Birational classification for algebraic tori (joint work with Aiichi Yamasaki)

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January 8, 2026

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2. Birational classification for algebraic k-tori T

   [HY] A. Hoshi, A. Yamasaki,
   Birational classification for algebraic tori, 210 pages,
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§1. Rationality problem for algebraic tori T (1/3)

- k: a base field which is NOT algebraically closed! (TODAY)
- T: algebraic k-torus, i.e. k-form of a split torus;
  an algebraic group over k (group k-scheme) with $T \times_k \overline{k} \simeq (\mathbb{G}_{m,\overline{k}})^n$.

Rationality problem for algebraic tori

Whether T is k-rational?, i.e. $T \approx \mathbb{P}^n$? (birationally k-equivalent)

Let $R^{(1)}_{K/k}(\mathbb{G}_m)$ be the norm one torus of K/k, i.e. the kernel of the norm map $N_{K/k} : R_{K/k}(\mathbb{G}_m) \to \mathbb{G}_m$ where $R_{K/k}$ is the Weil restriction:

$$1 \longrightarrow R^{(1)}_{K/k}(\mathbb{G}_m) \longrightarrow R_{K/k}(\mathbb{G}_m) \xrightarrow{N_{K/k}} \mathbb{G}_m \longrightarrow 1.$$

$$\dim \qquad n-1 \qquad\qquad n \qquad\qquad 1$$

- $\exists 2$ algebraic k-tori T with $\dim(T) = 1$;
  the trivial torus $\mathbb{G}_m$ and $R^{(1)}_{K/k}(\mathbb{G}_m)$ with $[K : k] = 2$, are k-rational.

Rationality problem for algebraic tori T (2/3)

- $\exists 13$ algebraic k-tori T with $\dim(T) = 2$.

Theorem (Voskresenskii 1967) 2-dim. algebraic tori T

T is k-rational.

- $\exists 73$ algebraic k-tori T with $\dim(T) = 3$.

Theorem (Kunyavskii 1990) 3-dim. algebraic tori T

(i) $\exists 58$ algebraic k-tori T which are k-rational;
(ii) $\exists 15$ algebraic k-tori T which are not k-rational.

- What happens in higher dimensions?

Algebraic k-tori T and G-lattices

- T: algebraic k-torus
  $\Longrightarrow \exists$ finite Galois extension L/k such that $T \times_k L \simeq (\mathbb{G}_{m,L})^n$.
- $G = \mathrm{Gal}(L/k)$ where L is the minimal splitting field.

Category of algebraic k-tori which split$/L \overset{\text{duality}}{\longleftrightarrow}$ Category of G-lattices
(i.e. finitely generated $\mathbb{Z}$-free $\mathbb{Z}[G]$-module)

- $T \mapsto$ the character group $\widehat{T} = \mathrm{Hom}(T, \mathbb{G}_m)$: G-lattice.
- $T = \mathrm{Spec}(L[M]^G)$ which splits$/L$ with $\widehat{T} \simeq M \hookleftarrow M$: G-lattice
- Tori of dimension $n \overset{1:1}{\longleftrightarrow}$ elements of the set $H^1(\mathcal{G}, \mathrm{GL}(n, \mathbb{Z}))$
  where $\mathcal{G} = \mathrm{Gal}(\overline{k}/k)$ since $\mathrm{Aut}(\mathbb{G}_m^n) = \mathrm{GL}(n, \mathbb{Z})$.
- k-torus T of dimension n is determined uniquely by the integral representation $h : \mathcal{G} \to \mathrm{GL}(n, \mathbb{Z})$ up to conjugacy, and the group $h(\mathcal{G})$ is a finite subgroup of $\mathrm{GL}(n, \mathbb{Z})$.
- The function field of $T \overset{\text{identified}}{\longleftrightarrow} L(M)^G$: invariant field.

Rationality problem for algebraic tori T (3/3)

- L/k: Galois extension with $G = \mathrm{Gal}(L/k)$.
- $M = \bigoplus_{1 \leq i \leq n} \mathbb{Z} \cdot u_i$: G-lattice with a $\mathbb{Z}$-basis $\{u_1, \ldots, u_n\}$.
- G acts on $L(x_1, \ldots, x_n)$ by

$$x_i^\sigma = \prod_{j=1}^{n} x_j^{a_{i,j}}, \quad 1 \leq i \leq n$$

  for any $\sigma \in G$, when $u_i^\sigma = \sum_{j=1}^{n} a_{i,j} u_j$, $a_{i,j} \in \mathbb{Z}$.
- $L(M) := L(x_1, \ldots, x_n)$ with this action of G.
- The function field of algebraic k-torus $T \overset{\text{identified}}{\longleftrightarrow} L(M)^G$

Rationality problem for algebraic tori T (2nd form)

Whether $L(M)^G$ is k-rational?
($=$ purely transcendental over k?; $L(M)^G = k(\exists t_1, \ldots, \exists t_n)$?)

Some definitions.

- K/k: a finite generated field extension.

Definition (stably rational)

K is called stably k-rational if $K(y_1, \ldots, y_m)$ is k-rational.

Definition (retract rational)

K is retract k-rational if $\exists k$-algebra (domain) $R \subset K$ such that
(i) K is the quotient field of R;
(ii) $\exists f \in k[x_1, \ldots, x_n]$ $\exists k$-algebra hom. $\varphi : R \to k[x_1, \ldots, x_n][1/f]$ and $\psi : k[x_1, \ldots, x_n][1/f] \to R$ satisfying $\psi \circ \varphi = 1_R$.

Definition (unirational)

K is k-unirational if $K \subset k(x_1, \ldots, x_n)$.

- k-rational $\Rightarrow$ stably k-rational $\Rightarrow$ retract k-rational $\Rightarrow$ k-unirational.
- $L(M)^G$ (resp. T) is always k-unirational.

Rationality problem for algebraic tori T (2-dim., 3-dim.)

- The function field of n-dim. $T \overset{\text{identified}}{\longleftrightarrow} L(M)^G$, $G \leq \mathrm{GL}(n,\mathbb{Z})$
- $\exists 13$ $\mathbb{Z}$-coujugacy subgroups $G \leq \mathrm{GL}(2,\mathbb{Z})$ ($\exists 13$ 2-dim. algebraic k-tori T).

Theorem (Voskresenskii 1967) 2-dim. algebraic tori T (restated)

T is k-rational.

- $\exists 73$ $\mathbb{Z}$-coujugacy subgroups $G \leq \mathrm{GL}(3,\mathbb{Z})$ ($\exists 73$ 3-dim. algebraic k-tori T).

Theorem (Kunyavskii 1990) 3-dim. algebraic tori T (precise form)

(i) T is k-rational $\iff$ T is stably k-rational $\iff$ T is retract k-rational $\iff$ $\exists G$: 58 groups;
(ii) T is not k-rational $\iff$ T is not stably k-rational $\iff$ T is not retract k-rational $\iff$ $\exists G$: 15 groups.

Rationality problem for algebraic tori T (4-dim.)

- The function field of n-dim. $T \stackrel{\text{identified}}{\longleftrightarrow} L(M)^G$, $G \leq \mathrm{GL}(n,\mathbb{Z})$
- $\exists 710$ $\mathbb{Z}$-coujugacy subgroups $G \leq \mathrm{GL}(4,\mathbb{Z})$ ($\exists 710$ 4-dim. algebraic k-tori T).

Theorem ([HY17]) 4-dim. algebraic tori T

(i) T is stably k-rational $\iff$ $\exists G$: 487 groups;
(ii) T is not stably but retract k-rational $\iff$ $\exists G$: 7 groups;
(iii) T is not retract k-rational $\iff$ $\exists G$: 216 groups.

- We do not know "k-rationality".
- Voskresenskii's conjecture: any stably k-rational torus is k-rational (Zariski problem).
- what happens for dimension 5?

Rationality problem for algebraic tori T (5-dim.)

- The function field of n-dim. $T \overset{\text{identified}}{\longleftrightarrow} L(M)^G$, $G \leq \mathrm{GL}(n, \mathbb{Z})$
- $\exists 6079$ $\mathbb{Z}$-coujugacy subgroups $G \leq \mathrm{GL}(5, \mathbb{Z})$ ($\exists 6079$ 5-dim. algebraic k-tori T).

Theorem ([HY17]) 5-dim. algebraic tori T

(i) T is stably k-rational $\iff$ $\exists G$: 3051 groups;
(ii) T is not stably but retract k-rational $\iff$ $\exists G$: 25 groups;
(iii) T is not retract k-rational $\iff$ $\exists G$: 3003 groups.

- what happens for dimension 6?
- BUT we do not know the answer for dimension 6.
- $\exists 85308$ $\mathbb{Z}$-coujugacy subgroups $G \leq \mathrm{GL}(6, \mathbb{Z})$ ($\exists 85308$ 6-dim. algebraic k-tori T).

Strategy: Flabby (Flasque) resolution

- M: G-lattice, i.e. f.g. $\mathbb{Z}$-free $\mathbb{Z}[G]$-module.

Definition

(i) M is permutation $\overset{\text{def}}{\iff}$ $M \simeq \oplus_{1\leq i\leq m}\mathbb{Z}[G/H_i]$.

(ii) M is stably permutation $\overset{\text{def}}{\iff}$ $M \oplus \exists P \simeq P'$, P, P': permutation.

(iii) M is invertible $\overset{\text{def}}{\iff}$ $M \oplus \exists M' \simeq P$: permutation.

(iv) M is coflabby $\overset{\text{def}}{\iff}$ $H^1(H, M) = 0$ $(\forall H \leq G)$.

(v) M is flabby $\overset{\text{def}}{\iff}$ $\widehat{H}^{-1}(H, M) = 0$ $(\forall H \leq G)$. ($\widehat{H}$: Tate cohomology)

- "permutation"
  $\implies$ "stably permutation"
  $\implies$ "invertible"
  $\implies$ "flabby and coflabby".

Commutative monoid $\mathcal{M}$

$M_1 \sim M_2 \overset{\text{def}}{\iff} M_1 \oplus P_1 \simeq M_2 \oplus P_2$ ($\exists P_1, \exists P_2$: permutation).
$\Longrightarrow$ commutative monoid $\mathcal{M}$: $[M_1] + [M_2] := [M_1 \oplus M_2]$, $0 = [P]$.

Theorem (Endo-Miyata 1974, Colliot-Thélène-Sansuc 1977)

$\exists P$: permutation, $\exists F$: flabby such that

$$0 \to M \to P \to F \to 0\text{: flabby resolution of } M.$$

- $[M]^{fl} := [F]$; flabby class of M.

Theorem (Endo-Miyata 1973, Voskresenskii 1974, Saltman 1984)

(EM73) $[M]^{fl} = 0 \iff L(M)^G$ is stably k-rational.
(Vos74) $[M]^{fl} = [M']^{fl} \iff L(M)^G(x_1, \ldots, x_m) \simeq L(M')^G(y_1, \ldots, y_n)$; stably k-isomorphic.
(Sal84) $[M]^{fl}$ is invertible $\iff L(M)^G$ is retract k-rational.

- $M = M_G \simeq \widehat{T} = \mathrm{Hom}(T, \mathbb{G}_m)$, $k(T) \simeq L(M)^G$, $G = \mathrm{Gal}(L/k)$.

Contributions of [HY17]

- We give a procedure to compute a flabby resolution of M, in particular $[M]^{fl} = [F]$, effectively (with smaller rank after base change) by computer software GAP.
- The function `IsFlabby` (resp. `IsCoflabby`) may determine whether M is flabby (resp. coflabby).
- The function `IsInvertibleF` may determine whether $[M]^{fl} = [F]$ is invertible ($\leftrightarrow$ whether $L(M)^G$ (resp. T) is retract rational).
- We provide some functions for checking a possibility of isomorphism

$$\left(\bigoplus_{i=1}^{r} a_i\,\mathbb{Z}[G/H_i]\right) \oplus a_{r+1}F \simeq \bigoplus_{i=1}^{r} b_i'\,\mathbb{Z}[G/H_i] \qquad (*)$$

  by computing some invariants (e.g. trace, $\widehat{Z}^0$, $\widehat{H}^0$) of both sides.
- [HY17, Example 10.7]. $G \simeq S_5 \leq \mathrm{GL}(5,\mathbb{Z})$ with number $(5,946,4)$
  $\Longrightarrow \mathrm{rank}(F) = 17$ and $\mathrm{rank}(*) = 88$ holds
  $\Longrightarrow [F] = 0 \Longrightarrow L(M)^G$ (resp. T) is stably rational over k.

Application to Krull-Schmidt

Corollary ($[F]=[M]^{fl}$: invertible case, $G\simeq S_5, F_{20}$)

$\exists T$, T'; 4-dim. not stably rational algebraic tori over k such that $T\not\sim T'$ (birational) and $T\times T'$: 8-dim. stably rational over k.
$\because -[M]^{fl}=[M']^{fl}\neq 0$.

Prop. ([HY17], Krull-Schmidt fails for permutation D_6-lattices)

$\{1\}$, $C_2^{(1)}$, $C_2^{(2)}$, $C_2^{(3)}$, C_3, V_4, C_6, $S_3^{(1)}$, $S_3^{(2)}$, D_6: conj. subgroups of D_6.

$$\mathbb{Z}[D_6]\oplus\mathbb{Z}[D_6/V_4]^{\oplus 2}\oplus\mathbb{Z}[D_6/C_6]\oplus\mathbb{Z}[D_6/S_3^{(1)}]\oplus\mathbb{Z}[D_6/S_3^{(2)}]$$
$$\simeq\mathbb{Z}[D_6/C_2^{(1)}]\oplus\mathbb{Z}[D_6/C_2^{(2)}]\oplus\mathbb{Z}[D_6/C_2^{(3)}]\oplus\mathbb{Z}[D_6/C_3]\oplus\mathbb{Z}^{\oplus 2}.$$

- D_6 is the smallest example exhibiting the failure of K-S:

Theorem (Dress 1973)

Krull-Schmidt holds for permutation G-lattices $\iff$ $G/O_p(G)$ is cyclic where $O_p(G)$ is the maximal normal p-subgroup of G.

Krull-Schmidt and Direct sum cancelation

Theorem (Hindman-Klingler-Odenthal 1998) Assume $G \neq D_8$

Krull-Schmidt holds for G-lattices $\iff$ (i) $G = C_p$ ($p \leq 19$; prime), (ii) $G = C_n$ ($n = 1, 4, 8, 9$), (iii) $G = V_4$ or (iv) $G = D_4$.

Theorem (Endo-Hironaka 1979)

Direct sum cancellation holds, i.e. $M_1 \oplus N \simeq M_2 \oplus N \Longrightarrow M_1 \simeq M_2$, $\Longrightarrow G$ is abelian, dihedral, A_4, S_4 or A_5 (*).

- via projective class group (see Swan 1988, Corollary 1.3, Section 7).
- Except for (*) $\Longrightarrow$ Direct sum cancelation fails $\Longrightarrow$ K-S fails

Theorem ([HY17]) $G \leq \mathrm{GL}(n, \mathbb{Z})$ (up to conjugacy)

(i) $n \leq 4 \Longrightarrow$ K-S holds.
(ii) $n = 5$. K-S fails $\iff$ 11 groups G (among 6079 groups).
(iii) $n = 6$. K-S fails $\iff$ 131 groups G (among 85308 groups).

§2. Birational classification for algebraic tori

- Two algebraic k-tori T and T' are stably birationally k-equivalent, denoted by $T \overset{\text{s.b.}}{\approx} T'$, if their function fields $k(T)$ and $k(T')$ are stably k-isomorphic, i.e. $k(T)(x_1, \ldots, x_m) \simeq k(T')(y_1, \ldots, y_n)$.
- $\mathcal{T}$: the category of algebraic k-tori.
- $\mathcal{T}_n$: the category of algebraic k-tori of dimension n.

Problem 1: Stably birational classification for algebraic tori

Determine the structure of $\mathcal{T}/\overset{\text{s.b.}}{\approx}$ (resp. $\mathcal{T}_n/\overset{\text{s.b.}}{\approx}$). In particular, for given two algebraic k-tori T and T' (resp. T and T' of dimension n) determine whether T and T' are stably birationally k-equivalent.

- $V_4 := C_2 \times C_2$.

Theorem (Colliot-Thélène and Sansuc 1977) $\dim(T) = \dim(T') = 3$

Let L/k and L'/k be Galois extensions with $\mathrm{Gal}(L/k) \simeq \mathrm{Gal}(L'/k) \simeq V_4$. Let $T = R^{(1)}_{L/k}(\mathbb{G}_m)$ and $T' = R^{(1)}_{L'/k}(\mathbb{G}_m)$ be the corresponding norm one tori. If $T \overset{\mathrm{s.b.}}{\approx} T'$ (stably birationally k-equivalent), then $L = L'$.

- In particular, if k is a number field, then there exist infinitely many stably birationally k-equivalent classes of (non-rational: $1\mathrm{st}/15$) k-tori which correspond to U_1 (cf. Main theorem 1, later).
- $T \overset{\mathrm{s.b.}}{\approx} T'$ (stably birationally k-equivalent) $\Longleftarrow L = L'$???

- $\overline{k}$: a fixed separable closure of k and $\mathcal{G} = \mathrm{Gal}(\overline{k}/k)$.
- X: a smooth k-compactification of T, i.e. smooth projective k-variety X containing T as a dense open subvariety.
- $\overline{X} = X \times_k \overline{k}$.

Theorem (Voskresenskii 1969, 1970)

There exists an exact sequence of $\mathcal{G}$-lattices

$$0 \to \widehat{T} \to \widehat{Q} \to \mathrm{Pic}\,\overline{X} \to 0$$

where $\widehat{Q}$ is permutation and $\mathrm{Pic}\,\overline{X}$ is flabby.

- $M_G \simeq \widehat{T}$, $[\widehat{T}]^{fl} = [\mathrm{Pic}\,\overline{X}]$ as $\mathcal{G}$-lattices.

Theorem (Voskresenskii 1970, 1973)

(i) T is stably k-rational if and only if $[\mathrm{Pic}\,\overline{X}] = 0$ as a $\mathcal{G}$-lattice.

(ii) $T \overset{\text{s.b.}}{\approx} T'$ (stably birationally k-equivalent) if and only if $[\mathrm{Pic}\,\overline{X}] = [\mathrm{Pic}\,\overline{X'}]$ as $\mathcal{G}$-lattices.

- From $\mathcal{G}$-lattice to G-lattice

Let L be the minimal splitting field of T with $G = \mathrm{Gal}(L/k) \simeq \mathcal{G}/\mathcal{H}$. We obtain a flabby resolution of $\widehat{T}$:

$$0 \to \widehat{T} \to \widehat{Q} \to \mathrm{Pic}\, X_L \to 0$$

with $[\widehat{T}]^{fl} = [\mathrm{Pic}\ X_L]$ as G-lattices. We get:

Theorem (Voskresenskii 1970, 1973)

(ii)′ $T \overset{\text{s.b.}}{\approx} T'$ (stably birationally k-equivalent) if and only if $[\mathrm{Pic}\, X_{\widetilde{L}}] = [\mathrm{Pic}\, X'_{\widetilde{L}}]$ as $\widetilde{H}$-lattices where $\widetilde{L} = LL'$ and $\widetilde{H} = \mathrm{Gal}(\widetilde{L}/k)$.

The group $\widetilde{H}$ becomes a *subdirect product* of $G = \mathrm{Gal}(L/k)$ and $G' = \mathrm{Gal}(L'/k)$, i.e. a subgroup $\widetilde{H}$ of $G \times G'$ with surjections $\varphi_1 : \widetilde{H} \twoheadrightarrow G$ and $\varphi_2 : \widetilde{H} \twoheadrightarrow G'$.

- This observation yields a concept of "*weak stably k-equivalence*".

Definition

(i) $[M]^{fl}$ and $[M']^{fl}$ are *weak stably k-equivalent*, if there exists a subdirect product $\widetilde{H} \leq G \times G'$ of G and G' with surjections $\varphi_1 : \widetilde{H} \twoheadrightarrow G$ and $\varphi_2 : \widetilde{H} \twoheadrightarrow G'$ such that $[M]^{fl} = [M']^{fl}$ as $\widetilde{H}$-lattices where $\widetilde{H}$ acts on M (resp. M') through the surjection φ_1 (resp. φ_2).
(ii) Algebraic k-tori T and T' are *weak stably birationally k-equivalent*, denoted by $T \overset{\text{s.b.}}{\sim} T'$, if $[\widehat{T}]^{fl}$ and $[\widehat{T'}]^{fl}$ are weak stably k-equivalent.

Remark

(1) $T \overset{\text{s.b.}}{\approx} T'$ (stably bir. k-equiv.) $\Rightarrow T \overset{\text{s.b.}}{\sim} T'$ (weak stably bir. k-equiv.).
(2) $\overset{\text{s.b.}}{\sim}$ becomes an equivalence relation and we call this equivalent class *the weak stably k-equivalent class* of $[\widehat{T}]^{fl}$ (or T) denoted by WSEC_r $(r \geq 0)$ with the stably k-rational class WSEC_0.

- Let L be the minimal splitting field of T with $G = \mathrm{Gal}(L/k) \simeq \mathcal{G}/\mathcal{H}$.
- By the inflation-restriction exact sequence $0 \to H^1(G, \mathrm{Pic}\, X_L) \xrightarrow{\mathrm{inf}} H^1(k, \mathrm{Pic}\, \overline{X}) \xrightarrow{\mathrm{res}} H^1(L, \mathrm{Pic}\, \overline{X})$, we get $\mathrm{inf} : H^1(G, \mathrm{Pic}\, X_L) \xrightarrow{\sim} H^1(k, \mathrm{Pic}\, \overline{X})$ because $H^1(L, \mathrm{Pic}\, \overline{X}) = 0$.
- $H^1(k, \mathrm{Pic}\, \overline{X}) \simeq \mathrm{Br}(X)/\mathrm{Br}(k) \simeq \mathrm{Br}_{\mathrm{nr}}(k(X)/k)/\mathrm{Br}(k)$
  by Colliot-Thélène-Sansuc 1987
  where $\mathrm{Br}(X)$ is the étale cohomological/Azumaya Brauer group of X and $\mathrm{Br}_{\mathrm{nr}}(k(X)/k)$ is the unramified Brauer group of $k(X)$ over k.

- Let G, $G' \leq \mathrm{GL}(n,\mathbb{Z})$ be $\mathrm{GL}(n,\mathbb{Z})$-conjugate. Then $\exists\psi : G \xrightarrow{\sim} G'$, $g \mapsto u^{-1}gu$ $(u \in \mathrm{GL}(n,\mathbb{Z}))$.
- Let T, T' be algebraic k-tori of dimension n with the minimal splitting fields L and L' and the character modules $\widehat{T} = M_G$ and $\widehat{T}' = M_{G'}$.
- Assume that $L = L'$. Then $G \simeq G' \simeq \mathrm{Gal}(L/k)$ and $\exists\varphi_1 : \mathrm{Gal}(L/k) \xrightarrow{\sim} G \leq \mathrm{GL}(n,\mathbb{Z})$, $f \mapsto \varphi_1(f)$, $\exists\varphi_2 : \mathrm{Gal}(L'/k) \xrightarrow{\sim} G' \leq \mathrm{GL}(n,\mathbb{Z})$, $f \mapsto \varphi_2(f)$, $\exists$ a subdirect product $\widetilde{H} = \{(\varphi_1(f), \varphi_2(f)) \mid f \in \mathrm{Gal}(L/k)\} \leq G \times G'$ $(\widetilde{H} \simeq G)$.
- Hence we can obtain $\sigma \in \mathrm{Aut}(G)$ such that $(\psi^{-1})(\varphi_2\varphi_1^{-1})(g) = g^\sigma$ $(\forall g \in G)$ and hence we can identify $\psi^{-1} : G' \simeq G^\sigma$ $(\sigma \in \mathrm{Aut}(G))$.
- Let T^σ be an algebraic k-torus of dimension n with the minimal splitting field L and the character module $\widehat{T}^\sigma = M_{G^\sigma}$ $(\sigma \in \mathrm{Aut}(G))$ with $G \simeq G^\sigma \simeq \mathrm{Gal}(L/k)$.
- Then the set

$$\{T^\sigma \mid \sigma \in \mathrm{Aut}(G)\}$$

gives all algebraic k-tori of dimension n with the minimal splitting field L and the character module $\widehat{T}^\sigma \simeq M_G$.

Definition (the groups X, Y, Z)

We define the following subgroups of $\mathrm{Aut}(G)$ for $G \leq \mathrm{GL}(n\,\mathbb{Z})$:

$$\mathrm{Inn}(G) \leq X \leq Y \leq Z \leq \mathrm{Aut}(G),$$

$$\begin{aligned}
X &= \mathrm{Aut}_{\mathrm{GL}(n,\mathbb{Z})}(G) \\
&= \{\sigma \in \mathrm{Aut}(G) \mid \exists u \in \mathrm{GL}(n,\mathbb{Z}) \text{ s.t. } u^{-1}gu = g^{\sigma} (\forall g \in G)\} \\
&\simeq N_{\mathrm{GL}(n,\mathbb{Z})}(G)/Z_{\mathrm{GL}(n,\mathbb{Z})}(G), \\
Y &= \{\sigma \in \mathrm{Aut}(G) \mid [M_G]^{fl} = [M_{G^{\sigma}}]^{fl} \text{ as } \widetilde{H}\text{-lattices} \\
&\quad \text{where } \widetilde{H} = \{(g, g^{\sigma}) \mid g \in G\} \simeq G\}, \\
Z &= \{\sigma \in \mathrm{Aut}(G) \mid [M_H]^{fl} \sim [M_{H^{\sigma}}]^{fl} \text{ for any } H \leq G\}.
\end{aligned}$$

Corollary

The set $\{T^{\sigma} \mid \sigma \in \mathrm{Aut}(G)\}$ gives all algebraic k-tori of dimension n with the minimal splitting field L and $\widehat{T}^{\sigma} \simeq M_G$ and splits into λ stably birationally k-equivalent classes consist of μ birationally k-equivalent classes where $\lambda = |Y\backslash\mathrm{Aut}(G)| \leq \mu \leq |X\backslash\mathrm{Aut}(G)|$.

Theorem ([HY, Theorem 1.26])

(1) For $\sigma \in \mathrm{Aut}(G)$, T and T^σ are weak stably birationally k-equivalent;
(2) For $\sigma \in \mathrm{Aut}(G)$, $\sigma \in X$ if and only if $M_G \simeq M_{G^\sigma}$ as $\mathrm{Gal}(L/k)$-lattices. In particular, T and T^σ are birationally k-equivalent, i.e. $k(T) \simeq L(M)^G \simeq L(M^\sigma)^{G^\sigma} \simeq k(T^\sigma)$;
(3) For $\sigma \in \mathrm{Aut}(G)$, $\sigma \in Y$ if and only if T and T^σ are stably birationally k-equivalent. In particular, $\{T^\sigma \mid \sigma \in \mathrm{Aut}(G)\}$ splits into λ stably birationally k-equivalent classes where $\lambda = |Y\backslash \mathrm{Aut}(G)|$;
(4) For $\sigma \in \mathrm{Aut}(G)$, $\sigma \in Z$ if and only if $T \times_k K$ and $T^\sigma \times_k K$ are weak stably birationally K-equivalent for any $k \subset K \subset L$.
In particular, (i) if $Y = Z$, then $\sigma \in Z$ if and only if T and T^σ are stably birationally k-equivalent; and (ii) if $X = Y = Z$ (resp. $X = Y$), then $\sigma \in Z$ (resp. $\sigma \in Y$) if and only if T and T^σ are birationally k-equivalent.

Rationality problem for 3-dimensional algebraic k-tori T was solved by Kunyavskii (1990). Stably/retract rationality for algebraic k-tori T of dimensions 4 and 5 are given in Hoshi and Yamasaki [HY17, Chapter 1].

Definition $(N_{3,i}, N_{31,i}, N_{4,i}, I_{4,i})$

(1) The 15 groups $G = N_{3,i} \leq \mathrm{GL}(3,\mathbb{Z})$ $(1 \leq i \leq 15)$ for which $k(T) \simeq L(M)^G$ is not retract k-rational are as in [HY, Table 6].
(2) The 64 groups $G = N_{31,i} \leq \mathrm{GL}(4,\mathbb{Z})$ $(1 \leq i \leq 64)$ for which $k(T) \simeq L(M)^G$ is not retract k-rational where $M \simeq M_1 \oplus M_2$ with $\mathrm{rank}\ M = 3+1$ are as in [HY, Table 7].
(3) The 152 groups $G = N_{4,i} \leq \mathrm{GL}(4,\mathbb{Z})$ $(1 \leq i \leq 152)$ for which $k(T) \simeq L(M)^G$ is not retract k-rational with $\mathrm{rank}\ M = 4$ are as in [HY, Table 8].
(4) The 7 groups $G = I_{4,i} \leq \mathrm{GL}(4,\mathbb{Z})$ $(1 \leq i \leq 7)$ for which $k(T) \simeq L(M)^G$ is not stably but retract k-rational with $\mathrm{rank}\ M = 4$ are as in [HY, Table 9].

Main Theorems 1, 2, 3, 4, 5, 6, 7

- Main theorem 1 $\dim(T) = 3$: up to $\overset{\text{s.b.}}{\sim}$ (weak)
- Main theorem 2 $\dim(T) = 3$: up to $\overset{\text{s.b.}}{\approx}$
- Main theorem 3 $\dim(T) = 4$: up to $\overset{\text{s.b.}}{\sim}$ (weak)
- Main theorem 4 $\dim(T) = 4$ $(N_{4,i})$: up to $\overset{\text{s.b.}}{\approx}$
- Main theorem 5 $\dim(T) = 4$ $(I_{4,i})$: up to $\overset{\text{s.b.}}{\approx}$
- Main theorem 6 $\dim(T) = 4$: $I_{4,i}$ cases $(1 \leq i \leq 7)$
- Main theorem 7 higher dimensional cases: $\dim(T) \geq 3$

Definition

The G-lattice M_G of rank n is defined to be the G-lattice with a $\mathbb{Z}$-basis $\{u_1, \ldots, u_n\}$ on which G acts by $u_i^\sigma = \sum_{j=1}^n a_{i,j} u_j$ for any $\sigma = [a_{i,j}] \in G \leq \mathrm{GL}(n, \mathbb{Z})$.

Main theorem 1 ([HY, Theorem 1.28]) $\dim(T) = 3$: up to $\overset{\text{s.b.}}{\sim}$

There exist exactly $14 = 1 + 13$ weak stably birationally k-equivalent classes of algebraic k-tori T of dimension 3 which consist of the stably rational class WSEC_0 and 13 classes WSEC_r $(1 \leq r \leq 13)$ for $[\widehat{T}]^{fl}$ with $\widehat{T} = M_G$ and $G = N_{3,i}$ $(1 \leq i \leq 15)$: (red $\leftrightarrow$ norm one tori)

r	$G = N_{3,i} : [\widehat{T}]^{fl} = [M_G]^{fl} \in \mathrm{WSEC}_r$	$\mathrm{G}_r \simeq G$	$\lambda_r = \lvert\mathrm{WSEC}_{r,L}\rvert$
1	$N_{3,1} \simeq U_1$ ([CTS 1977])	V_4	1
2	$N_{3,2} \simeq U_2$	C_2^3	**7**
3	$N_{3,3} \simeq W_2$	C_2^3	1
4	$N_{3,4} \simeq W_1$	$C_4 \times C_2$	1
5	$N_{3,5} \simeq U_3$, $N_{3,6} \simeq U_4$	D_4	**2**
6	$N_{3,7} \simeq U_6$	$D_4 \times C_2$	4
7	$N_{3,8} \simeq U_5$	A_4	1
8	$N_{3,9} \simeq U_7$	$A_4 \times C_2$	1
9	$N_{3,10} \simeq W_3$	$A_4 \times C_2$	1
10	$N_{3,11} \simeq U_9$, $N_{3,13} \simeq U_{10}$	S_4	1
11	$N_{3,12} \simeq U_8$	S_4	1
12	$N_{3,14} \simeq U_{12}$	$S_4 \times C_2$	1
13	$N_{3,15} \simeq U_{11}$	$S_4 \times C_2$	2

Main theorem 2 ([HY, Theorem 1.29]) $\dim(T)=3$: up to $\overset{\text{s.b.}}{\approx}$

Let T_i and T'_j $(1\leq i,j\leq 15)$ be algebraic k-tori of dimension 3 with the minimal splitting fields L_i and L'_j, and $\widehat{T}_i=M_G$ and $\widehat{T}'_j=M_{G'}$ which satisfy that G and G' are $\mathrm{GL}(3,\mathbb{Z})$-conjugate to $N_{3,i}$ and $N_{3,j}$ respectively. For $1\leq i,j\leq 15$, the following conditions are equivalent:

(1) $T_i \overset{\text{s.b.}}{\approx} T'_j$ (stably birationally k-equivalent);
(2) $G\simeq G'$, $L_i=L'_j$, $T_i\times_k K$ and $T'_j\times_k K$ are weak stably birationally K-equivalent for any $k\subset K\subset L_i$;
(3) $G\simeq G'$, $L_i=L'_j$, $T_i\times_k K$ and $T'_j\times_k K$ are weak stably birationally K-equivalent for any $k\subset K\subset L_i$ corresponding to WSEC_r $(r\geq 1)$;
(4) $G\simeq G'$, $L_i=L'_j$, $T_i\times_k K$ and $T'_j\times_k K$ are weak stably birationally K-equivalent for any $k\subset K\subset L_i$ corresponding to WSEC_r $(r\geq 1)$ with $[K:k]=d$ where

$$d=\begin{cases}1 & (i=1,3,4,8,9,10,11,12,13,14),\\ 1,2 & (i=2,5,6,7,15).\end{cases}$$

In particular, for $d=1$, (4) $\Leftrightarrow G\simeq G'$, $L_i=L'_j$, i.e. $\widetilde{H}\simeq G\simeq G'$.

Main theorem 2 ([HY, Theorem 1.29]) $\dim(T) = 3$: up to $\overset{\text{s.b.}}{\approx}$

Moreover, if $i = j$ with $G \simeq G'$, $L_i = L'_j$, i.e. $\widetilde{H} \simeq G \simeq G'$, then $Y = Z$ (which is equivalent to $(1) \Leftrightarrow (2)$)
and for $1 \leq r \leq 13$, we get the following disjoint union decompositions

$$\mathrm{WSEC}_r = \coprod_{\substack{L/k \\ \mathrm{Gal}(L/k) \simeq \mathrm{G}_r}} \mathrm{WSEC}_{r,L}, \ \ \mathrm{WSEC}_{r,L} = \coprod_{t=1}^{\lambda_r} \mathrm{SEC}_{r,L,t}$$

modulo $\overset{\text{s.b.}}{\approx}$ where $\mathrm{Gal}(L/k) \simeq \mathrm{G}_r \simeq N_{3,i}$ $(1 \leq r \leq 13)$ and
$\lambda_r = |\mathrm{WSEC}_{r,L}| = |Y \backslash \mathrm{Aut}(G)|$ is given as in Main theorem 1.
Furthermore, for the cases $G = N_{3,i}$ $(i = 1, 5, 6, 8, 9, 10, 11, 12, 13, 15)$ with $X = Y$, the following conditions are also equivalent:
(0) T_i and T'_i are birationally k-equivalent;
(1) T_i and T'_i are stably birationally k-equivalent.

Corollary (Stably birational classification for T with $\dim(T) = 3$)

Let $\mathcal{T}_3$ be the category of algebraic k-tori of dimension 3. We get a classification (disjoint union decomposition) of $\mathcal{T}_3$ with respect to the stably birationally k-equivalence $\overset{\text{s.b.}}{\approx}$:

$$\mathcal{T}_3 = \coprod_{r=0}^{13} \mathrm{WSEC}_r = \mathrm{SEC}_0 \coprod \left(\coprod_{r=1}^{13} \coprod_{\substack{L/k \\ \mathrm{Gal}(L/k) \simeq \mathrm{G}_r}} \coprod_{t=1}^{\lambda_r} \mathrm{SEC}_{r,L,t} \right)$$

modulo $\overset{\text{s.b.}}{\approx}$ where SEC_0 is the stably k-equivalent class consists of stably k-rational tori $T \in \mathcal{T}_3$, $\mathrm{Gal}(L/k) \simeq \mathrm{G}_r \simeq N_{3,i}$ $(1 \leq r \leq 13)$ and $\lambda_r = |\mathrm{WSEC}_{r,L}| = |Y \backslash \mathrm{Aut}(G)|$ is given as in Main theorem 1.

Example of Main theorem 2: WSEC_5; $G = N_{3,6} \simeq U_4 \simeq D_4$, $G' = N_{3,6} \simeq U_4 \simeq D_4$ with $\lambda_5 = 2$

- $k = \mathbb{Q}$, $K_4 = \mathbb{Q}(\sqrt[4]{2})$, $K_4' = \mathbb{Q}(\sqrt[4]{2}\zeta_8)$ with $[K_4 : \mathbb{Q}] = [K_4' : \mathbb{Q}] = 4$ and the same Galois closure $L = \mathbb{Q}(\sqrt[4]{2}, \sqrt{-1})$, $\mathrm{Gal}(L/\mathbb{Q}) \simeq D_4$.
- $T = R^{(1)}_{K_4/\mathbb{Q}}(\mathbb{G}_m)$, $T' = R^{(1)}_{K_4'/\mathbb{Q}}(\mathbb{G}_m)$: the corresponding norm one tori with $\widehat{T} = M_G$, $\widehat{T}' = M_{G'}$, $G = N_{3,6} \simeq U_4 \simeq D_4$, $G' = N_{3,6} \simeq U_4 \simeq D_4$, $[\widehat{T}]^{fl}, [\widehat{T}']^{fl} \in \mathrm{WSEC}_2$ (not retract $\mathbb{Q}$-rational).
- By Main theorem 2, T and T' are not stably birationally $\mathbb{Q}$-equivalent although both T and T' correspond to $N_{3,6} \simeq U_4 \simeq D_4$ with the same splitting field L.
- Because $\lambda_5 = 2$, if we take an algebraic $\mathbb{Q}$-torus T'' of dimension 3 with the minimal splitting field $L = \mathbb{Q}(\sqrt[4]{2}, \sqrt{-1})$ and $[T'']^{fl} \in N_{3,5} \simeq U_3$, then T'' is stably birationally $\mathbb{Q}$-equivalent to either $T = R^{(1)}_{K_4/\mathbb{Q}}(\mathbb{G}_m)$ or $T' = R^{(1)}_{K_4'/\mathbb{Q}}(\mathbb{G}_m)$.

Example of Main theorem 2: WSEC_2; $G = N_{3,2} \simeq U_2 \simeq C_2^3$, $G' = N_{3,2} \simeq U_2 \simeq C_2^3$ with $\lambda_2 = 7$

- $k = \mathbb{Q}$ and T is an algebraic $\mathbb{Q}$-torus with the minimal splitting field $L = \mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5})$, $[L : \mathbb{Q}] = 8$, $\widehat{T} = M_G$, $G = N_{3,2} \simeq U_2 \simeq C_2^3$, $[\widehat{T}]^{fl} \in \mathrm{WSEC}_2$ (not retract $\mathbb{Q}$-rational).
- $\exists$ exactly 7 subgroups $H_i \leq G$ $(1 \leq i \leq 7)$ with $[G : H_i] = 2$ and $\exists$ exactly 7 subgroups $H_i' \leq G$ $(1 \leq i \leq 7)$ with $[G : H_i'] = 4$.
- If $H_1 \simeq U_1 \simeq C_2^2$, then $[M_G\,|_{H_i}]^{fl} = [M_{H_i}]^{fl} = 0$ $(2 \leq i \leq 7)$ and $[M_G\,|_{H_i'}]^{fl} = [M_{H_i'}]^{fl} = 0$ $(1 \leq i \leq 7)$ (stably $\mathbb{Q}$-rational).
- $G' = G^\sigma \leq \mathrm{GL}(3, \mathbb{Z})$ with $G' \simeq G \simeq C_2^3 \simeq (\mathbb{F}_2)^3$ $(\sigma \in \mathrm{Aut}(G))$ where $\mathrm{Aut}(G) \simeq \mathrm{GL}(3, \mathbb{F}_2) \simeq \mathrm{PGL}(3, \mathbb{F}_2) \simeq \mathrm{SL}(3, \mathbb{F}_2) \simeq \mathrm{PSL}(3, \mathbb{F}_2) \simeq \mathrm{PSL}(2, \mathbb{F}_7)$ with $|\mathrm{Aut}(G)| = 168$.
- By Main theorem 2, we have $\lambda_2 = |\mathrm{WSEC}_{r,L}| = |Y\backslash\mathrm{Aut}(G)| = 7$ where $Y = \mathrm{Stab}_{H_1}(\mathrm{Aut}(G)) \simeq S_4$ with $|Y| = 24$.
- $\mathrm{WSEC}_{2,L} = \coprod_{t=1}^{7} \mathrm{SEC}_{2,L,t} = \{[\widehat{T}_i]^{fl} \mid 1 \leq i \leq 7\}$.
- $Y\backslash\mathrm{Aut}(G) \simeq \mathrm{Gr}_{\mathbb{F}_2}(2,3) \simeq \mathrm{Gr}_{\mathbb{F}_2}(1,3) \simeq \mathbb{P}^2_{\mathbb{F}_2}$ with $|\mathbb{P}^2_{\mathbb{F}_2}| = 7$ where $\mathrm{Gr}_{\mathbb{F}_2}(d,3)$ is the Grassmannian of d-dimensional subspaces of $(\mathbb{F}_2)^3$.

Main theorems 3, 4, 5: $\dim(T) = 4$

- $\exists G = N_{31,i} \leq \mathrm{GL}(4,\mathbb{Z})$ $(1 \leq i \leq 64)$ for which $k(T) \simeq L(M)^G$ is not retract k-rational where $M \simeq M_1 \oplus M_2$ with $\mathrm{rank}\ M = 3+1$.
- $\exists G = N_{4,i} \leq \mathrm{GL}(4,\mathbb{Z})$ $(1 \leq i \leq 152)$ for which $k(T) \simeq L(M)^G$ is not retract k-rational with $\mathrm{rank}\ M = 4$.
- $\exists G = I_{4,i} \leq \mathrm{GL}(4,\mathbb{Z})$ $(1 \leq i \leq 7)$ for which $k(T) \simeq L(M)^G$ is not stably but retract k-rational with $\mathrm{rank}\ M = 4$.

Main theorem 3 ([HY, Theorem 1.34]) $\dim(T) = 4$: up to $\overset{\text{s.b.}}{\sim}$

There exist exactly $129 = 1 + 121 + 7$ weak stably birationally k-equivalent classes of algebraic k-tori T of dimension 4 which consist of the stably rational class WSEC_0, 121 classes WSEC_r $(1 \leq r \leq 121)$ for $[\widehat{T}]^{fl}$ with $\widehat{T} = M_G$ and $G = N_{31,i}$ $(1 \leq i \leq 64)$ as in [HY, Table 3] and for $[\widehat{T}]^{fl}$ with $\widehat{T} = M_G$ and $G = N_{4,i}$ $(1 \leq i \leq 152)$ as in [HY, Table 4], and 7 classes WSEC_r $(122 \leq r \leq 128)$ for $[\widehat{T}]^{fl}$ with $\widehat{T} = M_G$ and $G = I_{4,i}$ $(1 \leq i \leq 7)$: (red $\leftrightarrow$ norm one tori)

r	$G = I_{4,i} : [\widehat{T}]^{fl} = [M_G]^{fl} \in \mathrm{WSEC}_r$	$\mathrm{G}_r \simeq G$	$\lambda_r = \lvert \mathrm{WSEC}_{r,L} \rvert$
122	$I_{4,1}$	F_{20}	1
123	$I_{4,2}$	F_{20}	1
124	$I_{4,3}$	$F_{20} \times C_2$	2
125	$I_{4,4}$	S_5	1
126	$I_{4,5}$	S_5	1
127	$I_{4,6}$	$S_5 \times C_2$	2
128	$I_{4,7}$	$C_3 \rtimes C_8$	2

Main theorem 4 ([HY, Theorem 1.36]) $\dim(T)=4$ $(N_{4,i})$: up to $\overset{\text{s.b.}}{\approx}$

Let T_i and T'_j $(1\leq i,j\leq 152)$ be algebraic k-tori of dimension 4 with the minimal splitting fields L_i and L'_j and the character modules $\widehat{T}_i=M_G$ and $\widehat{T}'_j=M_{G'}$ which satisfy that G and G' are $\mathrm{GL}(4,\mathbb{Z})$-conjugate to $N_{4,i}$ and $N_{4,j}$ respectively. For $1\leq i,j\leq 152$ except for the cases $i=j=137,139,145,147$, the following conditions are equivalent:

(1) $T_i \overset{\text{s.b.}}{\approx} T'_j$ (stably birationally k-equivalent);

(2) $G\simeq G'$, $L_i=L'_j$, $T_i\times_k K$ and $T'_j\times_k K$ are weak stably birationally K-equivalent for any $k\subset K\subset L_i$;

(3) $G\simeq G'$, $L_i=L'_j$, $T_i\times_k K$ and $T'_j\times_k K$ are weak stably birationally K-equivalent for any $k\subset K\subset L_i$ corresponding to WSEC_r $(r\geq 1)$;

(4) $G\simeq G'$, $L_i=L'_j$, $T_i\times_k K$ and $T'_j\times_k K$ are weak stably birationally K-equivalent for any $k\subset K\subset L_i$ corresponding to WSEC_r $(r\geq 1)$ with $[K:k]=d$ where d is given as in [HY, Theorem 1.26].

In particular, for $d=1$, $(4)\Leftrightarrow G\simeq G'$, $L_i=L'_j$, i.e. $\widetilde{H}\simeq G\simeq G'$.

Main theorem 4 ([HY, Theorem 1.36]) $\dim(T) = 4$ $(N_{4,i})$: up to $\overset{\text{s.b.}}{\approx}$

For the exceptional cases $i = j = 137, 139, 145, 147$ ($G \simeq Q_8 \times C_3$, $(Q_8 \times C_3) \rtimes C_2$, $\mathrm{SL}(2, \mathbb{F}_3) \rtimes C_4$, $(\mathrm{GL}(2, \mathbb{F}_3) \rtimes C_2) \rtimes C_2 \simeq (\mathrm{SL}(2, \mathbb{F}_3) \rtimes C_4) \rtimes C_2$), we have the implications $(1) \Rightarrow (2) \Leftrightarrow (3) \Leftrightarrow (4)$, there exists $\sigma \in \mathrm{Aut}(G)$ such that $G' = G^\sigma$ and $X = Y \lhd Z$ with $Z/Y \simeq C_2, C_2^2, C_2, C_2$ respectively ($Y = Z$ is equivalent to $(1) \Leftrightarrow (2)$) and we have $(1) \Leftrightarrow M_G \simeq M_{G^\sigma}$ as $\widetilde{H}$-lattices $\Leftrightarrow M_G \otimes_{\mathbb{Z}} \mathbb{F}_p \simeq M_{G^\sigma} \otimes_{\mathbb{Z}} \mathbb{F}_p$ as $\mathbb{F}_p[\widetilde{H}]$-lattices for $p = 2$ $(i = j = 137)$, for $p = 2$ and 3 $(i = j = 139)$, for $p = 3$ $(i = j = 145, 147)$.
Furthermore, for the cases $G = N_{4,i}$ with $X = Y$ (82 cases of 152), the following conditions are also equivalent:
(0) T_i and T_i' are birationally k-equivalent;
(1) T_i and T_i' are stably birationally k-equivalent.

Main theorem 5 ([HY, Theorem 1.39]) $\dim(T)=4$ ($I_{4,i}$): up to $\overset{\text{s.b.}}{\approx}$

Let T_i and T'_j $(1\leq i,j\leq 7)$ be algebraic k-tori of dimension 4 with the minimal splitting fields L_i and L'_j and the character modules $\widehat{T}_i=M_G$ and $\widehat{T}'_j=M_{G'}$ which satisfy that G and G' are $\mathrm{GL}(4,\mathbb{Z})$-conjugate to $I_{4,i}$ and $I_{4,j}$ respectively. For $1\leq i,j\leq 7$ except for the case $i=j=7$, the following conditions are equivalent:

(1) $T_i \overset{\text{s.b.}}{\approx} T'_j$ (stably birationally k-equivalent);
(2) $G\simeq G'$, $L_i=L'_j$, $T_i\times_k K$ and $T'_j\times_k K$ are weak stably birationally K-equivalent for any $k\subset K\subset L_i$;
(3) $G\simeq G'$, $L_i=L'_j$, $T_i\times_k K$ and $T'_j\times_k K$ are weak stably birationally K-equivalent for any $k\subset K\subset L_i$ corresponding to WSEC_r $(r\geq 1)$;
(4) $G\simeq G'$, $L_i=L'_j$, $T_i\times_k K$ and $T'_j\times_k K$ are weak stably birationally K-equivalent for any $k\subset K\subset L_i$ corresponding to WSEC_r $(r\geq 1)$ with $[K:k]=d$ where $d=1$ $(i=1,2,4,5,7)$, $d=1,2$ $(i=3,6)$.

In particular, for $d=1$, (4) $\Leftrightarrow$ $G\simeq G'$, $L_i=L'_j$, i.e. $\widetilde{H}\simeq G\simeq G'$.

Main theorem 5 ([HY, Theorem 1.39]) $\dim(T) = 4$ ($I_{4,i}$): up to $\overset{\text{s.b.}}{\approx}$

For the exceptional case $i = j = 7$ ($G \simeq C_3 \rtimes C_8$), we have the implications $(1) \Rightarrow (2) \Leftrightarrow (3) \Leftrightarrow (4)$, there exists $\sigma \in \mathrm{Aut}(G)$ such that $G' = G^\sigma$ and $D_6 \simeq X = Y \lhd Z$ with $Z/Y \simeq C_2$ ($Y = Z$ is equivalent to $(1) \Leftrightarrow (2)$) and we have $(1) \Leftrightarrow M_G \simeq M_{G^\sigma}$ as $\widetilde{H}$-lattices $\Leftrightarrow M_G \otimes_{\mathbb{Z}} \mathbb{F}_3 \simeq M_{G^\sigma} \otimes_{\mathbb{Z}} \mathbb{F}_3$ as $\mathbb{F}_3[\widetilde{H}]$-lattices.
Furthermore, we have $X = Y$ for all the cases $G = I_{4,i}$ $(1 \leq i \leq 7)$, and hence the following conditions are also equivalent:
(0) T_i and T_i' are birationally k-equivalent;
(1) T_i and T_i' are stably birationally k-equivalent.

Corollary (Stably birational classification for T with $\dim(T) = 4$)

Let $\mathcal{T}_4$ be the category of algebraic k-tori of dimension 4. We get a classification (disjoint union decomposition) of $\mathcal{T}_4$ with respect to the stably birationally k-equivalence $\overset{\text{s.b.}}{\approx}$:

$$\mathcal{T}_4 = \coprod_{r=0}^{128} \mathrm{WSEC}_r = \mathrm{SEC}_0 \coprod \left(\coprod_{r=1}^{128} \coprod_{\substack{L/k \\ \mathrm{Gal}(L/k)\simeq \mathrm{G}_r}} \coprod_{t=1}^{\lambda_r} \mathrm{SEC}_{r,L,t} \right)$$

modulo $\overset{\text{s.b.}}{\approx}$ where SEC_0 is the stably k-equivalent class consists of stably k-rational tori $T \in \mathcal{T}_4$, $\mathrm{Gal}(L/k) \simeq \mathrm{G}_r \simeq N_{4,i}$ $(1 \leq r \leq 128)$ and $\lambda_r = |\mathrm{WSEC}_{r,L}| = |Y\backslash \mathrm{Aut}(G)|$ is given as in [HY, Table 4] and Main theorem 3.

Main theorem 6 ([HY, Theorem 1.42]) $\dim(T)=4$: seven $I_{4,i}$ cases

Let T_i $(1\leq i\leq 7)$ be an algebraic k-torus of dimension 4 with the minimal splitting field L_i and the character module $\widehat{T}_i=M_{G_i}$ which satisfies that G_i is $\mathrm{GL}(4,\mathbb{Z})$-conjugate to $I_{4,i}$. Let T_i^σ be the algebraic k-torus with $\widehat{T}_i^\sigma=M_{G^\sigma}$ $(\sigma\in\mathrm{Aut}(G_i))$. Then T_i and T_i^σ are not stably but retract k-rational, i.e. $[M_{G_i}]^{fl}\neq 0$, $[M_{G_i^\sigma}]^{fl}\neq 0$ but invertible of infinite order. We have:

(1) [HY17, Theorem 1.27] If $L_1=L_2$, then $T_1\times_k T_2$ is stably k-rational;
(2) $T_3\times_k T_3^\sigma$ is stably k-rational for $\sigma\in\mathrm{Aut}(G_3)$ with $1\neq\overline{\sigma}\in\mathrm{Aut}(G_3)/\mathrm{Inn}(G_3)\simeq C_2$;
(3) [HY17, Theorem 1.27] If $L_4=L_5$, then $T_4\times_k T_5$ is stably k-rational;
(4) $T_6\times_k T_6^\sigma$ is stably k-rational for $\sigma\in\mathrm{Aut}(G_6)$ with $1\neq\overline{\sigma}\in\mathrm{Aut}(G_6)/\mathrm{Inn}(G_6)\simeq C_2$;
(5) $T_7\times_k T_7^\sigma$ is stably k-rational for $\sigma\in\mathrm{Aut}(G_7)$ with $1\neq\overline{\sigma}\in\mathrm{Aut}(G_7)/X\simeq C_2$ where

$$X=\mathrm{Aut}_{\mathrm{GL}(4,\mathbb{Z})}(G_7)=\{\sigma\in\mathrm{Aut}(G_7)\mid G_7 \text{ and } G_7^\sigma \text{ are conjugate in } \mathrm{GL}(4,\mathbb{Z})\}\simeq D_6.$$

Higher dimensional cases: $\dim(T) \geq 3$

The following theorem can answer Problem 1 for algebraic k-tori T and T' of dimensions $m \geq 3$ and $n \geq 3$ respectively with $[\widehat{T}]^{fl}, [\widehat{T}']^{fl} \in \mathrm{WSEC}_r$ $(1 \leq r \leq 128)$ via Main theorem 2, Main theorem 4, and Main theorem 5.

Main theorem 7 ([HY, Theorem 1.44]) higher dimensional cases

Let T be an algebraic k-torus of dimension $m \geq 3$ with the minimal splitting field L, $\widehat{T} = M_G$, $G \leq \mathrm{GL}(m, \mathbb{Z})$ and $[\widehat{T}]^{fl} \in \mathrm{WSEC}_r$ $(1 \leq r \leq 128)$. Then there exists an algebraic k-torus T'' of dimension 3 or 4 with the minimal splitting field L'', $\widehat{T}'' = M_{G''}$, and $G'' = N_{3,i}$ $(1 \leq i \leq 15)$, $G'' = N_{4,i}$ $(1 \leq i \leq 152)$ or $G'' = I_{4,i}$ $(1 \leq i \leq 7)$ such that T'' and T are stably birationally k-equivalent and $L'' \subset L$, i.e. $[M_{G''}]^{fl} = [M_G]^{fl}$ as G-lattices and G acts on $[M_{G''}]^{fl}$ through $G'' \simeq G/N$ for the corresponding normal subgroup $N \lhd G$.

Corollary (Stably birational classification for T with $[\widehat{T}]^{fl} \in \mathrm{WSEC}_r$ $(0 \leq r \leq 128)$)

Let $\mathcal{T}'$ be the category of algebraic k-tori T with $[\widehat{T}]^{fl} \in \mathrm{WSEC}_r$ $(0 \leq r \leq 128)$. We get a classification (disjoint union decomposition) of $\mathcal{T}'$ with respect to the stably birationally k-equivalence $\overset{\text{s.b.}}{\approx}$:

$$\mathcal{T}' = \coprod_{r=0}^{128} \mathrm{WSEC}_r = \mathrm{SEC}_0 \coprod \left(\coprod_{r=1}^{128} \coprod_{\substack{L/k \\ \mathrm{Gal}(L/k) \simeq \mathrm{G}_r}} \coprod_{t=1}^{\lambda_r} \mathrm{SEC}_{r,L,t} \right)$$

modulo $\overset{\text{s.b.}}{\approx}$ where SEC_0 is the stably k-equivalent class consists of stably k-rational tori $T \in \mathcal{T}'$, $\mathrm{Gal}(L/k) \simeq \mathrm{G}_r \simeq N_{3,i} \in \mathrm{WSEC}_r$ $(1 \leq r \leq 13)$, $N_{4,i} \in \mathrm{WSEC}_r$ $(14 \leq r \leq 121)$, $I_{4,i} \in \mathrm{WSEC}_r$ $(122 \leq r \leq 128)$ and $\lambda_r = |\mathrm{WSEC}_{r,L}| = |Y \backslash \mathrm{Aut}(G)|$ is given as in Main theorem 1, [HY, Table 4] and Main theorem 3.

Sketch: Proof of Main theorems 1, 2, 3, 4, 5, 6, 7

- For Main theorems 1 (and 3), we obtain WSEC_r use torus invariants and establish $[\widehat{T}_i]^{fl} = [\widehat{T}_j]^{fl}$ as $\widetilde{H}$-lattices for some $i < j$ for $\widehat{T}_i = M_{G_i}$, $G_i = N_{3,i}$ with $G_i \simeq G'_j$, $L_i = L'_j$, i.e. $\widetilde{H} \simeq G_i \simeq G'_j$.
- For Main theorems 2 (and 4, 5), $(1) \Rightarrow (2)$, we should show that $[\widehat{T}_i]^{fl} = [\widehat{T}_j]^{fl}$ as $\widetilde{H}$-lattices for $i = j$ with $G \simeq G'$, $L_i = L'_j$, i.e. $\widetilde{H} \simeq G \simeq G'$ via the p-part of the flabby class $[\widehat{T}]^{fl}$ $(p = 2, 3)$ as a (faithful and indecomposable) $\mathbb{Z}_p[\mathrm{Syl}_p(G)]$-lattice via p-adic analysis.
- For Main theorems 2 (and 4, 5), $(2) \Rightarrow (1)$, we should show that if $i = j$ with $G \simeq G'$, $L_i = L'_j$, i.e. $\widetilde{H} \simeq G \simeq G'$, then $Y = Z$ (which is equivalent to $(1) \Leftrightarrow (2)$) with some exceptional cases.
- For Main theorems 2 (and 4, 5), $(2) \Leftrightarrow (3) \Leftrightarrow (4)$, we should show that $(4) \Rightarrow (2)$ because we already have $(2) \Rightarrow (3) \Rightarrow (4)$.
- For Main theorem 6, we should show that $[M_{G_i}]^{fl} + [M_{G_i^\sigma}]^{fl} = 0$ $(i = 3, 6, 7)$.
- For Main theorem 7, we should show that $L'' \subset L$.

Torus invariants (1/2)

Define

$$Ш^i_\omega(G,M) := \mathrm{Ker}\left\{H^i(G,M) \xrightarrow{\mathrm{res}} \prod_{g\in G} H^i(\langle g\rangle, M)\right\} \quad (i \geq 1).$$

Theorem (Kunyavskii, Skorobogatov and Tsfasman, 1989)

Let M be a G-lattice. Then three groups

$$Ш^1_\omega(G,[M]^{fl}) \simeq Ш^2_\omega(G,M) \simeq H^1(G,[M]^{fl}),$$
$$Ш^2_\omega(G,([M]^{fl})^\circ) \simeq Ш^1_\omega(G,M^\circ) \simeq H^1(G,(([M]^{fl})^\circ)^{fl}),$$
$$Ш^2_\omega(G,[M]^{fl}) \simeq Ш^1_\omega(G,([M]^{fl})^{fl}) \simeq H^1(G,([M]^{fl})^{fl})$$

are invariants of the flabby class $[M]^{fl}$ of M.

Torus invariants (2/2)

Definition (Torus invariants)

Let $G \leq \mathrm{GL}(n,\mathbb{Z})$ and M_G be the corresponding G-lattice of $\mathbb{Z}$-rank n. The torus invariants $TI_G = [l_1, l_2, l_3, l_4]$ of $[M_G]^{fl}$ are defined to be

$$l_1 = \begin{cases} 0 & \text{if} \quad [M_G]^{fl} = 0, \\ 1 & \text{if} \quad [M_G]^{fl} \neq 0 \text{ but is invertible}, \\ 2 & \text{if} \quad [M_G]^{fl} \text{ is not invertible}, \end{cases}$$

l_2 (resp. l_3, l_4) is the abelian invariants of $\text{Ш}^1_\omega(G, [M_G]^{fl})$ (resp. $\text{Ш}^2_\omega(G, ([M_G]^{fl})^\circ)$, $\text{Ш}^2_\omega(G, [M_G]^{fl})$).

Definition (The p-part $\widetilde{N}_p$ of $[M]^{fl}$ as a $\mathbb{Z}_p[\mathrm{Syl}_p(G)]$-lattice)

Let G be a finite group, M be a G-lattice and

$$0 \to M \to P \to F \to 0$$

be a flabby resolution of M where P is permutation and F is flabby.
(1) (Kunyavskii, 1990) By tensering with $\mathbb{Z}_2$, we also get a flabby resolution of $\widetilde{M} = M \otimes_{\mathbb{Z}} \mathbb{Z}_2$ as $\mathbb{Z}_2[G]$-lattices:

$$0 \to \widetilde{M} \to \widetilde{P} \to \widetilde{F} \to 0$$

where $\widetilde{P} = P \otimes_{\mathbb{Z}} \mathbb{Z}_2$ is permutation and $\widetilde{F} = F \otimes_{\mathbb{Z}} \mathbb{Z}_2$ is flabby. Take a direct sum decomposition $\widetilde{F} \simeq \widetilde{N} \oplus \widetilde{Q}$ where $\widetilde{N}$ does not contain a permutation direct summand and $\widetilde{Q}$ is permutation.
(2) Take a flabby resolution of $M_p = M\mid_{\mathrm{Syl}_p(G)}$ as $\mathbb{Z}[\mathrm{Syl}_p(G)]$-lattices:

$$0 \to M_p \to P_p \to F_p \to 0$$

where P_p is permutation and F_p is flabby. Then $[F_p] = [F\mid_{\mathrm{Syl}_p(G)}]$ and take the direct sum decomposition $\widetilde{F}_p = F_p \otimes_{\mathbb{Z}} \mathbb{Z}_p \simeq \widetilde{N}_p \oplus \widetilde{Q}_p$ as $\mathbb{Z}_p[\mathrm{Syl}_p(G)]$-lattices where $\widetilde{N}_p$ does not contain a permutation direct summand and $\widetilde{Q}_p$ is permutation.

Example ($\widetilde{N}$ is not uniquely determined)

- $G = I_{4,1} \leq \mathrm{GL}(4,\mathbb{Z})$ with $G \simeq F_{20}$.
- M_G is the corresponding G-lattice with rank $M_G = 4$.
- Take $\widetilde{M} = M_G \otimes \mathbb{Z}_2$.
- Then we see that $\widetilde{M}$ is an indecomposable $\mathbb{Z}_2[G]$-lattice and $\widetilde{M} \oplus \mathbb{Z}_2 \simeq \mathbb{Z}_2[G/C_4]$ as $\mathbb{Z}_2[G]$-lattices.
- In particular, $\widetilde{M}^{\oplus r}$ does not contain a permutatin direct summand for any $r \geq 1$.
- We get that $F = [M_G]^{fl}$ with rank $F = 16$ and $\widetilde{F} = F \otimes_{\mathbb{Z}} \mathbb{Z}_2 \simeq \widetilde{M} \oplus \widetilde{M} \oplus \widetilde{M} \oplus \mathbb{Z}_2[G/C_5]$ as $\mathbb{Z}_2[G]$-lattices with $\widetilde{N} \simeq \widetilde{M} \oplus \widetilde{M} \oplus \widetilde{M}$.
- On the other hand, we also see that $\widetilde{F} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \simeq \mathbb{Z}_2[G/C_4] \oplus \mathbb{Z}_2[G/C_4] \oplus \mathbb{Z}_2[G/C_4] \oplus \mathbb{Z}_2[G/C_5]$ and hence $\widetilde{N} = 0$.
- $\widetilde{N}$ is not uniquely determined in general when G is not a 2-group.

- Retract non-rationality of an algebraic k-torus T can be detected by the non-vanishing of $\widetilde{N}_p$.

Lemma ([HY, Lemma 7.5])

Let M be a G-lattice and $\widetilde{N}_p$ be the p-part of $[M]^{fl}$ as a $\mathbb{Z}_p[\mathrm{Syl}_p(G)]$-lattice. If $\widetilde{N}_p \neq 0$, then $[M]^{fl}$ is not invertible. In particular, the corresponding torus T with $\widehat{T} = M$ is not retract k-rational.

- The following theorem is given by Kunyavskii (1990) except for the indecomposability of $\widetilde{N}_2$.

Theorem ([HY, Theorem 7.14], see also Kunyavskii (1990))

Let $G = N_{3,i}$ $(1 \leq i \leq 15)$ and M_G be the corresponding G-lattice. The 2-part $\widetilde{N}_2$ of $[M_G]^{fl}$ as a $\mathbb{Z}_2[\mathrm{Syl}_2(G)]$-lattice is a faithful and indecomposable $\mathbb{Z}_2[\mathrm{Syl}_2(G)]$-lattice and the $\mathbb{Z}_2$-rank of $\widetilde{N}_2$ is given as in [HY, Table 14].

Theorem ([HY, Theorem 7.15])

Let $G = N_{4,i}$ $(1 \leq i \leq 152)$ and M_G be the corresponding G-lattice. The 2-part $\widetilde{N}_2$ (resp. 3-part $\widetilde{N}_3$) of $[M_G]^{fl}$ as a $\mathbb{Z}_2[\mathrm{Syl}_2(G)]$-lattice (resp. $\mathbb{Z}_3[\mathrm{Syl}_3(G)]$-lattice) is a faithful and indecomposable $\mathbb{Z}_2[\mathrm{Syl}_2(G)]$-lattice (resp. $\mathbb{Z}_3[\mathrm{Syl}_3(G)]$-lattice) unless it vanishes and the $\mathbb{Z}_2$-rank of $\widetilde{N}_2$ (resp. the $\mathbb{Z}_3$-rank of $\widetilde{N}_3$) is given as in [HY, Table 15].