H^1(k, \text{Pic } \overline{X}) = 0$. In particular, $\text{III}(T) = 0$.

Examples of Theorem 3

Example ($G = 8T4 \simeq D_4$, $8T13 \simeq A_4 \times C_2$, $8T14 \simeq S_4$,
 $8T37 \simeq \text{PSL}_2(\mathbb{F}_7)$, $10T7 \simeq A_5$, $14T30 \simeq \text{PSL}_2(\mathbb{F}_{13})$)

$\text{III}(T) = 0 \iff \exists v \in V_k$ such that $V_4 \leq G_v$.

Example ($G = 10T26 \simeq \text{PSL}_2(\mathbb{F}_9)$)

$\text{III}(T) = 0 \iff \exists v \in V_k$ such that $D_4 \leq G_v$.

Example ($G = 10T32 \simeq S_6 \leq S_{10}$)

$\text{III}(T) = 0 \iff \exists v \in V_k$ such that

- (i) $V_4 \leq G_v$ where $N_{\tilde{G}}(V_4) \simeq C_8 \rtimes (C_2 \times C_2)$ for the normalizer $N_{\tilde{G}}(V_4)$ of V_4 in \tilde{G} with the normalizer $\tilde{G} = N_{S_{10}}(G) \simeq \text{Aut}(G)$ of G in S_{10} or
- (ii) $D_4 \leq G_v$ where $D_4 \leq [G, G] \simeq A_6$.

- ▶ 45/165 subgroups $V_4 \leq G$ satisfy (i).
- ▶ 45/180 subgroups $D_4 \leq G$ satisfy (ii).

Definition of some rationalities

- L/k : f.g. field extension. L is k -rational $\stackrel{\text{def}}{\iff} L \simeq k(x_1, \dots, x_n)$.

Definition (stably rational)

L is called **stably k -rational** if $L(y_1, \dots, y_m)$ is k -rational.

Definition (retract rational)

Let k be an infinite field.

L is called **retract k -rational** if $\exists k$ -algebra $R \subset L$ such that

- (i) L is the quotient field of R ;
- (ii) $\exists f \in k[x_1, \dots, x_n]$, $\exists k$ -algebra hom. $\varphi : R \rightarrow k[x_1, \dots, x_n][1/f]$ and $\psi : k[x_1, \dots, x_n][1/f] \rightarrow R$ satisfying $\psi \circ \varphi = 1_R$.

Definition (unirational)

L is called **k -unirational** if $L \subset k(t_1, \dots, t_n)$.

- “rational” \Rightarrow “stably rational” \Rightarrow “retract rational” \Rightarrow “unirational”.
- algebraic k -torus T is k -unirational.

§2 Rationality problem for algebraic tori (1/3)

Problem (Rationality problem for algebraic tori)

Whether an algebraic torus T is k -rational?

- ▶ $\exists 2$ algebraic tori with $\dim(T) = 1$; the trivial torus \mathbb{G}_m and $R_{K/k}^{(1)}(\mathbb{G}_m)$ with $[K : k] = 2$, which are k -rational.
- ▶ $\exists 13$ algebraic tori with $\dim(T) = 2$;

Theorem (Voskresenskii 1967)

All the algebraic tori T with $\dim(T) = 2$ are k -rational.

- ▶ $\exists 73$ algebraic tori with $\dim(T) = 3$;

Theorem (Kunyavskii 1990)

- $\exists 58$ algebraic tori T with $\dim(T) = 3$ which are k -rational;
- $\exists 15$ algebraic tori T with $\dim(T) = 3$ which are **not** k -rational;
- T is k -rational $\Leftrightarrow T$ is stably k -rational $\Leftrightarrow T$ is retract k -rational.

- ▶ $\exists 710$ algebraic tori with $\dim(T) = 4$;

Theorem (Hoshi-Yamasaki 2017)

- (i) $\exists 487$ algebraic tori T with $\dim(T) = 4$ which are stably k -rational;
- (ii) $\exists 7$ algebraic tori T with $\dim(T) = 4$ which are not stably k -rational but retract k -rational;
- (iii) $\exists 216$ algebraic tori T with $\dim(T) = 4$ which are not retract k -rational.

- ▶ $\exists 6079$ algebraic tori with $\dim(T) = 5$;

Theorem (Hoshi-Yamasaki 2017)

- (i) $\exists 3051$ algebraic tori T with $\dim(T) = 5$ which are stably k -rational;
- (ii) $\exists 25$ algebraic tori T with $\dim(T) = 5$ which are not stably k -rational but retract k -rational;
- (iii) $\exists 3003$ algebraic tori T with $\dim(T) = 5$ which are not retract k -rational.

- ▶ We do not know “ k -rationality”.
- ▶ **Voskresenskii's conjecture**: any stably k -rational torus is k -rational (Zariski problem).

Rationality problem for algebraic tori T (2/3)

- ▶ T : algebraic k -torus
 $\implies \exists$ finite Galois extension L/k such that $T \times_k L \simeq (\mathbb{G}_{m,L})^n$.
- ▶ $G = \text{Gal}(L/k)$ where L is the minimal splitting field.

Category of algebraic k -tori which split/ L $\xleftrightarrow{\text{duality}}$ Category of G -lattices
(i.e. finitely generated \mathbb{Z} -free $\mathbb{Z}[G]$ -module)

- ▶ $T \mapsto$ the character group $X(T) = \text{Hom}(T, \mathbb{G}_m)$: G -lattice.
- ▶ $T = \text{Spec}(L[M]^G)$ which splits/ L with $X(T) \simeq M \leftarrow M$: G -lattice.
- ▶ Tori of dimension $n \xleftrightarrow{1:1}$ elements of the set $H^1(\mathcal{G}, \text{GL}(n, \mathbb{Z}))$
where $\mathcal{G} = \text{Gal}(\bar{k}/k)$ since $\text{Aut}(\mathbb{G}_m^n) = \text{GL}(n, \mathbb{Z})$.
- ▶ k -torus T of dimension n is determined uniquely by the integral representation $h : \mathcal{G} \rightarrow \text{GL}(n, \mathbb{Z})$ up to conjugacy, and the group $h(\mathcal{G})$ is a finite subgroup of $\text{GL}(n, \mathbb{Z})$.
- ▶ The function field of $T \xleftrightarrow{\text{identified}} L(M)^G$: invariant field.

Rationality problem for algebraic tori T (3/3)

- ▶ L/k : Galois extension with $G = \text{Gal}(L/k)$.
- ▶ $M = \bigoplus_{1 \leq j \leq n} \mathbb{Z} \cdot u_j$: G -lattice with a \mathbb{Z} -basis $\{u_1, \dots, u_n\}$.
- ▶ G acts on $L(x_1, \dots, x_n)$ by

$$\sigma(x_i) = \prod_{j=1}^n x_j^{a_{i,j}}, \quad 1 \leq i \leq n$$

for any $\sigma \in G$, when $\sigma(u_i) = \sum_{j=1}^n a_{i,j} u_j$, $a_{i,j} \in \mathbb{Z}$.

- ▶ $L(M) := L(x_1, \dots, x_n)$ with this action of G .
- ▶ The function field of algebraic k -torus T $\overset{\text{identified}}{\longleftrightarrow} L(M)^G$

Rationality problem for algebraic tori T (2nd form)

Whether $L(M)^G$ is k -rational?

(= purely transcendental over k ?; $L(M)^G = k(\exists t_1, \dots, \exists t_n)$?)

Flabby (Flasque) resolution (1/3)

- ▶ M : G -lattice, i.e. f.g. \mathbb{Z} -free $\mathbb{Z}[G]$ -module.

Definition

- (i) M is **permutation** $\stackrel{\text{def}}{\iff} M \simeq \bigoplus_{1 \leq i \leq m} \mathbb{Z}[G/H_i]$;
- (ii) M is **stably permutation** $\stackrel{\text{def}}{\iff} M \oplus \exists P \simeq P'$, P, P' : permutation;
- (iii) M is **invertible** $\stackrel{\text{def}}{\iff} M \oplus \exists M' \simeq P$: permutation;
- (iv) M is **coflabby** $\stackrel{\text{def}}{\iff} H^1(H, M) = 0$ ($\forall H \leq G$);
- (v) M is **flabby** $\stackrel{\text{def}}{\iff} \widehat{H}^{-1}(H, M) = 0$ ($\forall H \leq G$).

- ▶ “permutation” \Rightarrow “stably permutation” \Rightarrow “invertible”
 \Rightarrow “flabby and coflabby”.

Definition (Commutative monoid of G -lattices mod. permutation)

$M_1 \sim M_2 \stackrel{\text{def}}{\iff} M_1 \oplus P_1 \simeq M_2 \oplus P_2$ ($\exists P_1, \exists P_2$: permutation)
 \implies commutative monoid $\mathcal{L} : [M_1] + [M_2] := [M_1 \oplus M_2], 0 = [P]$.

Flabby (Flasque) resolution (2/3)

Theorem (Endo-Miyata 1975, Colliot-Thélène and Sansuc 1977)

For any G -lattice M , there exists a short exact sequence of G -lattices

$$0 \rightarrow M \rightarrow P \rightarrow F \rightarrow 0$$

where P is permutation and F is flabby.

- ▶ called a **flabby resolution** of the G -lattice M .
- ▶ $[M]^{fl} := [F]$: **flabby class** of M (well-defined).

Theorem (Endo-Miyata 1973, Voskresenskii 1974, Saltman 1984)

(EM73) $[M]^{fl} = 0 \iff L(M)^G$ is stably k -rational.

(Vos74) $[M]^{fl} = [M']^{fl} \iff L(M)^G(x_1, \dots, x_m) \simeq L(M')^G(y_1, \dots, y_n)$.

(Sal84) $[M]^{fl}$ is invertible $\iff L(M)^G$ is retract k -rational.

Flabby (Flasque) resolution (3/3)

Theorem (Voskresenskii 1969)

Let k be a field and $\mathcal{G} = \text{Gal}(\bar{k}/k)$. Let T be an algebraic k -torus, X be a smooth k -compactification of T and $\bar{X} = X \times_k \bar{k}$. Then

$$0 \rightarrow \hat{T} \rightarrow \hat{Q} \rightarrow \text{Pic } \bar{X} \rightarrow 0$$

is an exact seq. of \mathcal{G} -lattice where \hat{Q} is permutation and $\text{Pic } \bar{X}$ is flabby.

► $[\hat{T}]^{fl} = [\text{Pic } \bar{X}]$; **flabby class** of \hat{T} .

Theorem (Endo-Miyata 1973, Voskresenskii 1974, Saltman 1984)

(EM73) $[\text{Pic } \bar{X}] = 0 \iff T$ is stably k -rational.

(Vos74) $[\text{Pic } \bar{X}] = [\text{Pic } \bar{X}'] \iff T$ and T' are stably bir. k -equivalent.

(Sal84) $[\text{Pic } \bar{X}]$ is invertible $\iff T$ is retract k -rational.

Voskresenskii's theorem (1969) (1/2)

Theorem (Voskresenskii 1969)

Let k be a global field, T be an algebraic k -torus and X be a smooth k -compactification of T . Then there exists an exact sequence

$$0 \rightarrow A(T) \rightarrow H^1(k, \text{Pic } \overline{X})^\vee \rightarrow \text{III}(T) \rightarrow 0$$

where $M^\vee = \text{Hom}(M, \mathbb{Q}/\mathbb{Z})$ is the Pontryagin dual of M .

- ▶ The group $A(T) := \left(\prod_{v \in V_k} T(k_v) \right) / \overline{T(k)}$ is called the kernel of the weak approximation of T .
- ▶ $T : \text{retract rational} \iff [\widehat{T}]^{fl} = [\text{Pic } \overline{X}]$ is invertible
 $\implies \text{Pic } \overline{X}$ is flabby and coflabby
 $\implies H^1(k, \text{Pic } \overline{X})^\vee = 0 \implies A(T) = \text{III}(T) = 0$.
- ▶ when $T = R_{K/k}^{(1)}(\mathbb{G}_m)$, by Ono's theorem $\text{III}(T) \simeq \text{Obs}(K/k)$,
 $T : \text{retract } k\text{-rational} \implies \text{Obs}(K/k) = 0$ (HNP for K/k holds).

Voskresenskii's theorem (1969) (2/2)

- ▶ when $T = R_{K/k}^{(1)}(\mathbb{G}_m)$, by Ono's theorem $\text{III}(T) \simeq \text{Obs}(K/k)$,
 T : retract k -rational $\implies \text{Obs}(K/k) = 0$ (HNP for K/k holds).
- ▶ when $T = R_{K/k}^{(1)}(\mathbb{G}_m)$, $\widehat{T} = J_{G/H}$ where
 $J_{G/H} = (I_{G/H})^\circ = \text{Hom}(I_{G/H}, \mathbb{Z})$ is the dual lattice of
 $I_{G/H} = \text{Ker}(\varepsilon)$ and $\varepsilon : \mathbb{Z}[G/H] \rightarrow \mathbb{Z}$ is the augmentation map.
- ▶ (Hoshi-Yamasaki, 2018, Hasegawa-Hoshi-Yamasaki, 2020)
For $[K : k] = n \leq 15$ except $9T27 \simeq \text{PSL}_2(\mathbb{F}_8)$, the classification of
stably/retract rational $R_{K/k}^{(1)}(\mathbb{G}_m)$ was given.
- ▶ when $T = R_{K/k}^{(1)}(\mathbb{G}_m)$, T : retract k -rational $\implies H^1(k, \text{Pic } \overline{X}) = 0$
(use this to get Theorem 2).
- ▶ $H^1(k, \text{Pic } \overline{X}) \simeq \text{Br}(X)/\text{Br}(k) \simeq \text{Br}_{\text{nr}}(k(X)/k)/\text{Br}(k)$
where $\text{Br}(X)$ is the étale cohomological/Azumaya Brauer group of X
by Colliot-Thélène-Sansuc 1987.

§3 Proof of Theorem 3

- ▶ We use Drakokhrust-Platonov's method :

Definition (first obstruction to the HNP)

Let $L \supset K \supset k$ be a tower of finite extensions where L is normal over k . We call the group

$$\text{Obs}_1(L/K/k) = (N_{K/k}(\mathbb{A}_K^\times) \cap k^\times) / ((N_{L/k}(\mathbb{A}_L^\times) \cap k^\times) N_{K/k}(K^\times))$$

the first obstruction to the HNP for K/k corresponding to the tower $L \supset K \supset k$.

- ▶ $\text{Obs}_1(L/K/k) = \text{Obs}(K/k) / (N_{L/k}(\mathbb{A}_L^\times) \cap k^\times)$.
- ▶ $\text{Obs}_1(L/K/k)$ is easier than $\text{Obs}(K/k)$.
- ▶ We use GAP. The related algorithms/functions we made are available from <https://doi.org/10.57723/289563>

(KURENAI: repository of Kyoto University).

Drakokhrust-Platonov's method (1/3)

Theorem (Drakokhrust-Platonov 1987)

Let $L \supset K \supset k$ be a tower of finite extensions where L is Galois over k . Let $G = \text{Gal}(L/k)$ and $H = \text{Gal}(L/K)$. Then

$$\text{Obs}_1(L/K/k) \simeq \text{Ker } \psi_1 / \varphi_1(\text{Ker } \psi_2)$$

where

$$\begin{array}{ccc} H/[H, H] & \xrightarrow{\psi_1: H \hookrightarrow G} & G/[G, G] \\ \uparrow \varphi_1: H_w \hookrightarrow H & & \uparrow \varphi_2: G_v \hookrightarrow G \\ \bigoplus_{v \in V_k} \left(\bigoplus_{w|v} H_w/[H_w, H_w] \right) & \xrightarrow{\psi_2} & \bigoplus_{v \in V_k} G_v/[G_v, G_v] \end{array}$$

and ψ_2 is defined by

$$\psi_2(h[H_w, H_w]) = x^{-1}hx[G_v, G_v]$$

for $h \in H_w = H \cap xG_vx^{-1}$ ($x \in G$).

Drakokhrust-Platonov's method (2/3)

- ▶ ψ_2^v : the restriction of the map ψ_2 to $\bigoplus_{w|v} H_w/[H_w, H_w]$.
- ▶ $\text{Obs}_1(L/K/k) = \text{Ker } \psi_1 / \varphi_1(\text{Ker } \psi_2^{\text{nr}}) \varphi_1(\text{Ker } \psi_2^r)$.

Proposition (Drakokhrust-Platonov 1987)

- (i) $G_{v_1} \leq G_{v_2} \implies \varphi_1(\text{Ker } \psi_2^{v_1}) \subset \varphi_1(\text{Ker } \psi_2^{v_2})$;
- (ii) $\varphi_1(\text{Ker } \psi_2^{\text{nr}}) = \langle [h, x] \mid h \in H \cap xHx^{-1}, x \in G \rangle / [H, H]$;
- (iii) Let $H_i \leq G_i \leq G$ ($1 \leq i \leq m$), $H_i \leq H \cap G_i$, $k_i = L^{G_i}$ and $K_i = L^{H_i}$. If $\text{Obs}(K_i/k_i) = 1$ for any $1 \leq i \leq m$ and

$$\bigoplus_{i=1}^m \hat{H}^{-3}(G_i, \mathbb{Z}) \xrightarrow{\text{cores}} \hat{H}^{-3}(G, \mathbb{Z})$$

is surjective, then $\text{Obs}(K/k) = \text{Obs}_1(L/K/k)$. In particular,

$$[K : k] = n \text{ is square-free} \implies \text{Obs}(K/k) = \text{Obs}_1(L/K/k).$$

Drakokhrust-Platonov's method (3/3)

Theorem (Drakokhrust 1989; Opolka 1980)

Let $\tilde{L} \supset L \supset k$ be a tower of Galois extensions with $\tilde{G} = \text{Gal}(\tilde{L}/k)$ and $\tilde{H} = \text{Gal}(\tilde{L}/K)$ which correspond to a central extension

$$1 \rightarrow A \rightarrow \tilde{G} \rightarrow G \rightarrow 1 \text{ with } A \cap [\tilde{G}, \tilde{G}] \simeq M(G) = H^2(G, \mathbb{C}^\times);$$

the Schur multiplier of G . Then

$$\text{Obs}(K/k) = \text{Obs}_1(\tilde{L}/K/k).$$

In particular, if \tilde{G} is a Schur cover of G , i.e. $A \simeq M(G)$, then $\text{Obs}(K/k) = \text{Obs}_1(\tilde{L}/K/k)$.

- ▶ This theorem is useful, but \tilde{G} may become large!
- ▶ We use GAP. The related algorithms/functions we made are available from <https://doi.org/10.57723/289563>

(KURENAI: repository of Kyoto University).

Example : $G = 12T261 \simeq (S_3)^4 \rtimes V_4 \simeq S_3 \wr V_4 (1/2)$

Example ($G = 12T261 \simeq (S_3)^4 \rtimes V_4 \simeq S_3 \wr V_4$)

$\text{III}(T) = 0 \iff$ there exists a place v of k such that

- (i) $V_4 \leq G_v$ where $V_4 \cap D(G) = 1$ for the unique characteristic subgroup $D(G) \simeq (C_3)^4 \rtimes (C_2)^3 \triangleleft G$,
- (ii) $C_4 \times C_2 \leq G_v$ where $(C_4 \times C_2) \cap D(G) \simeq C_2$ with $D(G) \simeq (C_3)^4 \rtimes (C_2)^3 \triangleleft G$,
- (iii) $D_4 \leq G_v$ where $D_4 \cap (S_3)^4 \simeq C_2$ with $(S_3)^4 \triangleleft G$,
- (iv) $Q_8 \leq G_v$, or
- (v) $(C_2)^3 \rtimes C_3 \leq G_v$.

▶ $H^1(k, \text{Pic } \overline{X}) \simeq \mathbb{Z}/2\mathbb{Z}$.

▶ $|G| = 6^4 \times 4 = 5184$.

▶ $H^3(G, \mathbb{Z}) \simeq M(G) \simeq (\mathbb{Z}/2\mathbb{Z})^{\oplus 4}$: Schur multiplier of G .

$\tilde{G} \leftarrow$ **too large !** $|\tilde{G}| = 6^4 \times 4 \times 2^4 = 82944$.

Example : $G = 12T261 \simeq (S_3)^4 \rtimes V_4 \simeq S_3 \wr V_4 (2/2)$

We can take a minimal stem ext. $\bar{G} = \tilde{G}/A'$ (i.e. $\bar{A} \leq Z(\bar{G}) \cap [\bar{G}, \bar{G}]$) of G in the commutative diagram

$$\begin{array}{ccccccc}
 1 & \longrightarrow & A = M(G) & \longrightarrow & \tilde{G} & \xrightarrow{\pi} & G \longrightarrow 1 \\
 & & \downarrow & & \downarrow & & \parallel \\
 1 & \longrightarrow & \bar{A} = A/A' & \longrightarrow & \bar{G} = \tilde{G}/A' & \xrightarrow{\bar{\pi}} & G \longrightarrow 1
 \end{array}$$

with $\bar{A} \simeq \mathbb{Z}/2\mathbb{Z}$. There exists 15 minimal stem extensions. Then we can find **exactly one** (1/15) minimal stem extension which satisfies that

$$\bigoplus_{i=1}^{m'} \hat{H}^{-3}(G_i, \mathbb{Z}) \xrightarrow{\text{cores}} \hat{H}^{-3}(\bar{G}, \mathbb{Z})$$

is surjective. By Drakorust-Platonov's Proposition (iii), we have

$$\text{Obs}(K/k) = \text{Obs}_1(\bar{L}_j/K/k).$$

- ▶ $\text{Ker } \psi_1 = (\mathbb{Z}/2\mathbb{Z})^{\oplus 4}$.
- ▶ $\varphi_1^{\text{nr}}(\text{Ker } \psi_2^{\text{nr}}) = (\mathbb{Z}/2\mathbb{Z})^{\oplus 3}$.
- ▶ $\varphi_1^{\text{r}}(\text{Ker } \psi_2^{\text{r}}) = \mathbb{Z}/2\mathbb{Z}$ (819/891 cases) or 0 (72/891 cases).

Sketch of the proof of Theorem 3 (1/2)

Step 1

- For $G = \text{Gal}(L/k) = nTm \leq S_n$ and $H = \text{Gal}(L/K) \leq G$ with $[G : H] = n$, determine $T = R_{K/k}^{(1)}(\mathbb{G}_m)$ satisfying $H^1(k, \text{Pic } \overline{X}) \neq 0$. (Make Table 1)

- ▶ We should treat $n = (4, 6), 8, 9, 10, 12, 14, 15$ because $H^1(k, \text{Pic } \overline{X}) = 0$ when $n = p$: prime.

Step 2

- For the cases in Table 1, determine $\text{III}(T) \simeq \text{Obs}(K/k)$.

(2-1)

(a) $n = pq$ ($p \neq q$: primes) $\rightarrow \text{Obs}(K/k) \simeq \text{Obs}_1(L/K/k)$.

(b) otherwise \rightarrow Find a Schur cover \tilde{G} .

Then we get \tilde{L}/k s.t. $\text{Obs}(K/k) \simeq \text{Obs}_1(\tilde{L}/K/k)$.

(2-2) **Calculation** $\text{Obs}_1(\overline{L}/K/k)$ for suitable $\overline{L} \subset \tilde{L}$.

Sketch of the proof of Theorem 3 (2/2)

(2-2) Calculation $\text{Obs}_1(\overline{L}/K/k)$.

By Drakokhrust-Platonov's Theorem,

$$\text{Obs}_1(\overline{L}/K/k) \simeq \text{Ker } \psi_1 / \varphi_1(\text{Ker } \psi_2^{\text{nr}}) \varphi_1(\text{Ker } \psi_2^{\text{r}}),$$

$$\begin{array}{ccc} H/[H, H] & \xrightarrow{\psi_1} & G/[G, G] \\ \uparrow \varphi_1 & & \uparrow \varphi_2 \\ \bigoplus_{v \in V_k} \left(\bigoplus_{w|v} H_w/[H_w, H_w] \right) & \xrightarrow{\psi_2} & \bigoplus_{v \in V_k} G_v/[G_v, G_v]. \end{array}$$

We compute the following:

- (i) $\text{Ker } \psi_1 = (\overline{H} \cap [\overline{G}, \overline{G}]) / [\overline{H}, \overline{H}]$;
- (ii) $\varphi_1(\text{Ker } \psi_2^{\text{nr}}) = \langle [h, x] \mid h \in \overline{H} \cap x\overline{H}x^{-1}, x \in G \rangle / [\overline{H}, \overline{H}]$;
(by Drakokhrust-Platonov's Proposition (ii))
- (iii) $\varphi_1(\text{Ker } \psi_2^{\text{r}})$ (in terms of G_v).

§4 Application 1: R -equivalence in algebraic k -tori (1/2)

Definition (R -equivalence, Manin 1974, in *Cubic Forms*)

- ▶ $f : Z \rightarrow X$: rational map of k -varieties covers a point $x \in X(k)$.
 $\stackrel{\text{def}}{\iff}$ there exists a point $z \in Z(k)$ such that f is defined at z and $f(z) = x$.
- ▶ $x, y \in X(k)$ are R -equivalent.
 $\stackrel{\text{def}}{\iff}$ there exist a fin. seq. of points $x = x_1, \dots, x_r = y$ and rational maps $f_i : \mathbb{P}^1 \rightarrow X$ ($1 \leq i \leq r-1$) such that f_i covers x_i, x_{i+1} .

Theorem (Colliot-Thélène and Sansuc 1977)

Let k be a field, T be an algebraic k -torus and $1 \rightarrow S \rightarrow Q \rightarrow T \rightarrow 1$ be a flabby resolution of T . Then $T(k) = H^0(k, T) \xrightarrow{\delta} H^1(k, S)$ induces

$$T(k)/R \simeq H^1(k, S).$$

Application 1: R -equivalence in algebraic k -tori (2/2)

- ▶ Let k be a local field. Using Tate-Nakayama duality, we have

$$T(k)/R \simeq H^1(k, S) \simeq H^1(k, \widehat{S}) \simeq H^1(k, \text{Pic } \overline{X})$$

for norm one tori $T = R_{K/k}^{(1)}(\mathbb{G}_m)$ where $[K : k] = n \leq 15$.

Theorem ([HKY22], [HKY23])

Let $2 \leq n \leq 15$ be an integer. Let k be a local field, K/k be a separable field extension of degree n , and $T = R_{K/k}^{(1)}(\mathbb{G}_m)$ be the norm one torus of K/k . Then, $T(k)/R \simeq H^1(k, \text{Pic } \overline{X}) \neq 0 \iff G$ is given as in [HKY22, Table 1] of [HKY23, Table 1].

Application 2: Tamagawa number of k -tori (1/2)

Theorem (Ono 1963)

Let k be a global field, T be an algebraic k -torus and $\tau(T)$ be the Tamagawa number of T . Then

$$\tau(T) = \frac{|H^1(k, \widehat{T})|}{|\text{III}(T)|}.$$

In particular, if T is retract k -rational, then $\tau(T) = |H^1(k, \widehat{T})|$.

- ▶ Let k be a number field, K/k be a field extension of degree n , $T = R_{K/k}^{(1)}(\mathbb{G}_m)$ be the norm one torus of K/k . By Ono's formula, we can calculate Tamagawa number of T explicitly.
- ▶ Example. $G = 15T9 \Rightarrow \tau(T) = \frac{3}{5}$ or 3 because $\text{III}(T) \leq \mathbb{Z}/5\mathbb{Z}$.

Application 2: Tamagawa number of k -tori (2/2)

► $\tau(T) = |H^1(k, \widehat{T})|/|\text{III}(T)|.$

Theorem ([HKY22, Theorem 8.2])

Let k be a global field and T be an algebraic k -torus of dimension 4 (resp. 5). Among 710 (reps. 6079) cases of algebraic k -tori T , if T is one of the 688 (resp. 5805) cases with $H^1(k, \text{Pic } \overline{X}) = 0$, then $\tau(T) = |H^1(k, \widehat{T})|.$

Theorem ([HKY22, Theorem 8.3], [HKY23, Remark 1.4])

Let $2 \leq n \leq 15$ be an integer. Let k be a number field, K/k be a field extension of degree n , L/k be the Galois closure of K/k , and $T = R_{K/k}^{(1)}(\mathbb{G}_m)$ be the norm one torus of K/k . Then $\tau(T) = |H^1(k, \widehat{T})|$ except for the cases in [HKY22, Table 1] and [HKY23, Table 1]. For the exceptional cases, we have $\tau(T) = |H^1(G, J_{G/H})|/|\text{III}(T)|.$

Sporadic simple group cases: M_{11} and J_1 (1/3)

- ▶ k : a number field.
- ▶ K/k : a separable field extension of $[K : k] = n$ (**not fixed**).
- ▶ L/k : Galois closure of K/k with $G = \text{Gal}(L/k)$ and $H = \text{Gal}(L/K) \leq G$ with $[G : H] = n$.
- ▶ $T = R_{K/k}^{(1)}(\mathbb{G}_m)$ with $\dim(T) = n - 1$.
- ▶ X : a smooth k -compactification of T .
- ▶ $G \simeq M_{11}$ with $|G| = 7920 = 2^4 \cdot 3^2 \cdot 5 \cdot 11$ or
 $G \simeq J_1$ with $|G| = 175560 = 2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19$
 $\Rightarrow M(G) \simeq H^3(G, \mathbb{Z}) = 0$: Schur multiplier of G .

Theorem ([HKY2, Theorem 1.6]) $G \simeq M_{11}$

Assume that $G \simeq M_{11}$ and $H = \text{Gal}(L/K) \leq G$.

$$H^1(k, \text{Pic } \overline{X}) = \begin{cases} 0 & \text{if } \text{Syl}_2(H) \not\simeq C_2, C_4, C_8, \\ \mathbb{Z}/2\mathbb{Z} & \text{if } \text{Syl}_2(H) \simeq C_2, C_4, C_8. \end{cases}$$

Sporadic simple group cases: M_{11} and J_1 (2/3)

Theorem ([HKY2, Theorem 1.8]) $G \simeq M_{11}$

Assume that $G \simeq M_{11}$ and $H = \text{Gal}(L/K) \not\leq G$.

(1) If $\text{Syl}_2(H) \not\cong C_2, C_4, C_8$, then $A(T) \simeq \text{III}(T) \simeq H^1(k, \text{Pic } \overline{X}) = 0$.

(2) If $\text{Syl}_2(H) \simeq C_2, C_4, C_8$, then either

(a) $A(T) = 0$ and $\text{III}(T) \simeq H^1(k, \text{Pic } \overline{X}) \simeq \mathbb{Z}/2\mathbb{Z}$ or

(b) $A(T) \simeq H^1(k, \text{Pic } \overline{X}) \simeq \mathbb{Z}/2\mathbb{Z}$ and $\text{III}(T) = 0$,

and the condition (b) is equivalent to:

(c) there exists a place v of k such that

$$\begin{cases} V_4 \leq G_v \text{ or } Q_8 \leq G_v & \text{if } \text{Syl}_2(H) \simeq C_2, \\ D_4 \leq G_v \text{ or } Q_8 \leq G_v & \text{if } \text{Syl}_2(H) \simeq C_4, \\ QD_8 \leq G_v & \text{if } \text{Syl}_2(H) \simeq C_8 \end{cases}$$

where G_v is the decomposition group of G at a place v of k .

► $0 \rightarrow A(T) \rightarrow H^1(k, \text{Pic } \overline{X})^\vee \rightarrow \text{III}(T) \rightarrow 0$ (Voskresenskii 1969).

Sporadic simple group cases: M_{11} and J_1 (3/3)

Theorem ([HKY2, Theorem 1.7]) $G \simeq J_1$

Assume that $G \simeq J_1$ and $H = \text{Gal}(L/K) \not\leq G$.

$$H^1(k, \text{Pic } \overline{X}) = \begin{cases} 0 & \text{if } \text{Syl}_2(H) \not\simeq C_2, \\ \mathbb{Z}/2\mathbb{Z} & \text{if } \text{Syl}_2(H) \simeq C_2. \end{cases}$$

Theorem ([HKY2, Theorem 1.9]) $G \simeq J_1$

Assume that $G \simeq J_1$ and $H = \text{Gal}(L/K) \not\leq G$.

(1) If $\text{Syl}_2(H) \not\simeq C_2$, then $A(T) \simeq \text{III}(T) \simeq H^1(k, \text{Pic } \overline{X}) = 0$.

(2) If $\text{Syl}_2(H) \simeq C_2$, then either

(a) $A(T) = 0$ and $\text{III}(T) \simeq H^1(k, \text{Pic } \overline{X}) \simeq \mathbb{Z}/2\mathbb{Z}$ or

(b) $A(T) \simeq H^1(k, \text{Pic } \overline{X}) \simeq \mathbb{Z}/2\mathbb{Z}$ and $\text{III}(T) = 0$,

and the condition (b) is equivalent to:

(c) there exists a place v of k such that $V_4 \leq G_v$ where G_v is the decomposition group of G at a place v of k .

Proof: Macedo and Newton [MN22, Corollary 3.4]

- ▶ $L/K/k$: a tower of finite extensions with L/k Galois.
- ▶ $G = \text{Gal}(L/k)$ and $H = \text{Gal}(L/K) \leq G$.
- ▶ $H_p = \text{Syl}_p(H)$ and $K_p = L^{H_p}$.
- ▶ X and X_p : smooth compactifications of $T = R_{K/k}^{(1)}(\mathbb{G}_m)$ and $T_p = R_{K_p/k}^{(1)}(\mathbb{G}_m)$ respectively.

Theorem (Macedo and Newton [MN22, Corollary 3.4])

We obtain a commutative diagram with exact rows as follows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & A(T)_{(p)} & \longrightarrow & H^1(k, \text{Pic } \overline{X})_{(p)}^\vee & \longrightarrow & \text{III}(T)_{(p)} \longrightarrow 0 \\ & & \cong \downarrow & & \cong \downarrow & & \cong \downarrow \\ 0 & \longrightarrow & A(T_p)_{(p)} & \longrightarrow & H^1(k, \text{Pic } \overline{X}_p)_{(p)}^\vee & \longrightarrow & \text{III}(T_p)_{(p)} \longrightarrow 0 \end{array}$$

where (p) stands for the p -primary part and the vertical isomorphisms are induced by the natural inclusion $T \hookrightarrow T_p$.