Rationality problem for fields of invariants

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Table of contents

$\mathrm{Br}_{\mathrm{nr}}(X/\mathbb{C}) \simeq H^3(X,\mathbb{Z})_{\mathrm{tors}}$; Artin-Mumford invariant $(X : RC)$

$H^3_{\mathrm{nr}}(X,\mathbb{Q}/\mathbb{Z}) \simeq \mathrm{Hdg}^4(X,\mathbb{Z})/\mathrm{Hdg}^4(X,\mathbb{Z})_{\mathrm{alg}} \leftrightarrow$ integral Hodge conjecture

cf. Colliot-Thélène and Voisin, Duke Math. J. **161** (2012) 735–801.

§0. Introduction

Inverse Galois problem (IGP)

Does every finite group occur as a quotient group of the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$?

- Related to rationality problem (Emmy Noether's strategy: 1913)

A finite group $G \curvearrowright k(x_g \mid g \in G)$: rational function field over k by permutation $h(x_g) = x_{hg}$ (for any $g, h \in G$)

$k(x_g \mid g \in G)^G$ is rational over k, i.e. $k(x_g \mid g \in G)^G \simeq k(t_1, \ldots, t_n)$ (Noether's problem has an affirmative answer)

$\Longrightarrow k(x_g \mid g \in G)^G$ is retract rational over k (weaker concept)

$\Longleftrightarrow \exists$ generic extension (polynomial) for (G, k) (Saltman's sense)

$\overset{k\text{:Hilbertian}}{\Longrightarrow}$ IGP for (k, G) has an affirmative answer

Rationality problem for quasi-monomial actions

Definition (quasi-monomial action)

Let K/k be a finite field extension and $G \leq \mathrm{Aut}_k(K(x_1,\ldots,x_n))$; finite where $K(x_1,\ldots,x_n)$ is the rational function field of n variables over K. The action of G on $K(x_1,\ldots,x_n)$ is called quasi-monomial if
(i) $\sigma(K) \subset K$ for any $\sigma \in G$;
(ii) $K^G = k$;
(iii) for any $\sigma \in G$,
$$\sigma(x_j) = c_j(\sigma)\prod_{i=1}^{n} x_i^{a_{ij}}$$
where $c_j(\sigma) \in K^\times$, $1 \leq j \leq n$, $[a_{i,j}]_{1\leq i,j\leq n} \in GL_n(\mathbb{Z})$.

Rationality problem

Under what situation the fixed field $K(x_1,\ldots,x_n)^G$ is rational over k, i.e. $K(x_1,\ldots,x_n)^G \simeq k(t_1,\ldots,t_n)$ (=purely transcendental over k), if G acts on $K(x_1,\ldots,x_n)$ by quasi-monomial k-automorphisms.

Rationality problem for quasi-monomial actions

Definition (quasi-monomial action)

Let K/k be a finite field extension and $G \leq \mathrm{Aut}_k(K(x_1,\ldots,x_n))$; finite where $K(x_1,\ldots,x_n)$ is the rational function field of n variables over K. The action of G on $K(x_1,\ldots,x_n)$ is called quasi-monomial if

(i) $\sigma(K) \subset K$ for any $\sigma \in G$;

(ii) $K^G = k$;

(iii) for any $\sigma \in G$, $\qquad \sigma(x_j) = c_j(\sigma) \prod_{i=1}^{n} x_i^{a_{ij}}$

where $c_j(\sigma) \in K^\times$, $1 \leq j \leq n$, $[a_{i,j}]_{1\leq i,j\leq n} \in GL_n(\mathbb{Z})$.

- When $G \curvearrowright K$; trivial (i.e. $K = k$), called (just) monomial action.
- When $G \curvearrowright K$; trivial and permutation $\leftrightarrow$ Noether's problem.
- When $c_j(\sigma) = 1$ $(\forall \sigma \in G, \forall j)$, called purely (quasi-)monomial.
- $G = \mathrm{Gal}(K/k)$ and purely $\leftrightarrow$ Rationality problem for algebraic tori.

Exercises (1/2): Noether's problem

- $S_n \curvearrowright \mathbb{Q}(x_1, \ldots, x_n)$; permutation
  Q. Is $\mathbb{Q}(x_1, \ldots, x_n)^{S_n}$ rational over $\mathbb{Q}$? Ans. Yes!
  $\mathbb{Q}(x_1, \ldots, x_n)^{S_n} = \mathbb{Q}(s_1, \ldots, s_n)$; s_i, ith elementary symmetric
  $\Longrightarrow$ IGP for $(\mathbb{Q}, S_n)$ has affirmative solution.
- $A_n \curvearrowright \mathbb{Q}(x_1, \ldots, x_n)$; permutation
  Q. Is $\mathbb{Q}(x_1, \ldots, x_n)^{A_n}$ rational over $\mathbb{Q}$? Ans. Yes? ?? ??
  $\mathbb{Q}(x_1, \ldots, x_n)^{A_n} = \mathbb{Q}(s_1, \ldots, s_n, \Delta)$; but ...

  Open problem Is $\mathbb{Q}(x_1, \ldots, x_n)^{A_n}$ rational over $\mathbb{Q}$? $(n \geq 6)$
- $\mathbb{Q}(x_1, \ldots, x_5)^{A_5}$ is rational over $\mathbb{Q}$ (Maeda, 1989).

Exercises (2/2): Noether's problem

- $\mathbb{Q}(x_1,x_2,x_3)^{A_3}=\mathbb{Q}(x_1,x_2,x_3)^{C_3}=\mathbb{Q}(t_1,t_2,t_3)$, **Q.** t_1,t_2,t_3?
  $(C_3: x_1\mapsto x_2\mapsto x_3\mapsto x_1)$
- **Ans.** $\mathbb{Q}(x_1,x_2,x_3)^{C_3}=\mathbb{Q}(t_1,t_2,t_3)$ where

$$t_1=x_1+x_2+x_3,$$
$$t_2=\frac{x_1x_2^2+x_2x_3^2+x_3x_1^2-3x_1x_2x_3}{x_1^2+x_2^2+x_3^2-x_1x_2-x_2x_3-x_3x_1},$$
$$t_3=\frac{x_1^2x_2+x_2^2x_3+x_3^2x_1-3x_1x_2x_3}{x_1^2+x_2^2+x_3^2-x_1x_2-x_2x_3-x_3x_1}.$$

- $\mathbb{Q}(x_1,x_2,\ldots,x_8)^{C_8}=\mathbb{Q}(t_1,t_2,\ldots,t_8)$, **Q.** $t_1,t_2,\ldots,t_8$?
  $(C_8: x_1\mapsto x_2\mapsto x_3\mapsto\cdots\mapsto x_8\mapsto x_1)$
- **Ans.** None: $\mathbb{Q}(x_1,x_2,\ldots,x_8)^{C_8}$ is not rational over $\mathbb{Q}$!

Today's talk (1/2)

Definition (quasi-monomial action)

Let K/k be a finite field extension and $G \leq \mathrm{Aut}_k(K(x_1,\ldots,x_n))$; finite where $K(x_1,\ldots,x_n)$ is the rational function field of n variables over K. The action of G on $K(x_1,\ldots,x_n)$ is called quasi-monomial if
(i) $\sigma(K) \subset K$ for any $\sigma \in G$;
(ii) $K^G = k$;
(iii) for any $\sigma \in G$, $$\sigma(x_j) = c_j(\sigma)\prod_{i=1}^{n} x_i^{a_{ij}}$$
where $c_j(\sigma) \in K^\times$, $1 \leq j \leq n$, $[a_{i,j}]_{1\leq i,j\leq n} \in GL_n(\mathbb{Z})$.

§1. $G \curvearrowright K$; trivial: monomial action & Noether's problem
§2. $G \curvearrowright K$; trivial and permutation: Noether's problem over $\mathbb{C}$
§3. (general) quasi-monomial actions (1-dim. and 2-dim. cases)
§4. $G = \mathrm{Gal}(K/k)$ and purely: rationality problem for algebraic tori

Today's talk (2/2)

§1. $G \curvearrowright K$; trivial: monomial action & Noether's problem

Hoshi-Kitayama-Yamasaki, J. Algebra **341** (2011) 45–108.

§2. $G \curvearrowright K$; trivial and permutation: Noether's problem over $\mathbb{C}$

Hoshi-Kang-Kunyavskii, Asian J. Math. **17** (2013) 689–714.

Chu-Hoshi-Hu-Kang, J. Algebra **442** (2015) 233–259.

Hoshi, J. Algebra **445** (2016) 394–432.

Hoshi-Kang-Yamasaki, J. Algebra **458** (2016) 120–133.

Hoshi-Kang-Yamasaki, J. Algebra **544** (2020) 262–301.

Hoshi-Kang-Yamasaki, Mem. AMS **283** (2023) no. 1403, 137 pp.

§3. (general) quasi-monomial actions (1-dim. and 2-dim. cases)

Hoshi-Kang-Kitayama, J. Algebra **403** (2014) 363–400.

§4. $G = \mathrm{Gal}(K/k)$ and purely: rationality problem for algebraic tori

Hoshi-Yamasaki, Mem. AMS **248** (2017) no. 1176, 215 pp.

Various rationalities: definitions

$k \subset L$; f.g. field extension, L is rational over k $\overset{\text{def}}{\Longleftrightarrow}$ $L \simeq k(x_1, \ldots, x_n)$.

Definition (stably rational)

L is called stably rational over k $\overset{\text{def}}{\Longleftrightarrow}$ $L(y_1, \ldots, y_m)$ is rational over k.

Definition (retract rational)

L is retract rational over k $\overset{\text{def}}{\Longleftrightarrow}$ $\exists k$-algebra $R \subset L$ such that
(i) L is the quotient field of R;
(ii) $\exists f \in k[x_1, \ldots, x_n]$ $\exists k$-algebra hom. $\varphi : R \to k[x_1, \ldots, x_n][1/f]$ and $\psi : k[x_1, \ldots, x_n][1/f] \to R$ satisfying $\psi \circ \varphi = 1_R$.

Definition (unirational)

L is unirational over k $\overset{\text{def}}{\Longleftrightarrow}$ $L \subset k(t_1, \ldots, t_n)$.

- Assume $L_1(x_1, \ldots, x_n) \simeq L_2(y_1, \ldots, y_m)$; stably isomorphic. If L_1 is retract rational over k, then so is L_2 over k.
- "rational" $\Longrightarrow$ "stably rational" $\Longrightarrow$ "retract rational" $\Longrightarrow$ "unirational"

"rational" $\Longrightarrow$ "stably rational" $\Longrightarrow$ "retract rational" $\Longrightarrow$ "unirational"

- The direction of the implication cannot be reversed.
- (Lüroth's problem) "unirational" $\Longrightarrow$ "rational" ? YES if trdeg$=1$
- (Castelnuovo, 1894)
  L is unirational over $\mathbb{C}$ and $\mathrm{trdeg}_{\mathbb{C}} L = 2 \Longrightarrow L$ is rational over $\mathbb{C}$.
- (Zariski, 1958) Let k be an alg. closed field and $k \subset L \subset k(x,y)$. If $k(x,y)$ is separable algebraic over L, then L is rational over k.
- (Zariski cancellation problem) $V_1 \times \mathbb{P}^n \approx V_2 \times \mathbb{P}^n \Longrightarrow V_1 \approx V_2$?
  In particular, "stably rational" $\Longrightarrow$ "rational"?
- (Beauville, Colliot-Thélène, Sansuc, Swinnerton-Dyer, 1985, Ann. Math.)
  $L = \mathbb{Q}(x,y,t)$ with $x^2+3y^2=t^3-2$ (Châtelet surface)
  $\Longrightarrow L$ is not rational but stably rational over $\mathbb{Q}$.
  Indeed, $L(y_1,y_2,y_3)$ is rational over $\mathbb{Q}$.
- $L(y_1,y_2)$ is rational over $\mathbb{Q}$ (Shepherd-Barron, 2002, Fano Conf.).
- $\mathbb{Q}(x_1,\ldots,x_{47})^{C_{47}}$ is not stably but retract rational over $\mathbb{Q}$.
- $\mathbb{Q}(x_1,\ldots,x_8)^{C_8}$ is not retract but unirational over $\mathbb{Q}$.

Châtelet surface as an invariant field

- (Beauville, Colliot-Thélène, Sansuc, Swinnerton-Dyer, 1985, Ann. Math.) $L = \mathbb{Q}(x, y, t)$ with $x^2 + 3y^2 = t^3 - 2$ (Châtelet surface) $\Longrightarrow L$ is not rational but stably rational over $\mathbb{Q}$.
- $L = \mathbb{Q}(x, y, t) = \mathbb{Q}(\sqrt{-3})(X, Y)^{\langle\sigma\rangle}$ where

$$\sigma : \sqrt{-3} \mapsto -\sqrt{-3}, X \mapsto X, Y \mapsto \frac{X^3 - 2}{Y}.$$

Indeed, we have

$$x = \frac{1}{2}\left(Y + \frac{X^3 - 2}{Y}\right),$$
$$y = \frac{1}{2\sqrt{-3}}\left(Y - \frac{X^3 - 2}{Y}\right),$$
$$t = X.$$

Retract rationality and generic extension

Theorem (Saltman, 1982, DeMeyer)

Let k be an infinite field and G be a finite group.
The following are equivalent:
(i) $k(x_g \mid g \in G)^G$ is retract rational over k.
(ii) There is a generic G-Galois extension over k;
(iii) There exists a generic G-polynomial over k.

- related to Inverse Galois Problem (IGP). (i) $\Longrightarrow$ IGP(G/k): true

Definition (generic polynomial)

A polynomial $f(t_1,\ldots,t_n;X) \in k(t_1,\ldots,t_n)[X]$ is generic for G over k if
(1) $\mathrm{Gal}(f/k(t_1,\ldots,t_n)) \simeq G$;
(2) $\forall L/M \supset k$ with $\mathrm{Gal}(L/M) \simeq G$,
$\exists a_1,\ldots,a_n \in M$ such that $L = \mathrm{Spl}(f(a_1,\ldots,a_n;X)/M)$.

- By Hilbert's irreducibility theorem, $\exists L/\mathbb{Q}$ such that $\mathrm{Gal}(L/\mathbb{Q}) \simeq G$.

§1. Monomial action & Noether's problem

Definition (monomial action) $G \curvearrowright K$; trivial, $k = K^G = K$

An action of G on $k(x_1, \ldots, x_n)$ is monomial $\overset{\text{def}}{\Longleftrightarrow}$

$$\sigma(x_j) = c_j(\sigma) \prod_{i=1}^{n} x_i^{a_{i,j}},\ 1 \leq j \leq n, \forall \sigma \in G$$

where $[a_{i,j}]_{1 \leq i,j \leq n} \in \mathrm{GL}_n(\mathbb{Z})$, $c_j(\sigma) \in k^\times := k \setminus \{0\}$.

If $c_j(\sigma) = 1$ for any $1 \leq j \leq n$ then σ is called purely monomial.

- Application to Noether's problem (permutation action)

Noether's problem (1/3) [$G = A$; abelian case]

- k; field, G; finite group
- $G \curvearrowright k$; trivial, $G \curvearrowright k(x_g \mid g \in G)$; permutation.
- $k(G) := k(x_g \mid g \in G)^G$; invariant field

Noether's problem (Emmy Noether, 1913)

Is $k(G)$ rational over k?, i.e. $k(G) \simeq k(t_1, \ldots, t_n)$?

- Is the quotient variety $\mathbb{P}^n/G$ rational over k?

- Assume $G = A$; abelian group.
- (Fisher, 1915) $\mathbb{C}(A)$ is rational over $\mathbb{C}$.
- (Masuda, 1955, 1968) $\mathbb{Q}(C_p)$ is rational over $\mathbb{Q}$ for $p \leq 11$.
- (Swan, 1969, Invent. Math.)
  $\mathbb{Q}(C_{47}), \mathbb{Q}(C_{113}), \mathbb{Q}(C_{233})$ are not rational over $\mathbb{Q}$.
- S. Endo and T. Miyata (1973), V.E. Voskresenskii (1973), ...
  e.g. $\mathbb{Q}(C_8)$ is not rational over $\mathbb{Q}$.
- (Lenstra, 1974, Invent. Math.)
  $k(A)$ is rational over k $\iff$ some condition;

Noether's problem (2/3) [$G = A$; abelian case]

- (Endo-Miyata, 1973) $\mathbb{Q}(C_{p^r})$ is rational over $\mathbb{Q}$
  $\iff \exists \alpha \in \mathbb{Z}[\zeta_{\varphi(p^r)}]$ such that $N_{\mathbb{Q}(\zeta_{\varphi(p^r)})/\mathbb{Q}}(\alpha) = \pm p$
- $h(\mathbb{Q}(\zeta_m)) = 1$ if $m < 23$
  $\Longrightarrow \mathbb{Q}(C_p)$ is rational over $\mathbb{Q}$ for $p \leq 43$ and $p = 61, 67, 71$.
- (Endo-Miyata, 1973) For $p = 47, 79, 113, 137, 167, \ldots,$
  $\mathbb{Q}(C_p)$ is not rational over $\mathbb{Q}$.
- However, for $p = 59, 83, 89, 97, 107, 163, \ldots$, unknown.
  Under the GRH, $\mathbb{Q}(C_p)$ is not rational for the above primes.
  But it was unknown for $p = 251, 347, 587, 2459, \ldots$
- For $p \leq 20000$, see speaker's paper (using PARI/GP):
  Proc. Japan Acad. Ser. A 91 (2015) 39-44.

Theorem (Plans, 2017, Proc. AMS)

$\mathbb{Q}(C_p)$ is rational over $\mathbb{Q} \iff p \leq 43$ or $p = 61, 67, 71$.

- Using lower bound of height, $\mathbb{Q}(C_p)$ is rational $\Rightarrow p < 173$.

Noether's problem (3/3) [G; non-abelian case]

Noether's problem (Emmy Noether, 1913)

Is $k(G)$ rational over k?, i.e. $k(G) \simeq k(t_1, \ldots, t_n)$?

- Assume G; non-abelian group.
- (Maeda, 1989) $k(A_5)$ is rational over k;
- (Rikuna, 2003; Plans, 2007) $k(GL_2(\mathbb{F}_3))$ and $k(SL_2(\mathbb{F}_3))$ is rational over k;
- (Serre, 2003) if 2-Sylow subgroup of $G \simeq C_{8m}$, then $\mathbb{Q}(G)$ is not rational over $\mathbb{Q}$; if 2-Sylow subgroup of $G \simeq Q_{16}$, then $\mathbb{Q}(G)$ is not rational over $\mathbb{Q}$; e.g. $G = Q_{16}, SL_2(\mathbb{F}_7), SL_2(\mathbb{F}_9)$,
  $SL_2(\mathbb{F}_q)$ with $q \equiv 7$ or $9 \pmod{16}$.

From Noether's problem to monomial actions (1/2)

- $k(G) := k(x_g \mid g \in G)^G$; invariant field

Noether's problem (Emmy Noether, 1913)

Is $k(G)$ rational over k?, i.e. $k(G) \simeq k(t_1, \ldots, t_n)$?

By Hilbert 90, we have:

No-name lemma (e.g. Miyata, 1971, Remark 3)

Let G act faithfully on k-vector space V, $W \subset V$ faithful $k[G]$-submodule. Then $K(V)^G = K(W)^G(t_1, \ldots, t_m)$.

Rationality problem: linear action

Let G act on finite-dimensional k-vector space V and $\rho : G \to GL(V)$ be a representation. Whether $k(V)^G$ is rational over k?

- the quotient variety V/G is rational over k?

From Noether's problem to monomial actions (2/2)

- For $\rho : G \to GL(V)$; monomial representation, i.e. matrix rep. has exactly one non-zero entry in each row and each column, G acts on $k(\mathbb{P}(V)) = k(\frac{w_1}{w_n}, \ldots, \frac{w_{n-1}}{w_n})$ by monomial action

By Hilbert 90, we have:

Lemma (e.g. Miyata, 1971, Lemma)

$k(V)^G = k(\mathbb{P}(V))^G(t)$.

- $V/G \approx \mathbb{P}(V)/G \times \mathbb{P}^1$ (birational)
- $k(\mathbb{P}(V))^G$ (monomial action) is rational over k
  $\Longrightarrow k(V)^G$ (linear action) is rational over k
  $\Longrightarrow k(G)$ (permutation action) is rational over k
  (Noether's problem has an affirmative answer)

Example: Noether's problem for $GL_2(\mathbb{F}_3)$ and $SL_2(\mathbb{F}_3)$

- $G = GL_2(\mathbb{F}_3) = \langle A, B, C, D \rangle \subset GL_4(\mathbb{Q})$, $|G| = 48$,
- $H = SL_2(\mathbb{F}_3) = \langle A, B, C \rangle \subset GL_4(\mathbb{Q})$, $|H| = 24$, where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, C = \begin{bmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

- G and H act on $k(V) = k(w_1, w_2, w_3, w_4)$ by

$$A : w_1 \mapsto -w_2 \mapsto -w_1 \mapsto w_2 \mapsto w_1,\ w_3 \mapsto -w_4 \mapsto -w_3 \mapsto w_4 \mapsto w_3,$$
$$B : w_1 \mapsto -w_3 \mapsto -w_1 \mapsto w_3 \mapsto w_1,\ w_2 \mapsto w_4 \mapsto -w_2 \mapsto -w_4 \mapsto w_2,$$
$$C : w_1 \mapsto -w_2 \mapsto w_3 \mapsto w_1,\ w_4 \mapsto w_4, \quad D : w_1 \mapsto w_1,\ w_2 \mapsto -w_2,\ w_3 \leftrightarrow w_4.$$

- $k(\mathbb{P}(V)) = k(x, y, z)$, $x = w_1/w_4$, $y = w_2/w_4$, $z = w_3/w_4$.
- G and H act on $k(x, y, z)$ as $G/Z(G) \simeq S_4$ and $H/Z(H) \simeq A_4$:

$$A : x \mapsto \frac{y}{z},\ y \mapsto \frac{-x}{z},\ z \mapsto \frac{-1}{z},\ B : x \mapsto \frac{-z}{y},\ y \mapsto \frac{-1}{y},\ z \mapsto \frac{x}{y},$$
$$C : x \mapsto y \mapsto z \mapsto x,\ D : x \mapsto \frac{x}{z},\ y \mapsto \frac{-y}{z},\ z \mapsto \frac{1}{z}.$$

- $k(\mathbb{P}(V))^G$: rational $\Longrightarrow k(V)^G$: rational $\Longrightarrow k(G)$: rational.

Monomial action (1/3) [3-dim. case]

Theorem (Hajja,1987) 2-dim. monomial action

$k(x_1, x_2)^G$ is rational over k.

Theorem (Hajja-Kang 1994, Hoshi-Rikuna 2008) 3-dim. purely monomial

$k(x_1, x_2, x_3)^G$ is rational over k.

Theorem (Prokhorov, 2010) 3-dim. monomial action over $k = \mathbb{C}$

$\mathbb{C}(x_1, x_2, x_3)^G$ is rational over $\mathbb{C}$.

However,
$\mathbb{Q}(x_1, x_2, x_3)^{\langle\sigma\rangle}$, $\sigma : x_1 \mapsto x_2 \mapsto x_3 \mapsto \frac{-1}{x_1 x_2 x_3}$ is not rational over $\mathbb{Q}$ (Hajja,1983).

Monomial action (2/3) [3-dim. case]

Theorem (Saltman, 2000) char $k \neq 2$

If $[k(\sqrt{a_1}, \sqrt{a_2}, \sqrt{a_3}) : k] = 8$, then $k(x_1, x_2, x_3)^{\langle \sigma \rangle}$,

$$\sigma : x_1 \mapsto \frac{a_1}{x_1}, x_2 \mapsto \frac{a_2}{x_2}, x_3 \mapsto \frac{a_3}{x_3}$$

is not retract rational over k (hence not rational over k).

Theorem (Kang, 2004)

$k(x_1, x_2, x_3)^{\langle \sigma \rangle}$, σ: $x_1 \mapsto x_2 \mapsto x_3 \mapsto \dfrac{c}{x_1 x_2 x_3} \mapsto x_1$, is rational over k
$\iff$ at least one of the following conditions is satisfied:
(i) char $k = 2$; (ii) $c \in k^2$; (iii) $-4c \in k^4$; (iv) $-1 \in k^2$.
If $k(x, y, z)^{\langle \sigma \rangle}$ is not rational over k, then it is not retract rational over k.

Recall that

- "rational" $\Longrightarrow$ "stably rational" $\Longrightarrow$ "retract rational" $\Longrightarrow$ "unirational"

Monomial action (3/3) [3-dim. case] (char $k \neq 2$)

Theorem (Yamasaki, 2012) 3-dim. monomial

$\exists$ 8 cases $G \leq GL_3(\mathbb{Z})$ s.t $k(x_1, x_2, x_3)^G$ is not retract rational over k.
Moreover, the necessary and sufficient conditions are given.

- Two of 8 cases are Saltman's and Kang's cases.
- $\exists G \leq GL_3(\mathbb{Z})$; 73 finite subgroups (up to conjugacy)

Theorem (Hoshi-Kitayama-Yamasaki, 2011) 3-dim. monomial

$k(x_1, x_2, x_3)^G$ is rational over k except for the 8 cases and $G = A_4$.
For $G = A_4$, if $[k(\sqrt{a}, \sqrt{-1}) : k] \leq 2$, then it is rational over k.

Corollary

$\exists L = k(\sqrt{a})$ such that $L(x_1, x_2, x_3)^G$ is rational over L.

- However, $\exists$4-dim. $\mathbb{C}(x_1, x_2, x_3, x_4)^{C_2 \times C_2}$ is not retract rational.

§2. Noether's problem over $\mathbb{C}$ (1/3)

Let G be a p-group. $\mathbb{C}(G) := \mathbb{C}(x_g \mid g \in G)^G$.

- (Fisher, 1915) $\mathbb{C}(A)$ is rational over $\mathbb{C}$ if A; finite abelian group.
- (Saltman, 1984, Invent. Math.)
  For $\forall p$; prime, $\exists$ meta-abelian p-group G of order p^9
  such that $\mathbb{C}(G)$ is not retract rational over $\mathbb{C}$.
- (Bogomolov, 1988)
  For $\forall p$; prime, $\exists$ p-group G of order p^6
  such that $\mathbb{C}(G)$ is not retract rational over $\mathbb{C}$.

Indeed they showed $\mathrm{Br}_{\mathrm{nr}}(\mathbb{C}(G)/\mathbb{C}) \neq 0$; unramified Brauer group

- rational $\Longrightarrow$ stably rational $\Longrightarrow$ retract rational $\Longrightarrow$ $\mathrm{Br}_{\mathrm{nr}}(\mathbb{C}(G)) = 0$.

not rational $\Leftarrow$ not stably rational $\Leftarrow$ not retract rational $\Leftarrow$ $\mathrm{Br}_{\mathrm{nr}}(\mathbb{C}(G)) \neq 0$.

- $k(G)$; retract rational $\Longrightarrow$ IGP for (k, G) has an affirmative answer.

Unramified Brauer group

Definition (Unramified Brauer group) Saltman (1984)

Let $k \subset K$ be an extension of fields.
$\mathrm{Br}_{\mathrm{nr}}(K/k) = \cap_R \mathrm{Image}\{\mathrm{Br}(R) \to \mathrm{Br}(K)\}$ where $\mathrm{Br}(R) \to \mathrm{Br}(K)$ is the natural map of Brauer groups and R runs over all the DVR such that $k \subset R \subset K$ and $K = \mathrm{Quot}(R)$.

- If K is retract rational over k, then $\mathrm{Br}(k) \xrightarrow{\sim} \mathrm{Br}_{\mathrm{nr}}(K/k)$.
  In particular, if K is retract rational over $\mathbb{C}$, then $\mathrm{Br}_{\mathrm{nr}}(K/\mathbb{C}) = 0$.
- For a smooth projective variety X over $\mathbb{C}$ with function field K, $\mathrm{Br}_{\mathrm{nr}}(K/\mathbb{C}) \simeq H^3(X, \mathbb{Z})_{\mathrm{tors}}$ which is given by Artin-Mumford (1972).

Theorem (Bogomolov 1988, Saltman 1990) $\mathrm{Br}_{\mathrm{nr}}(\mathbb{C}(G)/\mathbb{C}) \simeq B_0(G)$

Let G be a finite group. Then $\mathrm{Br}_{\mathrm{nr}}(\mathbb{C}(G)/\mathbb{C})$ is isomorphic to

$$B_0(G) = \bigcap_{A} \mathrm{Ker}\{\mathrm{res} : H^2(G, \mathbb{Q}/\mathbb{Z}) \to H^2(A, \mathbb{Q}/\mathbb{Z})\}$$

where A runs over all the bicyclic subgroups of G
(bicyclic $=$ cyclic or direct product of two cyclic groups).

- $\mathbb{C}(G)$: "retract rational" $\Longrightarrow B_0(G) = 0$.
  $B_0(G) \neq 0 \Longrightarrow \mathbb{C}(G)$: not (retract) rational over k.
- $B_0(G) \leq H^2(G, \mu) \simeq H_2(G, \mathbb{Z})$; Schur multiplier.
- $B_0(G)$ is called Bogomolov multiplier.

Noether's problem over $\mathbb{C}$ (2/3)

- (Chu-Kang, 2001) G is p-group ($|G| \leq p^4$) $\Longrightarrow \mathbb{C}(G)$ is rational.

Theorem (Moravec, 2012, Amer. J. Math.)

Assume $|G| = 3^5 = 243$. $B_0(G) \neq 0 \iff G = G(243, i)$, $28 \leq i \leq 30$.
In particular, $\exists 3$ groups G such that $\mathbb{C}(G)$ is not retract rational over $\mathbb{C}$.

- $\exists G$: 67 groups such that $|G| = 243$.

Theorem (Hoshi-Kang-Kunyavskii, 2013, Asian J. Math.)

Assume $|G| = p^5$ where p is odd prime.
$B_0(G) \neq 0 \iff G$ belongs to the isoclinism family Φ_{10}.
In particular, $\exists$ gcd$(4, p-1)+$ gcd$(3, p-1) + 1$ (resp. $\exists 3$) groups G of order p^5 $(p \geq 5)$ (resp. $p = 3$) s.t. $\mathbb{C}(G)$ is not retract rational over $\mathbb{C}$.

- $\exists 2p + 61+$ gcd$(4, p-1) + 2$ gcd$(3, p-1)$ groups such that $|G| = p^5 (p \geq 5)$. $(\exists \Phi_1, \ldots, \Phi_{10})$

From the proof (1/3)

Definition (isoclinic)

p-groups G_1 and G_2 are isoclinic $\overset{\text{def}}{\Longleftrightarrow}$
isom. $\theta : G_1/Z(G_1) \overset{\sim}{\to} G_2/Z(G_2)$, $\phi : [G_1, G_1] \overset{\sim}{\to} [G_2, G_2]$ such that

$$\begin{array}{ccc} G_1/Z(G_1) \times G_1/Z(G_1) & \xrightarrow[\simeq]{(\theta,\theta)} & G_2/Z(G_2) \times G_2/Z(G_2) \\ {\scriptstyle [\ ,\ ]}\downarrow & \circlearrowleft & \downarrow{\scriptstyle [\ ,\ ]} \\ [G_1, G_1] & \xrightarrow[\simeq]{\phi} & [G_2, G_2] \end{array}$$

Invariants

- lower central series
- \# of conj. classes with precisely p^i members
- \# of irr. complex rep. of G of degree p^i

From the proof (2/3)

- $|G| = p^4 (p > 2)$. $\exists 15$ groups (Φ_1, Φ_2, Φ_3)
- $|G| = 2^4 = 16$. $\exists 14$ groups (Φ_1, Φ_2, Φ_3)
- $|G| = p^5 (p > 3)$. $\exists 2p + 61 + (4, p-1) + 2 \times (3, p-1)$ groups $(\Phi_1, \dots, \Phi_{10})$

	Φ_1	Φ_2	Φ_3	Φ_4	Φ_5	Φ_6	Φ_7	Φ_8
#	7	15	13	$p+8$	2	$p+7$	5	1
$(p=3)$						7		

	Φ_9	Φ_{10}
#	$2 + (3, p-1)$	$1 + (4, p-1) + (3, p-1)$
$(p=3)$		3

From the proof (3/3)

[HKK, Question 1.11] (2013) (arXiv:1202.5812)

Let G_1 and G_2 be isoclinic p-groups.
Is it true that the fields $k(G_1)$ and $k(G_2)$ are stably isomorphic, or, at least, that $B_0(G_1)$ is isomorphic to $B_0(G_2)$?

Theorem (Moravec, 2013) (arXiv:1203.2422)

G_1 and G_2 are isoclinic $\Longrightarrow B_0(G_1) \simeq B_0(G_2)$.

Theorem (Bogomolov-Böhning, 2013) (arXiv: 1204.4747)

G_1 and G_2 are isoclinic $\Longrightarrow \mathbb{C}(G_1)$ and $\mathbb{C}(G_2)$ are stably isomorphic.

Proof (Φ_{10}): $B_0(G) \neq 0$

Lemma 1. $N \lhd G$.

(i) $\mathrm{tr}\colon H^1(N,\mathbb{Q}/\mathbb{Z})^G \to H^2(G/N,\mathbb{Q}/\mathbb{Z})$ is not surjective
where tr is the transgression map.
(ii) $AN/N \leq G/N$ is cyclic ($\forall A \leq G$; bicyclic).
$\Longrightarrow B_0(G) \neq 0$.

Proof. Consider the Hochschild-Serre 5-term exact sequence

$$0 \to H^1(G/N,\mathbb{Q}/\mathbb{Z}) \to H^1(G,\mathbb{Q}/\mathbb{Z}) \to H^1(N,\mathbb{Q}/\mathbb{Z})^G$$
$$\xrightarrow{\mathrm{tr}} H^2(G/N,\mathbb{Q}/\mathbb{Z}) \xrightarrow{\psi} H^2(G,\mathbb{Q}/\mathbb{Z})$$

where ψ is an inflation map.

(i) $\Longrightarrow$ ψ is not zero-map $\Longrightarrow \mathrm{Image}(\psi) \neq 0$.

We will show that $\mathrm{Image}(\psi) \subset B_0(G)$ by (ii).

It suffices to show that $H^2(G/N,\mathbb{Q}/\mathbb{Z}) \xrightarrow{\psi} H^2(G,\mathbb{Q}/\mathbb{Z}) \xrightarrow{\mathrm{res}} H^2(A,\mathbb{Q}/\mathbb{Z})$ is zero-map ($\forall A \leq G$: bicyclic).

Consider the following commutative diagram:

$$\begin{array}{ccccc} H^2(G/N,\mathbb{Q}/\mathbb{Z}) & \xrightarrow{\psi} & H^2(G,\mathbb{Q}/\mathbb{Z}) & \xrightarrow{\mathrm{res}} & H^2(A,\mathbb{Q}/\mathbb{Z}) \\ \psi_0 \downarrow & & & & \uparrow \psi_1 \\ H^2(AN/N,\mathbb{Q}/\mathbb{Z}) & & \overset{\widetilde{\psi}}{\simeq} & & H^2(A/A\cap N,\mathbb{Q}/\mathbb{Z}) \end{array}$$

where ψ_0 is the restriction map, ψ_1 is the inflation map, $\widetilde{\psi}$ is the natural isomorphism.

(ii) $\Longrightarrow AN/N \simeq C_m \Longrightarrow H^2(C_m,\mathbb{Q}/\mathbb{Z}) = 0$.

$\Longrightarrow \psi_0$ is zero-map.

$\Longrightarrow$ res $\circ\, \psi\colon H^2(G/N,\mathbb{Q}/\mathbb{Z}) \to H^2(A,\mathbb{Q}/\mathbb{Z})$ is zero-map.

$\therefore \mathrm{Image}(\psi) \subset B_0(G)$

$\mathrm{Image}(\psi) \subset B_0(G)$ and $\mathrm{Image}(\psi) \neq 0$ (by (i)) $\Longrightarrow B_0(G) \neq 0$. □

Proof (Φ_6): $B_0(G) = 0$

- $G = \Phi_6(211)a = \langle f_1, f_2, f_0, h_1, f_2 \rangle, f_1^p = h_1, f_2^p = h_2,$
  $Z(G) = \langle h_1, h_2 \rangle, f_0^p = h_1^p = h_2^p = 1$
  $[f_1, f_2] = f_0, [f_0, f_1] = h_1, [f_0, f_2] = h_2$

Hochschild-Serre 5-term exact sequence:

$$0 \to H^1(G/N, \mathbb{Q}/\mathbb{Z}) \to H^1(G, \mathbb{Q}/\mathbb{Z}) \to H^1(N, \mathbb{Q}/\mathbb{Z})^G \xrightarrow{\mathrm{tr}} H^2(G/N, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\psi} H^2(G, \mathbb{Q}/\mathbb{Z})$$

Proof (Φ_6): $B_0(G) = 0$

- $G = \Phi_6(211)a = \langle f_1, f_2, f_0, h_1, f_2\rangle, f_1^p = h_1, f_2^p = h_2,$
  $Z(G) = \langle h_1, h_2\rangle, f_0^p = h_1^p = h_2^p = 1$
  $[f_1, f_2] = f_0, [f_0, f_1] = h_1, [f_0, f_2] = h_2$

Hochschild-Serre 5-term exact sequence:

$$0 \to H^1(G/N, \mathbb{Q}/\mathbb{Z}) \to H^1(G, \mathbb{Q}/\mathbb{Z}) \to H^1(N, \mathbb{Q}/\mathbb{Z})^G \xrightarrow{\mathrm{tr}} H^2(G/N, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\psi} H^2(G, \mathbb{Q}/\mathbb{Z})$$

$$\downarrow$$

$$\mathrm{Ker}\{H^2(G, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\mathrm{res}} H^2(N, \mathbb{Q}/\mathbb{Z})\} =: H^2(G, \mathbb{Q}/\mathbb{Z})_1$$

$$\downarrow$$

$$H^1(G/N, H^1(N, \mathbb{Q}/\mathbb{Z}))$$

$$\lambda \downarrow$$

$$H^3(G/N, \mathbb{Q}/\mathbb{Z})$$

- Explicit formula for λ is given
  by Dekimpe-Hartl-Wauters (2012)
- $N := \langle f_1, f_0, h_1, h_2\rangle \Longrightarrow G/N \simeq C_p \Longrightarrow H^2(G/N, \mathbb{Q}/\mathbb{Z}) = 0$
- $B_0(G) \subset H^2(G, \mathbb{Q}/\mathbb{Z})_1$
- We should show $H^2(G, \mathbb{Q}/\mathbb{Z})_1 = 0$ ($\iff$ λ: injective)

Noether's problem over $\mathbb{C}$ (3/3)

Theorem (Hoshi-Kang-Kunyavskii, 2013, Asian J. Math.)

Assume $|G| = p^5$ where p is odd prime.
$B_0(G) \neq 0 \iff G$ belongs to the isoclinism family Φ_{10}.

Theorem (Chu-Hoshi-Hu-Kang, 2015, J. Algebra) $|G| = 3^5 = 243$

If $B_0(G) = 0$, then $\mathbb{C}(G)$ is rational over $\mathbb{C}$ except for Φ_7.

- Non-rationality of Φ_7 is detected by $H^3_{\mathrm{nr}}(\mathbb{C}(G), \mathbb{Q}/\mathbb{Z})$ (later).
- Φ_5 and Φ_7 are very similar: $C = 1$ (Φ_5), $C = \omega$ (Φ_7).

$\mathbb{C}(G)$ is stably isomorphic to $\mathbb{C}(z_1, z_2, z_3, z_4, z_5, z_6, z_7, z_8, z_9)^{\langle f_1, f_2 \rangle}$

$$f_1 : z_1 \mapsto z_2, z_2 \mapsto \frac{1}{z_1 z_2}, z_3 \mapsto z_4, z_4 \mapsto \frac{1}{z_3 z_4},$$
$$z_5 \mapsto \frac{z_5}{z_1^2 z_3}, z_6 \mapsto \frac{z_1 z_6}{z_3}, z_7 \mapsto z_8, z_8 \mapsto \frac{1}{z_7 z_8}, z_9 \mapsto \frac{z_4 z_9}{z_1},$$
$$f_2 : z_1 \mapsto z_3, z_2 \mapsto z_4, z_3 \mapsto \frac{1}{z_1 z_3}, z_4 \mapsto \frac{1}{z_2 z_4},$$
$$z_5 \mapsto z_6, z_6 \mapsto \frac{1}{z_5 z_6}, z_7 \mapsto C \frac{z_4 z_7}{z_3}, z_8 \mapsto C \frac{z_8}{z_3 z_4^2}, z_9 \mapsto \frac{z_4 z_9}{z_1}.$$

Unramified Brauer group: purely monomial case (1/3)

Theorem (Hoshi-Kang-Yamasaki, 2023, Mem AMS) purely monomial

Let G be a finite group and M be a faithful G-lattice.
(1) If $\mathrm{rank}_{\mathbb{Z}} M \leq 3$, then $\mathrm{Br}_{\mathrm{nr}}(\mathbb{C}(M)^G) = 0$.
(2) When $\mathrm{rank}_{\mathbb{Z}} M = 4$, $\exists$ 5 M's with $\mathrm{Br}_{\mathrm{nr}}(\mathbb{C}(M)^G) \neq 0$.
(3) When $\mathrm{rank}_{\mathbb{Z}} M = 5$, $\exists$ 46 M's with $\mathrm{Br}_{\mathrm{nr}}(\mathbb{C}(M)^G) \neq 0$.
(4) When $\mathrm{rank}_{\mathbb{Z}} M = 6$, $\exists$ 1073 M's with $\mathrm{Br}_{\mathrm{nr}}(\mathbb{C}(M)^G) \neq 0$.

rank	# of G-lattices	# of unramified Brauer groups $\neq 0$
1	2	0
2	13	0
3	73	0
4	710	5
5	6079	46
6	85308	1073

- If M is of rank ≤ 6 and $\mathrm{Br}_{\mathrm{nr}}(\mathbb{C}(M^G)) \neq 0$, then G is solvable and non-abelian, and $\mathrm{Br}_{\mathrm{nr}}(\mathbb{C}(M)^G) \simeq \mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/3\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

Unramified Brauer group: purely monomial case (2/3)

Theorem (Hoshi-Kang-Yamasaki, 2023, Mem. AMS) $G = A_6$: simple

Embed $A_6 \simeq PSL_2(\mathbb{F}_9) \hookrightarrow S_{10}$. Let $N = \oplus_{1\leq i\leq 10}\mathbb{Z}\cdot x_i$ be the S_{10}-lattice defined by $\sigma \cdot x_i = x_{\sigma(i)}$ for any $\sigma \in S_{10}$; it becomes an A_6-lattice by restricting the action of S_{10} to A_6. Define $M = N/(\mathbb{Z}\cdot\sum_{i=1}^{10} x_i)$ with $\mathrm{rank}_{\mathbb{Z}} M = 9$. $\exists A_6$-lattices $M = M_1$, $M_2, \ldots, M_6$ which are $\mathbb{Q}$-conjugate but not $\mathbb{Z}$-conjugate to each other; in fact, all these M_i form a single $\mathbb{Q}$-class, but this $\mathbb{Q}$-class consists of six $\mathbb{Z}$-classes. Then we have

$$H^2_{\mathrm{nr}}(A_6, M_1) \simeq H^2_{\mathrm{nr}}(A_6, M_3) \simeq \mathbb{Z}/2\mathbb{Z},\ H^2_{\mathrm{nr}}(A_6, M_i) = 0 \text{ for } i = 2, 4, 5, 6.$$

In particular, $\mathbb{C}(M_1)^{A_6}$ and $\mathbb{C}(M_3)^{A_6}$ are not retract rational over $\mathbb{C}$. Furthermore, M_1 and M_3 may be distinguished by Tate cohomologies:

$$H^1(A_6, M_1) = 0, \qquad \widehat{H}^{-1}(A_6, M_1) = \mathbb{Z}/10\mathbb{Z},$$
$$H^1(A_6, M_3) = \mathbb{Z}/5\mathbb{Z}, \qquad \widehat{H}^{-1}(A_6, M_3) = \mathbb{Z}/2\mathbb{Z}.$$

Unramified Brauer group: purely monomial case (1/3)

By using a result of Saltman (1987, J. Algebra, Corollary 3.3), as a corollary of Theorem above, we can get:

Theorem (Hoshi-Kang-Yamasaki, 2023, Mem. AMS) $G = A_6$: simple

Let $N_1 \simeq (C_{10})^9$ and $N_3 \simeq (C_2)^8 \times C_{10}$. Then, for $i = 1, 3$, $\mathrm{Br}_u(\mathbb{C}(N_i \rtimes A_6)) \simeq \mathbb{Z}/2\mathbb{Z}$ and Noether's problem for $N_i \rtimes A_6$ over $\mathbb{C}$ has a negative answer. Moreover, $\mathbb{C}(N_i \rtimes A_6)$ $(i = 1, 3)$ is not retract (stably) rational over $\mathbb{C}$.

- Noether's problem for A_6 over $\mathbb{Q}$ (resp. over $\mathbb{C}$) is still unsolved!

Unramified cohomology (1/4)

Colliot-Thélène and Ojanguren (1989) generalized the notion of the unramified Brauer group $\mathrm{Br}_{\mathrm{nr}}(K/\mathbb{C})$ to the unramified cohomology $H^i_{\mathrm{nr}}(K/\mathbb{C}, \mu_n^{\otimes j})$ of degree $i \geq 1$:

Definition (Colliot-Thélène and Ojanguren, 1989, Invent. Math.)

Let $K/\mathbb{C}$ be a function field, that is finitely generated as a field over $\mathbb{C}$. The unramified cohomology group $H^i_{\mathrm{nr}}(K/\mathbb{C}, \mu_n^{\otimes j})$ of K over $\mathbb{C}$ of degree $i \geq 1$ is defined to be

$$H^i_{\mathrm{nr}}(K/\mathbb{C}, \mu_n^{\otimes j}) = \bigcap_R \mathrm{Ker}\{r_R : H^i(K, \mu_n^{\otimes j}) \to H^{i-1}(\Bbbk_R, \mu_n^{\otimes (j-1)})\}$$

where R runs over all the DVR of rank one such that $\mathbb{C} \subset R \subset K$ and $K = \mathrm{Quot}(R)$ and r_R is the residue map.

- Note that ${}_n\mathrm{Br}_{\mathrm{nr}}(K/\mathbb{C}) \simeq H^2_{\mathrm{nr}}(K/\mathbb{C}, \mu_n)$.

Proposition (Colliot-Thélène and Ojanguren, 1989)

If K and L are stably $\mathbb{C}$-isomorphic, then
$H^i_{\rm nr}(K/\mathbb{C},\mu_n^{\otimes j}) \xrightarrow{\sim} H^i_{\rm nr}(L/\mathbb{C},\mu_n^{\otimes j})$.
In particular, K is stably rational over $\mathbb{C}$, then $H^i_{\rm nr}(K/\mathbb{C},\mu_n^{\otimes j})=0$.

- Moreover, if K is retract rational over $\mathbb{C}$, then $H^i_{\rm nr}(K/\mathbb{C},\mu_n^{\otimes j})=0$.
- CTO (1989) $\exists$ $\mathbb{C}$-unirational field K with $\mathrm{trdeg}_{\mathbb{C}}K=6$ s.t. $H^3_{\rm nr}(K/\mathbb{C},\mu_2^{\otimes 3})\neq 0$ and $\mathrm{Br}_{\rm nr}(K/\mathbb{C})=0$.
- Peyre (1993) gave a sufficient condition for $H^i_{\rm nr}(K/\mathbb{C},\mu_p^{\otimes i})\neq 0$:
- $\exists K$ s.t. $H^3_{\rm nr}(K/\mathbb{C},\mu_p^{\otimes 3})\neq 0$ and $\mathrm{Br}_{\rm nr}(K/\mathbb{C})=0$;
- $\exists K$ s.t. $H^4_{\rm nr}(K/\mathbb{C},\mu_2^{\otimes 4})\neq 0$ and $\mathrm{Br}_{\rm nr}(K/\mathbb{C})=0$.

Unramified cohomology (2/4)

Take the direct limit with respect to n:

$$H^i(K/\mathbb{C}, \mathbb{Q}/\mathbb{Z}(j)) = \varinjlim_{n} H^i(K/\mathbb{C}, \mu_n^{\otimes j})$$

and we also define the unramified cohomology group

$$\begin{aligned} &H^i_{\mathrm{nr}}(K/\mathbb{C}, \mathbb{Q}/\mathbb{Z}(j)) \\ &= \bigcap_R \mathrm{Ker}\{r_R : H^i(K/\mathbb{C}, \mathbb{Q}/\mathbb{Z}(j)) \to H^{i-1}(\Bbbk_R, \mathbb{Q}/\mathbb{Z}(j-1))\}. \end{aligned}$$

Then we have $\mathrm{Br}_{\mathrm{nr}}(K/\mathbb{C}) \simeq H^2_{\mathrm{nr}}(K/\mathbb{C}, \mathbb{Q}/\mathbb{Z}(1))$.

- The case $K = \mathbb{C}(G)$:

Theorem (Peyre, 2008, Invent. Math.) p: odd prime

$\exists$ p-group G of order p^{12} such that $B_0(G) = 0$ and $H^3_{\mathrm{nr}}(\mathbb{C}(G), \mathbb{Q}/\mathbb{Z}) \neq 0$.
In particular, $\mathbb{C}(G)$ is not (retract, stably) rational over $\mathbb{C}$.

- Asok (2013) generalized Peyre's argument (1993):

Theorem (Asok, 2013, Compos. Math.)

(1) For any $n > 0$, $\exists$ a smooth projective complex variety X that is $\mathbb{C}$-unirational, for which $H^i_{\rm nr}(\mathbb{C}(X), \mu_2^{\otimes i}) = 0$ for each $i < n$, yet $H^n_{\rm nr}(\mathbb{C}(X), \mu_2^{\otimes n}) \neq 0$, and so
X is not $\mathbb{A}^1$-connected, nor (retract, stably) rational over $\mathbb{C}$;
(2) For any prime l and any $n \geq 2$, $\exists$ a smooth projective rationally connected complex variety Y such that $H^n_{\rm nr}(\mathbb{C}(Y), \mu_l^{\otimes n}) \neq 0$.
In particular, Y is not $\mathbb{A}^1$-connected, nor (retract, stably) rational over $\mathbb{C}$.

- Namely, the triviality of the unramified Brauer group or the unramified cohomology of higher degree is just a necessary condition of $\mathbb{C}$-rationality of fields.
- It is interesting to consider an analog of above Theorem for quotient varieties V/G, e.g. $\mathbb{C}(V_{\rm reg}/G) = \mathbb{C}(G)$.

Unramified cohomology (3/4)

Theorem (Peyre, 2008, Invent. Math.) p: odd prime

$\exists$ p-group G of order p^{12} such that $B_0(G) = 0$ and $H^3_{\mathrm{nr}}(\mathbb{C}(G), \mathbb{Q}/\mathbb{Z}) \neq 0$.
In particular, $\mathbb{C}(G)$ is not (retract, stably) rational over $\mathbb{C}$.

Using Peyre's method, we improve this result:

Theorem (Hoshi-Kang-Yamasaki, 2016, J. Algebra) p: odd prime

$\exists$ p-group G of order p^9 such that $B_0(G) = 0$ and $H^3_{\mathrm{nr}}(\mathbb{C}(G), \mathbb{Q}/\mathbb{Z}) \neq 0$.
In particular, $\mathbb{C}(G)$ is not (retract, stably) rational over $\mathbb{C}$.

On the other hand, CT and Voisin proved: ($\leftrightarrow$ integral Hodge conjecture)

Theorem (Colliot-Thélène and Voisin, 2012, Duke Math. J.)

Let X be a smooth projective rationally connected complex variety. Then
$H^3_{\mathrm{nr}}(X, \mathbb{Q}/\mathbb{Z}) \simeq \mathrm{Hdg}^4(X, \mathbb{Z})/\mathrm{Hdg}^4(X, \mathbb{Z})_{\mathrm{alg}}$.

Unramified cohomology (4/4)

- Using Peyre's formula [Peyre, 2008, Invent. Math.], we get:

Theorem (Hoshi-Kang-Yamasaki, 2020, J. Algebra) $|G| = 3^5$

$H^3_{\rm nr}(\mathbb{C}(G), \mathbb{Q}/\mathbb{Z}) \neq 0 \iff G$ belongs to the isoclinism family Φ_7.
In particular, $\mathbb{C}(G)$ is not rational over $\mathbb{C} \iff G$ belongs to Φ_7, Φ_{10}.

$\|G\| = 3^5$	Φ_1	Φ_2	Φ_3	Φ_4	Φ_5	Φ_6	Φ_7	Φ_8	Φ_9	Φ_{10}
$H^2_{\rm nr}(\mathbb{C}(G), \mathbb{Q}/\mathbb{Z})$	0	0	0	0	0	0	0	0	0	$\mathbb{Z}/3\mathbb{Z}$
$H^3_{\rm nr}(\mathbb{C}(G), \mathbb{Q}/\mathbb{Z})$	0	0	0	0	0	0	$\mathbb{Z}/3\mathbb{Z}$	0	0	0

Theorem (Hoshi-Kang-Yamasaki, 2020, J. Algebra) $|G| = 5^5$ or 7^5

$H^3_{\rm nr}(\mathbb{C}(G), \mathbb{Q}/\mathbb{Z}) \neq 0 \iff G$ belongs to Φ_6, Φ_7 or Φ_{10}.

$\|G\| = p^5$ $(p = 5, 7)$	Φ_1	Φ_2	Φ_3	Φ_4	Φ_5	Φ_6	Φ_7	Φ_8	Φ_9	Φ_{10}
$H^2_{\rm nr}(\mathbb{C}(G), \mathbb{Q}/\mathbb{Z})$	0	0	0	0	0	0	0	0	0	$\mathbb{Z}/p\mathbb{Z}$
$H^3_{\rm nr}(\mathbb{C}(G), \mathbb{Q}/\mathbb{Z})$	0	0	0	0	0	$\mathbb{Z}/p\mathbb{Z}$	$\mathbb{Z}/p\mathbb{Z}$	0	0	$\mathbb{Z}/p\mathbb{Z}$

Noether's problem over $\mathbb{C}$ for 2-groups

- (Chu-Kang, 2001) G is p-group ($|G| \leq p^4$) $\Longrightarrow \mathbb{C}(G)$ is rational.
- (Chu-Hu-Kang-Prokhorov, 2008)
  $|G| = 32 = 2^5 \Longrightarrow \mathbb{C}(G)$ is rational.
- $\exists 267$ groups G of order $64 = 2^6$ which are classified into 27 isoclinism families $\Phi_1, \ldots, \Phi_{27}$.

Theorem (Chu-Hu-Kang-Kunyavskii, 2010) $|G| = 64 = 2^6$

(1) $B_0(G) \neq 0 \iff G$ belongs to Φ_{16}. ($\exists 9$ such G's)
Moreover, if $B_0(G) \neq 0$, then $B_0(G) \simeq C_2$.
(2) If $B_0(G) = 0$, then $\mathbb{C}(G)$ is rational except for Φ_{13}. ($\exists 5$ such G's)

- ([CHKK10], [HY14]) ($B_0(G) = 0$, but rationality unknown)
  If G belongs to Φ_{13}, then $\mathbb{C}(G)$ is stably $\mathbb{C}$-isomorphic to $L_{\mathbb{C}}^{(0)}$.
- ([CHKK10], [HKK14]) ($B_0(G) \simeq C_2$, not retract rational)
  If G belongs to Φ_{16}, then $\mathbb{C}(G)$ is stably $\mathbb{C}$-isomorphic to $L_{\mathbb{C}}^{(1)}$.

- ([CHKK10], [HY14]) ($B_0(G)=0$, but rationality unknown)
  If G belongs to Φ_{13}, then $\mathbb{C}(G)$ is stably $\mathbb{C}$-isomorphic to $L_{\mathbb{C}}^{(0)}$.
- ([CHKK10], [HKK14]) ($B_0(G)\simeq C_2$, not retract rational)
  If G belongs to Φ_{16}, then $\mathbb{C}(G)$ is stably $\mathbb{C}$-isomorphic to $L_{\mathbb{C}}^{(1)}$.

Definition (The fields $L_{\mathbb{C}}^{(0)}$ and $L_{\mathbb{C}}^{(1)}$)

(i) The field $L_{\mathbb{C}}^{(0)}$ is defined to be $\mathbb{C}(X_1,X_2,X_3,X_4,X_5,X_6)^H$ where $H=\langle\sigma_1,\sigma_2\rangle\simeq C_2\times C_2$ act on $\mathbb{C}(X_1,X_2,X_3,X_4,X_5,X_6)$ by

$$\sigma_1: X_1\mapsto X_3,\ X_2\mapsto \frac{1}{X_1X_2X_3},\ X_3\mapsto X_1,\ X_4\mapsto X_6,\ X_5\mapsto \frac{1}{X_4X_5X_6},\ X_6\mapsto X_4,$$

$$\sigma_2: X_1\mapsto X_2,\ X_2\mapsto X_1,\ X_3\mapsto \frac{1}{X_1X_2X_3},\ X_4\mapsto X_5,\ X_5\mapsto X_4,\ X_6\mapsto \frac{1}{X_4X_5X_6}.$$

(ii) The field $L_{\mathbb{C}}^{(1)}$ is defined to be $\mathbb{C}(X_1,X_2,X_3,X_4)^{\langle\tau\rangle}$ where $\langle\tau\rangle\simeq C_2$ acts on $\mathbb{C}(X_1,X_2,X_3,X_4)$ by

$$\tau: X_1\mapsto -X_1,\ X_2\mapsto \frac{X_4}{X_2},\ X_3\mapsto \frac{(X_4-1)(X_4-X_1^2)}{X_3},\ X_4\mapsto X_4.$$

- ([CHKK10], [HY14]) ($B_0(G)=0$, but rationality unknown)
  If G belongs to Φ_{13}, then $\mathbb{C}(G)$ is stably $\mathbb{C}$-isomorphic to $L_{\mathbb{C}}^{(0)}$.
- ([CHKK10], [HKK14]) ($B_0(G)\simeq C_2$, not retract rational)
  If G belongs to Φ_{16}, then $\mathbb{C}(G)$ is stably $\mathbb{C}$-isomorphic to $L_{\mathbb{C}}^{(1)}$.

- $L_{\mathbb{C}}^{(0)}=\mathbb{C}(z_1,z_2,z_3,z_4,u_4,u_5,u_6)$ where
$$(z_1^2-a)(z_4^2-d)=(z_2^2-b)(z_3^2-c),$$
$$a=u_4(u_4-1), b=u_4-1, c=u_4(u_4-u_6^2), d=u_5^2(u_4-u_6^2).$$
- $L_{\mathbb{C}}^{(0)}=\mathbb{C}(u,v,t,w_3,w_4,w_5,w_6)$ where
$$u^2-tv^2=-\left(w_4^2(w_5^2-1)t^2+(w_3^2-w_3^2w_5^2+1)t-w_5^2\right)$$
$$\cdot\left(w_4^2w_6^2t^2-(w_4^2+w_3^2w_6^2)t+w_3^2-w_6^2+1\right).$$
- $L_{\mathbb{C}}^{(0)}=\mathbb{C}(m_0,\ldots,m_6)$ where
$$m_0^2=(4m_3+m_3m_4^2+m_4^2)(m_3-m_5^2+1)$$
$$\cdot(m_1^2m_3+m_6^2-1)(4m_3+m_1^2m_2^2m_3+m_2^2m_6^2).$$
- $L_{\mathbb{C}}^{(1)}=\mathbb{C}(u,v,t,w_3,w_4)$ where
$$u^2-tv^2=(tw_4^2-w_3^2+1)(t+tw_4^2-w_3^2).$$

- $\exists 2328$ groups G of order $128 = 2^7$ which are classified into 115 isoclinism families $\Phi_1, \ldots, \Phi_{115}$.

Theorem (Moravec, 2012, Amer. J. Math.) $|G| = 128 = 2^7$

$B_0(G) \neq 0$ if and only if G belongs to the isoclinism family Φ_{16}, Φ_{30}, Φ_{31}, Φ_{37}, Φ_{39}, Φ_{43}, Φ_{58}, Φ_{60}, Φ_{80}, Φ_{106} or Φ_{114}. If $B_0(G) \neq 0$, then

$$B_0(G) \simeq \begin{cases} C_2 & (\Phi_{16}, \Phi_{31}, \Phi_{37}, \Phi_{39}, \Phi_{43}, \Phi_{58}, \Phi_{60}, \Phi_{80}, \Phi_{106}, \Phi_{114}) \\ C_2 \times C_2 & (\Phi_{30}). \end{cases}$$

In particular, $\mathbb{C}(G)$ is not (retract, stably) rational over $\mathbb{C}$.

	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	Total
Family	Φ_{16}	Φ_{31}	Φ_{37}	Φ_{39}	Φ_{43}	Φ_{58}	Φ_{60}	Φ_{80}	Φ_{106}	Φ_{114}	Φ_{30}	
$B_0(G)$					C_2						$C_2 \times C_2$	
$\#$ G's	48	55	18	6	26	20	10	9	2	2	34	220

- Q. Birational classification of $\mathbb{C}(G)$?
  In particular, what happens when $B_0(G) \neq 0$?
  How many $\mathbb{C}(G)$'s exist up to stably $\mathbb{C}$-isomorphism?

Theorem (Hoshi, 2016, J. Algebra) $|G| = 128 = 2^7$

Assume that $B_0(G) \neq 0$.
Then $\mathbb{C}(G)$ and $L_{\mathbb{C}}^{(m)}$ are stably $\mathbb{C}$-isomorphic where

$$m = \begin{cases} 1 & \text{if } G \text{ belongs to } \Phi_{16}, \Phi_{31}, \Phi_{37}, \Phi_{39}, \Phi_{43}, \Phi_{58}, \Phi_{60} \text{ or } \Phi_{80}, \\ 2 & \text{if } G \text{ belongs to } \Phi_{106} \text{ or } \Phi_{114}, \\ 3 & \text{if } G \text{ belongs to } \Phi_{30}. \end{cases}$$

In particular, $\mathrm{Br}_{\mathrm{nr}}(L_{\mathbb{C}}^{(1)}) \simeq \mathrm{Br}_{\mathrm{nr}}(L_{\mathbb{C}}^{(2)}) \simeq C_2$ and $\mathrm{Br}_{\mathrm{nr}}(L_{\mathbb{C}}^{(3)}) \simeq C_2 \times C_2$ and hence $L_{\mathbb{C}}^{(1)}$, $L_{\mathbb{C}}^{(2)}$ and $L_{\mathbb{C}}^{(3)}$ are not (retract, stably) rational over $\mathbb{C}$.

- $L_{\mathbb{C}}^{(1)} \nsim L_{\mathbb{C}}^{(3)}$, $L_{\mathbb{C}}^{(2)} \nsim L_{\mathbb{C}}^{(3)}$ (not stably $\mathbb{C}$-isomorphic) because their unramified Brauer groups are not isomorphic.
- However, we do not know whether $L_{\mathbb{C}}^{(1)} \sim L_{\mathbb{C}}^{(2)}$.
- If not, evaluate the higher unramified cohomologies $H_{\mathrm{nr}}^i (i \geq 3)$? (Peyre's formula can not work for $|G| = 2^m$)

Definition (The fields $L_{\mathbb{C}}^{(2)}$ and $L_{\mathbb{C}}^{(3)}$)

(i) The field $L_{\mathbb{C}}^{(2)}$ is defined to be $\mathbb{C}(X_1,X_2,X_3,X_4,X_5,X_6)^{\langle\rho\rangle}$ where $\langle\rho\rangle\simeq C_4$ acts on $\mathbb{C}(X_1,X_2,X_3,X_4,X_5,X_6)$ by

$$\rho : X_1\mapsto X_2, X_2\mapsto -X_1, X_3\mapsto X_4, X_4\mapsto X_3,$$
$$X_5\mapsto X_6, X_6\mapsto \frac{(X_1^2X_2^2-1)(X_1^2X_3^2+X_2^2-X_3^2-1)}{X_5}.$$

(ii) The field $L_{\mathbb{C}}^{(3)}$ is defined to be $\mathbb{C}(X_1,X_2,X_3,X_4,X_5,X_6,X_7)^{\langle\lambda_1,\lambda_2\rangle}$ where $\langle\lambda_1,\lambda_2\rangle\simeq C_2\times C_2$ acts on $\mathbb{C}(X_1,X_2,X_3,X_4,X_5,X_6,X_7)$ by

$$\lambda_1 : X_1\mapsto X_1, X_2\mapsto \frac{X_1}{X_2}, X_3\mapsto \frac{1}{X_1X_3}, X_4\mapsto \frac{X_2X_4}{X_1X_3},$$
$$X_5\mapsto -\frac{X_1X_6^2-1}{X_5}, X_6\mapsto -X_6, X_7\mapsto X_7,$$
$$\lambda_2 : X_1\mapsto \frac{1}{X_1}, X_2\mapsto X_3, X_3\mapsto X_2, X_4\mapsto \frac{(X_1X_6^2-1)(X_1X_7^2-1)}{X_4},$$
$$X_5\mapsto -X_5, X_6\mapsto -X_1X_6, X_7\mapsto -X_1X_7.$$

§3. (general) quasi-monomial actions

Notion of "quasi-monomial" actions
is defined in Hoshi-Kang-Kitayama [HKK14], J. Algebra (2014).

Theorem ([HKK14]) 1-dim. quasi-monomial actions

(1) purely quasi-monomial $\Longrightarrow K(x)^G$ is rational over k.
(2) $K(x)^G$ is rational over k excpet for the case: $\exists N \leq G$ such that
(i) $G/N = \langle \sigma \rangle \simeq C_2$;
(ii) $K(x)^N = k(\alpha)(y)$, $\alpha^2 = a \in K^\times$, $\sigma(\alpha) = -\alpha$ (if char k $\neq 2$),
$\alpha^2 + \alpha = a \in K$, $\sigma(\alpha) = \alpha + 1$ (if char $k = 2$);
(iii) $\sigma \cdot y = b/y$ for some $b \in k^\times$.
For the exceptional case, $K(x)^G = k(\alpha)(y)^{G/N}$ is rational over $k \iff$
Hilbert symbol $(a, b)_k = 0$ (if char $k \neq 2$), $[a, b)_k = 0$ (if char $k = 2$).
Moreover, $K(x)^G$ is not rational over $k \Longrightarrow$ not unirational over k.

Theorem ([HKK14]) 2-dim. purely quasi-monomial actions

$N=\{\sigma\in G\mid \sigma(x)=x,\ \sigma(y)=y\}$, $H=\{\sigma\in G\mid \sigma(\alpha)=\alpha(\forall\alpha\in K)\}$.
$K(x,y)^G$ is rational over k except for:
(1) char $k\neq 2$ and (2) (i) $(G/N,HN/N)\simeq(C_4,C_2)$ or (ii) (D_4,C_2).
For the exceptional case, we have $k(x,y)=k(u,v)$:
(i) $(G/N,HN/N)\simeq(C_4,C_2)$,
$K^N=k(\sqrt{a})$, $G/N=\langle\sigma\rangle\simeq C_4$, $\sigma:\sqrt{a}\mapsto-\sqrt{a}$, $u\mapsto\frac{1}{u}$, $v\mapsto-\frac{1}{v}$;
(ii) $(G/N,HN/N)\simeq(D_4,C_2)$;
$K^N=k(\sqrt{a},\sqrt{b})$, $G/N=\langle\sigma,\tau\rangle\simeq D_4$, $\sigma:\sqrt{a}\mapsto-\sqrt{a}$, $\sqrt{b}\mapsto\sqrt{b}$,
$u\mapsto\frac{1}{u}$, $v\mapsto-\frac{1}{v}$, $\tau:\sqrt{a}\mapsto\sqrt{a}$, $\sqrt{b}\mapsto-\sqrt{b}$, $u\mapsto u$, $v\mapsto -v$.
Case (i), $K(x,y)^G$ is rational over $k\iff$ Hilbert symbol $(a,-1)_k=0$.
Case (ii), $K(x,y)^G$ is rational over $k\iff$ Hilbert symbol $(a,-b)_k=0$.
Moreover, $K(x,y)^G$ is not rational over $k\Longrightarrow$
$\mathrm{Br}(k)\neq 0$ and $K(x,y)^G$ is not unirational over k.

Galois-theoretic interpretation:
(i) rational over $k\iff k(\sqrt{a})$ may be embedded into C_4-ext. of k.
(ii) rational over $k\iff k(\sqrt{a},\sqrt{b})$ may be embedded into D_4-ext. of k.

Application to purely monomial actions (1/2)

Theorem ([HKK14]), 4-dim. purely monomial

Let M be a G-lattice with $\mathrm{rank}_{\mathbb{Z}} M = 4$ and G act on $k(M)$ by purely monomial k-automorphisms. If M is decomposable, i.e. $M = M_1 \oplus M_2$ as $\mathbb{Z}[G]$-modules where $1 \leq \mathrm{rank}_{\mathbb{Z}} M_1 \leq 3$, then $k(M)^G$ is rational over k.

- When $\mathrm{rank}_{\mathbb{Z}} M_1 = 1, \mathrm{rank}_{\mathbb{Z}} M_2 = 3$, it is easy to see $k(M)^G$ is rational.
- When $\mathrm{rank}_{\mathbb{Z}} M_1 = \mathrm{rank}_{\mathbb{Z}} M_2 = 2$, we may apply Theorem of 2-dim. to $k(M) = k(x_1, x_2, y_1, y_2) = k(x_1, x_2)(y_1, y_2) = K(y_1, y_2)$.

Theorem ([HKK14]) char $k \neq 2$

Let $C_2 = \langle \tau \rangle$ act on the rational function field $k(x_1, x_2, x_3, x_4)$ by k-automorphisms defined as

$$\tau : x_1 \mapsto -x_1,\ x_2 \mapsto \tfrac{x_4}{x_2},\ x_3 \mapsto \tfrac{(x_4-1)(x_4-x_1^2)}{x_3},\ x_4 \mapsto x_4.$$

Then $k(x_1, x_2, x_3, x_4)^{C_2}$ is not retract rational over k.
In particular, it is not rational over k.

Theorem A ([HKK14]) char $k \neq 2$, 5-dim. purely monomial

Let $D_4 = \langle \rho, \tau \rangle$ act on the rational function field $k(x_1, x_2, x_3, x_4, x_5)$ by k-automorphisms defined as

$$\rho : x_1 \mapsto x_2,\ x_2 \mapsto x_1,\ x_3 \mapsto \tfrac{1}{x_1 x_2 x_3},\ x_4 \mapsto x_5,\ x_5 \mapsto \tfrac{1}{x_4},$$
$$\tau : x_1 \mapsto x_3,\ x_2 \mapsto \tfrac{1}{x_1 x_2 x_3},\ x_3 \mapsto x_1,\ x_4 \mapsto x_5,\ x_5 \mapsto x_4.$$

Then $k(x_1, x_2, x_3, x_4, x_5)^{D_4}$ is not retract rational over k.
In particular, it is not rational over k.

Application to purely monomial actions (2/2)

Theorem ([HKK14]), 5-dim. purely monomial

Let M be a G-lattice and G act on $k(M)$ by purely monomial k-automorphisms. Assume that
(i) $M = M_1 \oplus M_2$ as $\mathbb{Z}[G]$-modules where $\mathrm{rank}_{\mathbb{Z}} M_1 = 3$ and $\mathrm{rank}_{\mathbb{Z}} M_2 = 2$,
(ii) either M_1 or M_2 is a faithful G-lattice.
Then $k(M)^G$ is rational over k except for the case as in Theorem A.

- we may apply Theorem of 2-dim. to $k(M) = k(x_1, x_2, x_3, y_1, y_2) = k(x_1, x_2, x_3)(y_1, y_2) = K(y_1, y_2)$.

More recent results

- 3-dim. purely quasi-monomial actions (Hoshi-Kitayama, 2020, Kyoto J. Math.)

§4. Rationality problem for algebraic tori (2-dim., 3-dim.)

$G \simeq \mathrm{Gal}(K/k) \curvearrowright K(x_1,\ldots,x_n)$: purely quasi-monomial, $K(x_1,\ldots,x_n)^G$ may be regarded as the function field of algebraic torus T over k which splits over K ($T \otimes_k K \simeq \mathbb{G}_m^n$).

- T is unirational over k, i.e. $K(x_1,\ldots,x_n)^G \subset k(t_1,\ldots,t_n)$.
- $\exists 13$ $\mathbb{Z}$-coujugacy subgroups $G \leq \mathrm{GL}_2(\mathbb{Z})$.

Theorem (Voskresenskii, 1967) 2-dim. algebraic tori T

T is rational over k.

- $\exists 73$ $\mathbb{Z}$-coujugacy subgroups $G \leq \mathrm{GL}_3(\mathbb{Z})$.

Theorem (Kunyavskii, 1990) 3-dim. algebraic tori T

(i) T is rational over k $\iff$ T is stably rational over k $\iff$ T is retract rational over k $\iff$ $\exists G$: 58 groups;
(ii) T is not rational over k $\iff$ T is not stably rational over k $\iff$ T is not retract rational over k $\iff$ $\exists G$: 15 groups.

Rationality of algebraic tori (4-dim., 5-dim.)

- $\exists 710$ $\mathbb{Z}$-coujugacy subgroups $G \leq \mathrm{GL}_4(\mathbb{Z})$.

Theorem (Hoshi-Yamasaki, 2017, Mem. AMS) 4-dim. alg. tori T

(i) T is stably rational over k $\Longleftrightarrow$ $\exists G$: 487 groups;
(ii) T is not stably but retract rational over k $\Longleftrightarrow$ $\exists G$: 7 groups;
(iii) T is not retract rational over k $\Longleftrightarrow$ $\exists G$: 216 groups.

- $\exists 6079$ $\mathbb{Z}$-coujugacy subgroups $G \leq \mathrm{GL}_5(\mathbb{Z})$.

Theorem (Hoshi-Yamasaki, 2017, Mem. AMS) 5-dim. alg. tori T

(i) T is stably rational over k $\Longleftrightarrow$ $\exists G$: 3051 groups;
(ii) T is not stably but retract rational over k $\Longleftrightarrow$ $\exists G$: 25 groups;
(iii) T is not retract rational over k $\Longleftrightarrow$ $\exists G$: 3003 groups.

- (Voskresenskii's conjecture) any stably rational torus is rational.
- $\exists 85308$ $\mathbb{Z}$-coujugacy subgroups $G \leq \mathrm{GL}_6(\mathbb{Z})$!

Proof: Flabby (Flasque) resolution (1/2)

- The function field of n-dim. $T \overset{\text{identified}}{\longleftrightarrow} L(M)^G$, $G \leq \mathrm{GL}(n,\mathbb{Z})$
- M: G-lattice, i.e. f.g. $\mathbb{Z}$-free $\mathbb{Z}[G]$-module.

Definition

(i) M is permutation $\overset{\text{def}}{\Longleftrightarrow}$ $M \simeq \oplus_{1\leq i\leq m}\mathbb{Z}[G/H_i]$.

(ii) M is stably permutation $\overset{\text{def}}{\Longleftrightarrow}$ $M \oplus \exists P \simeq P'$, P, P': permutation.

(iii) M is invertible $\overset{\text{def}}{\Longleftrightarrow}$ $M \oplus \exists M' \simeq P$: permutation.

(iv) M is coflabby $\overset{\text{def}}{\Longleftrightarrow}$ $H^1(H,M) = 0$ $(\forall H \leq G)$.

(v) M is flabby $\overset{\text{def}}{\Longleftrightarrow}$ $\widehat{H}^{-1}(H,M) = 0$ $(\forall H \leq G)$. ($\widehat{H}$: Tate cohomology)

- "permutation"
  $\Longrightarrow$ "stably permutation"
  $\Longrightarrow$ "invertible"
  $\Longrightarrow$ "flabby and coflabby".

Proof: Flabby (Flasque) resolution (2/2)

Commutative monoid $\mathcal{M}$

$M_1 \sim M_2 \overset{\text{def}}{\iff} M_1 \oplus P_1 \simeq M_2 \oplus P_2$ ($\exists P_1, \exists P_2$: permutation).
$\Longrightarrow$ commutative monoid $\mathcal{M}$: $[M_1] + [M_2] := [M_1 \oplus M_2]$, $0 = [P]$.

Theorem (Endo-Miyata, 1974, Colliot-Thélène-Sansuc, 1977)

$\exists P$: permutation, $\exists F$: flabby such that

$$0 \to M \to P \to F \to 0\text{: flabby resolution of } M.$$

$[M]^{fl} := [F]$, $[M]^{fl}$ is invertible $\overset{\text{def}}{\iff} [M]^{fl} = [E]$ ($\exists E$: invertible).

Theorem (Endo-Miyata, 1973, Voskresenskii, 1974, Saltman, 1984)

(EM73) $[M]^{fl} = 0 \iff L(M)^G$ is stably rational over k.
(Vos74) $[M]^{fl} = [M']^{fl} \iff L(M)^G(x_1, \ldots, x_m) \simeq L(M')^G(y_1, \ldots, y_n)$.
(Sal84) $[M]^{fl}$ is invertible $\iff L(M)^G$ is retract rational over k.

Our contribution

- We give a procedure to compute a flabby resolution of M, in particular $[M]^{fl} = [F]$, effectively (with smaller rank after base change) by computer software GAP.
- The function `IsFlabby` (resp. `IsCoflabby`) may determine whether M is flabby (resp. coflabby).
- The function `IsInvertibleF` may determine whether $[M]^{fl} = [F]$ is invertible ($\leftrightarrow$ whether $L(M)^G$ (resp. T) is retract rational).
- We provide some functions for checking a possibility of isomorphism

$$\left(\bigoplus_{i=1}^{r} a_i\,\mathbb{Z}[G/H_i]\right) \oplus a_{r+1}F \simeq \bigoplus_{i=1}^{r} b_i'\,\mathbb{Z}[G/H_i] \qquad (*)$$

by computing some invariants (e.g. trace, $\widehat{Z}^0$, $\widehat{H}^0$) of both sides.

- [HY17, Example 10.7]. $G \simeq S_5 \leq \mathrm{GL}(5,\mathbb{Z})$ with number $(5,946,4)$ $\Longrightarrow \mathrm{rank}(F) = 17$ and $\mathrm{rank}(*) = 88$ holds $\Longrightarrow [F] = 0 \Longrightarrow L(M)^G$ (resp. T) is stably rational over k.

Application

Corollary ($[F]=[M]^{fl}$: invertible case, $G\simeq S_5, F_{20}$)

$\exists T$, T'; 4-dim. not stably rational algebraic tori over k such that $T\not\approx T'$ (birational) and $T\times T'$: 8-dim. stably rational over k.
$\because -[M]^{fl}=[M']^{fl}\neq 0$.

Prop. ([HY17], Krull-Schmidt fails for permutation D_6-lattices)

$\{1\}$, $C_2^{(1)}$, $C_2^{(2)}$, $C_2^{(3)}$, C_3, C_2^2, C_6, $S_3^{(1)}$, $S_3^{(2)}$, D_6: conj. subgroups of D_6.

$$\mathbb{Z}[D_6]\oplus\mathbb{Z}[D_6/C_2^2]^{\oplus 2}\oplus\mathbb{Z}[D_6/C_6]\oplus\mathbb{Z}[D_6/S_3^{(1)}]\oplus\mathbb{Z}[D_6/S_3^{(2)}]$$
$$\simeq\mathbb{Z}[D_6/C_2^{(1)}]\oplus\mathbb{Z}[D_6/C_2^{(2)}]\oplus\mathbb{Z}[D_6/C_2^{(3)}]\oplus\mathbb{Z}[D_6/C_3]\oplus\mathbb{Z}^{\oplus 2}.$$

- D_6 is the smallest example exhibiting the failure of K-S:

Theorem (Dress, 1973)

Krull-Schmidt holds for permutation G-lattices $\iff$ $G/O_p(G)$ is cyclic where $O_p(G)$ is the maximal normal p-subgroup of G.

Krull-Schmidt and Direct sum cancelation

Theorem (Hindman-Klingler-Odenthal, 1998) Assume $G \neq D_8$

Krull-Schmidt holds for G-lattices $\iff$ (i) $G = C_p$ ($p \leq 19$; prime), (ii) $G = C_n$ ($n = 1, 4, 8, 9$), (iii) $G = V_4$ or (iv) $G = D_4$.

Theorem (Endo-Hironaka, 1979)

Direct sum cancellation holds, i.e. $M_1 \oplus N \simeq M_2 \oplus N \Longrightarrow M_1 \simeq M_2$, $\Longrightarrow G$ is abelian, dihedral, A_4, S_4 or A_5 (*).

- via projective class group (see Swan (1988) Corollary 1.3, Section 7).
- Except for (*) $\Longrightarrow$ Direct sum cancelation fails $\Longrightarrow$ K-S fails

Theorem ([HY17]) $G \leq \mathrm{GL}(n, \mathbb{Z})$ (up to conjugacy)

(i) $n \leq 4 \Longrightarrow$ K-S holds.
(ii) $n = 5$. K-S fails $\iff$ 11 groups G (among 6079 groups).
(iii) $n = 6$. K-S fails $\iff$ 131 groups G (among 85308 groups).

Special case: $T = R^{(1)}_{K/k}(\mathbb{G}_m)$; norm one tori (1/5)

- Rationality problem for $T = R^{(1)}_{K/k}(\mathbb{G}_m)$ is investigated by S. Endo, Colliot-Thélène and Sansuc, W. Hürlimann, L. Le Bruyn, A. Cortella and B. Kunyavskii, N. Lemire and M. Lorenz, M. Florence, etc.

Theorem (Endo-Miyata, 1974), (Saltman, 1984)

Let K/k be a finite Galois field extension and $G = \mathrm{Gal}(K/k)$.
(i) T is retract k-rational $\iff$ all the Sylow subgroups of G are cyclic;
(ii) T is stably k-rational $\iff$ G is a cyclic group, or a direct product of a cyclic group of order m and a group $\langle \sigma, \tau \,|\, \sigma^n = \tau^{2^d} = 1, \tau\sigma\tau^{-1} = \sigma^{-1} \rangle$, where $d, m \geq 1, n \geq 3, m, n$: odd, and $(m, n) = 1$.

Theorem (Endo, 2011)

Let K/k be a finite non-Galois, separable field extension and L/k be the Galois closure of K/k. Assume that the Galois group of L/k is nilpotent. Then the norm one torus $T = R^{(1)}_{K/k}(\mathbb{G}_m)$ is not retract k-rational.

Special case: $T = R^{(1)}_{K/k}(\mathbb{G}_m)$; norm one tori (2/5)

- Let K/k be a finite non-Galois, separable field extension
- Let L/k be the Galois closure of K/k.
- Let $G = \mathrm{Gal}(L/k)$ and $H = \mathrm{Gal}(L/K) \leq G$.

Theorem (Endo, 2011)

Assume that all the Sylow subgroups of G are cyclic.
Then T is retract k-rational.
$T = R^{(1)}_{K/k}(\mathbb{G}_m)$ is stably k-rational $\iff$ $G = D_n$, n odd $(n \geq 3)$ or $C_m \times D_n$, m, n odd $(m, n \geq 3)$, $(m, n) = 1$, $H \leq D_n$ with $|H| = 2$.

Special case: $T = R^{(1)}_{K/k}(\mathbb{G}_m)$; norm one tori (3/5)

Theorem (Endo, 2011) $\dim T = n - 1$

Assume that $\mathrm{Gal}(L/k) = S_n$, $n \geq 3$, and $\mathrm{Gal}(L/K) = S_{n-1}$ is the stabilizer of one of the letters in S_n.
(i) $R^{(1)}_{K/k}(\mathbb{G}_m)$ is retract k-rational $\iff$ n is a prime;
(ii) $R^{(1)}_{K/k}(\mathbb{G}_m)$ is (stably) k-rational $\iff$ $n = 3$.

Theorem (Endo, 2011) $\dim T = n - 1$

Assume that $\mathrm{Gal}(L/k) = A_n$, $n \geq 4$, and $\mathrm{Gal}(L/K) = A_{n-1}$ is the stabilizer of one of the letters in A_n.
(i) $R^{(1)}_{K/k}(\mathbb{G}_m)$ is retract k-rational $\iff$ n is a prime;
(ii) $\exists t \in \mathbb{N}$ s.t. $[R^{(1)}_{K/k}(\mathbb{G}_m)]^{(t)}$ is stably k-rational $\iff$ $n = 5$.

- $[R^{(1)}_{K/k}(\mathbb{G}_m)]^{(t)}$: the product of t copies of $R^{(1)}_{K/k}(\mathbb{G}_m)$.

Special case: $T = R^{(1)}_{K/k}(\mathbb{G}_m)$; norm one tori (4/5)

Theorem ([HY17], Rationality for $R^{(1)}_{K/k}(\mathbb{G}_m)$ (dim. 4, $[K:k]=5$))

Let K/k be a separable field extension of degree 5 and L/k be the Galois closure of K/k. Assume that $G = \mathrm{Gal}(L/k)$ is a transitive subgroup of S_5 and $H = \mathrm{Gal}(L/K)$ is the stabilizer of one of the letters in G. Then the rationality of $R^{(1)}_{K/k}(\mathbb{G}_m)$ is given by

G		$L(M) = L(x_1, x_2, x_3, x_4)^G$
$5T1$	C_5	stably k-rational
$5T2$	D_5	stably k-rational
$5T3$	F_{20}	not stably but retract k-rational
$5T4$	A_5	stably k-rational
$5T5$	S_5	not stably but retract k-rational

- This theorem is already known except for the case of A_5 (Endo).
- Stably k-rationality for the case A_5 is asked by S. Endo (2011).

Special case: $T = R^{(1)}_{K/k}(\mathbb{G}_m)$; norm one tori (5/5)

Corollary of (Endo, 2011) and [HY17]

Assume that $\mathrm{Gal}(L/k) = A_n$, $n \geq 4$, and $\mathrm{Gal}(L/K) = A_{n-1}$ is the stabilizer of one of the letters in A_n. Then
$R^{(1)}_{K/k}(\mathbb{G}_m)$ is stably k-rational $\iff$ $n = 5$.

More recent results on stably/retract k-rational classification for T

- $G \leq S_n$ $(n \leq 10)$ and $G \neq 9T27 \simeq PSL_2(\mathbb{F}_8)$,
  $G \leq S_p$ and $G \neq PSL_2(\mathbb{F}_{2^e})$ $(p = 2^e + 1 \geq 17$; Fermat prime$)$
  (Hoshi-Yamasaki, 2021, Israel J. Math.)
- $G \leq S_n$ $(n = 12, 14, 15)$ $(n = 2^e)$
  (Hoshi-Hasegawa-Yamasaki, 2020, Math. Comp.)

$Ш(T)$ and Hasse norm principle over number fields k

- (Hoshi-Kanai-Yamasaki, 2022, Math. Comp., 2023, J. Number Theory, 2024, J. Algebra, and arXiv:2210.09119)