Birational classification for algebraic tori (joint work with Aiichi Yamasaki)

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1. Rationality problem for algebraic k-tori T

   [HY17] A. Hoshi, A. Yamasaki,
   Rationality problem for algebraic tori,
   Mem. Amer. Math. Soc. 248 (2017), no. 1176, v+215 pp.

   $+$ Hasse norm principle (HNP) for K/k (via T. Ono's theorem)
   [HKY22], [HKY23] A. Hoshi, K. Kanai, A. Yamasaki.

2. Birational classification for algebraic k-tori T

   [HY] A. Hoshi, A. Yamasaki,
   Birational classification for algebraic tori, 175 pages,
   arXiv:2112.02280.

§1. Rationality problem for algebraic tori T (1/3)

- k: a base field which is NOT algebraically closed! (TODAY)
- T: algebraic k-torus, i.e. k-form of a split torus;
  an algebraic group over k (group k-scheme) with $T \times_k \overline{k} \simeq (\mathbb{G}_{m,\overline{k}})^n$.

Rationality problem for algebraic tori

Whether T is k-rational?, i.e. $T \approx \mathbb{P}^n$? (birationally k-equivalent)

Let $R^{(1)}_{K/k}(\mathbb{G}_m)$ be the norm one torus of K/k, i.e. the kernel of the norm map $N_{K/k} : R_{K/k}(\mathbb{G}_m) \to \mathbb{G}_m$ where $R_{K/k}$ is the Weil restriction:

$$1 \longrightarrow R^{(1)}_{K/k}(\mathbb{G}_m) \longrightarrow R_{K/k}(\mathbb{G}_m) \xrightarrow{N_{K/k}} \mathbb{G}_m \longrightarrow 1.$$

$$\dim \qquad n-1 \qquad\qquad n \qquad\qquad 1$$

- $\exists 2$ algebraic k-tori T with $\dim(T) = 1$;
  the trivial torus $\mathbb{G}_m$ and $R^{(1)}_{K/k}(\mathbb{G}_m)$ with $[K : k] = 2$, are k-rational.

Rationality problem for algebraic tori T (2/3)

- $\exists 13$ algebraic k-tori T with $\dim(T) = 2$.

Theorem (Voskresenskii 1967) 2-dim. algebraic tori T

T is k-rational.

- $\exists 73$ algebraic k-tori T with $\dim(T) = 3$.

Theorem (Kunyavskii 1990) 3-dim. algebraic tori T

(i) $\exists 58$ algebraic k-tori T which are k-rational;
(ii) $\exists 15$ algebraic k-tori T which are not k-rational.

- What happens in higher dimensions?

Algebraic k-tori T and G-lattices

- T: algebraic k-torus
  $\Longrightarrow \exists$ finite Galois extension L/k such that $T \times_k L \simeq (\mathbb{G}_{m,L})^n$.
- $G = \mathrm{Gal}(L/k)$ where L is the minimal splitting field.

Category of algebraic k-tori which split$/L$ $\overset{\text{duality}}{\longleftrightarrow}$ Category of G-lattices
(i.e. finitely generated $\mathbb{Z}$-free $\mathbb{Z}[G]$-module)

- $T \mapsto$ the character group $\widehat{T} = \mathrm{Hom}(T, \mathbb{G}_m)$: G-lattice.
- $T = \mathrm{Spec}(L[M]^G)$ which splits$/L$ with $\widehat{T} \simeq M \hookleftarrow M$: G-lattice
- Tori of dimension $n \overset{1:1}{\longleftrightarrow}$ elements of the set $H^1(\mathcal{G}, \mathrm{GL}(n, \mathbb{Z}))$
  where $\mathcal{G} = \mathrm{Gal}(\overline{k}/k)$ since $\mathrm{Aut}(\mathbb{G}_m^n) = \mathrm{GL}(n, \mathbb{Z})$.
- k-torus T of dimension n is determined uniquely by the integral representation $h : \mathcal{G} \to \mathrm{GL}(n, \mathbb{Z})$ up to conjugacy, and the group $h(\mathcal{G})$ is a finite subgroup of $\mathrm{GL}(n, \mathbb{Z})$.
- The function field of T $\overset{\text{identified}}{\longleftrightarrow}$ $L(M)^G$: invariant field.

Rationality problem for algebraic tori T (3/3)

- L/k: Galois extension with $G = \mathrm{Gal}(L/k)$.
- $M = \bigoplus_{1 \leq j \leq n} \mathbb{Z} \cdot u_j$: G-lattice with a $\mathbb{Z}$-basis $\{u_1, \ldots, u_n\}$.
- G acts on $L(x_1, \ldots, x_n)$ by

$$\sigma(x_j) = \prod_{i=1}^{n} x_i^{a_{i,j}}, \quad 1 \leq j \leq n$$

for any $\sigma \in G$, when $\sigma(u_j) = \sum_{i=1}^{n} a_{i,j} u_i$, $a_{i,j} \in \mathbb{Z}$.
- $L(M) := L(x_1, \ldots, x_n)$ with this action of G.
- $\boxed{\text{The function field of algebraic } k\text{-torus } T \overset{\text{identified}}{\longleftrightarrow} L(M)^G}$

Rationality problem for algebraic tori T (2nd form)

Whether $L(M)^G$ is k-rational?
($=$ purely transcendental over k?; $L(M)^G = k(\exists t_1, \ldots, \exists t_n)$?)

Some definitions.

- K/k: a finite generated field extension.

Definition (stably rational)

K is called stably k-rational if $K(y_1, \ldots, y_m)$ is k-rational.

Definition (retract rational)

K is retract k-rational if $\exists k$-algebra (domain) $R \subset K$ such that
(i) K is the quotient field of R;
(ii) $\exists f \in k[x_1, \ldots, x_n]$ $\exists k$-algebra hom. $\varphi : R \to k[x_1, \ldots, x_n][1/f]$ and $\psi : k[x_1, \ldots, x_n][1/f] \to R$ satisfying $\psi \circ \varphi = 1_R$.

Definition (unirational)

K is k-unirational if $K \subset k(x_1, \ldots, x_n)$.

- k-rational $\Rightarrow$ stably k-rational $\Rightarrow$ retract k-rational $\Rightarrow$ k-unirational.
- $L(M)^G$ (resp. T) is always k-unirational.

Rationality problem for algebraic tori T (2-dim., 3-dim.)

- $\boxed{\text{The function field of } n\text{-dim. } T \overset{\text{identified}}{\longleftrightarrow} L(M)^G,\ G \leq \mathrm{GL}(n,\mathbb{Z})}$
- $\exists 13$ $\mathbb{Z}$-coujugacy subgroups $G \leq \mathrm{GL}(2,\mathbb{Z})$ ($\exists 13$ 2-dim. algebraic k-tori T).

Theorem (Voskresenskii 1967) 2-dim. algebraic tori T (restated)

T is k-rational.

- $\exists 73$ $\mathbb{Z}$-coujugacy subgroups $G \leq \mathrm{GL}(3,\mathbb{Z})$ ($\exists 73$ 3-dim. algebraic k-tori T).

Theorem (Kunyavskii 1990) 3-dim. algebraic tori T (precise form)

(i) T is k-rational $\iff$ T is stably k-rational
$\iff$ T is retract k-rational $\iff$ $\exists G$: 58 groups;
(ii) T is not k-rational $\iff$ T is not stably k-rational
$\iff$ T is not retract k-rational $\iff$ $\exists G$: 15 groups.

Rationality problem for algebraic tori T (4-dim.)

- $\boxed{\text{The function field of } n\text{-dim. } T \overset{\text{identified}}{\longleftrightarrow} L(M)^G,\ G \leq \mathrm{GL}(n,\mathbb{Z})}$
- $\exists 710$ $\mathbb{Z}$-coujugacy subgroups $G \leq \mathrm{GL}(4,\mathbb{Z})$
  ($\exists 710$ 4-dim. algebraic k-tori T).

Theorem ([HY17]) 4-dim. algebraic tori T

(i) T is stably k-rational $\iff$ $\exists G$: 487 groups;
(ii) T is not stably but retract k-rational $\iff$ $\exists G$: 7 groups;
(iii) T is not retract k-rational $\iff$ $\exists G$: 216 groups.

- We do not know "k-rationality".
- Voskresenskii's conjecture:
  any stably k-rational torus is k-rational (Zariski problem).
- what happens for dimension 5?

Rationality problem for algebraic tori T (5-dim.)

- $\boxed{\text{The function field of } n\text{-dim. } T \overset{\text{identified}}{\longleftrightarrow} L(M)^G,\ G \leq \mathrm{GL}(n,\mathbb{Z})}$
- $\exists 6079$ $\mathbb{Z}$-coujugacy subgroups $G \leq \mathrm{GL}(5,\mathbb{Z})$ ($\exists 6079$ 5-dim. algebraic k-tori T).

Theorem ([HY17]) 5-dim. algebraic tori T

(i) T is stably k-rational $\iff$ $\exists G$: 3051 groups;
(ii) T is not stably but retract k-rational $\iff$ $\exists G$: 25 groups;
(iii) T is not retract k-rational $\iff$ $\exists G$: 3003 groups.

- what happens for dimension 6?
- BUT we do not know the answer for dimension 6.
- $\exists 85308$ $\mathbb{Z}$-coujugacy subgroups $G \leq \mathrm{GL}(6,\mathbb{Z})$ ($\exists 85308$ 6-dim. algebraic k-tori T).

Flabby (Flasque) resolution

- M: G-lattice, i.e. f.g. $\mathbb{Z}$-free $\mathbb{Z}[G]$-module.

Definition

(i) M is permutation $\overset{\text{def}}{\iff}$ $M \simeq \oplus_{1\leq i\leq m}\mathbb{Z}[G/H_i]$.

(ii) M is stably permutation $\overset{\text{def}}{\iff}$ $M \oplus \exists P \simeq P'$, P, P': permutation.

(iii) M is invertible $\overset{\text{def}}{\iff}$ $M \oplus \exists M' \simeq P$: permutation.

(iv) M is coflabby $\overset{\text{def}}{\iff}$ $H^1(H,M)=0$ $(\forall H \leq G)$.

(v) M is flabby $\overset{\text{def}}{\iff}$ $\widehat{H}^{-1}(H,M)=0$ $(\forall H \leq G)$. ($\widehat{H}$: Tate cohomology)

- "permutation"
  $\Longrightarrow$ "stably permutation"
  $\Longrightarrow$ "invertible"
  $\Longrightarrow$ "flabby and coflabby".

Commutative monoid $\mathcal{M}$

$M_1 \sim M_2 \overset{\text{def}}{\iff} M_1 \oplus P_1 \simeq M_2 \oplus P_2$ ($\exists P_1, \exists P_2$: permutation).
$\Longrightarrow$ commutative monoid $\mathcal{M}$: $[M_1] + [M_2] := [M_1 \oplus M_2]$, $0 = [P]$.

Theorem (Endo-Miyata 1974, Colliot-Thélène-Sansuc 1977)

$\exists P$: permutation, $\exists F$: flabby such that

$0 \to M \to P \to F \to 0$: flabby resolution of M.

- $[M]^{fl} := [F]$; flabby class of M

Theorem (Endo-Miyata 1973, Voskresenskii 1974, Saltman 1984)

(EM73) $[M]^{fl} = 0 \iff L(M)^G$ is stably k-rational.
(Vos74) $[M]^{fl} = [M']^{fl} \iff L(M)^G(x_1, \ldots, x_m) \simeq L(M')^G(y_1, \ldots, y_n)$;
stably k-equivalent.
(Sal84) $[M]^{fl}$ is invertible $\iff L(M)^G$ is retract k-rational.

- $M = M_G \simeq \widehat{T} = \mathrm{Hom}(T, \mathbb{G}_m)$, $k(T) \simeq L(M)^G$, $G = \mathrm{Gal}(L/k)$

Contributions of [HY17]

- We give a procedure to compute a flabby resolution of M, in particular $[M]^{fl} = [F]$, effectively (with smaller rank after base change) by computer software GAP.
- The function `IsFlabby` (resp. `IsCoflabby`) may determine whether M is flabby (resp. coflabby).
- The function `IsInvertibleF` may determine whether $[M]^{fl} = [F]$ is invertible ($\leftrightarrow$ whether $L(M)^G$ (resp. T) is retract rational).
- We provide some functions for checking a possibility of isomorphism

$$\left(\bigoplus_{i=1}^{r} a_i\,\mathbb{Z}[G/H_i]\right) \oplus a_{r+1}F \simeq \bigoplus_{i=1}^{r} b_i'\,\mathbb{Z}[G/H_i] \qquad (*)$$

  by computing some invariants (e.g. trace, $\widehat{Z}^0$, $\widehat{H}^0$) of both sides.
- [HY17, Example 10.7]. $G \simeq S_5 \leq \mathrm{GL}(5,\mathbb{Z})$ with number $(5,946,4)$
  $\Longrightarrow \mathrm{rank}(F) = 17$ and $\mathrm{rank}(*) = 88$ holds
  $\Longrightarrow [F] = 0 \Longrightarrow L(M)^G$ (resp. T) is stably rational over k.

Application to Krull-Schmidt

Corollary ($[F]=[M]^{fl}$: invertible case, $G\simeq S_5, F_{20}$)

$\exists T$, T'; 4-dim. not stably rational algebraic tori over k such that $T\not\approx T'$ (birational) and $T\times T'$: 8-dim. stably rational over k.
$\because -[M]^{fl}=[M']^{fl}\neq 0$.

Prop. ([HY17], Krull-Schmidt fails for permutation D_6-lattices)

$\{1\}$, $C_2^{(1)}$, $C_2^{(2)}$, $C_2^{(3)}$, C_3, V_4, C_6, $S_3^{(1)}$, $S_3^{(2)}$, D_6: conj. subgroups of D_6.

$$\mathbb{Z}[D_6]\oplus\mathbb{Z}[D_6/V_4]^{\oplus 2}\oplus\mathbb{Z}[D_6/C_6]\oplus\mathbb{Z}[D_6/S_3^{(1)}]\oplus\mathbb{Z}[D_6/S_3^{(2)}]$$
$$\simeq\mathbb{Z}[D_6/C_2^{(1)}]\oplus\mathbb{Z}[D_6/C_2^{(2)}]\oplus\mathbb{Z}[D_6/C_2^{(3)}]\oplus\mathbb{Z}[D_6/C_3]\oplus\mathbb{Z}^{\oplus 2}.$$

- D_6 is the smallest example exhibiting the failure of K-S:

Theorem (Dress 1973)

Krull-Schmidt holds for permutation G-lattices $\iff$ $G/O_p(G)$ is cyclic where $O_p(G)$ is the maximal normal p-subgroup of G.

Krull-Schmidt and Direct sum cancelation

Theorem (Hindman-Klingler-Odenthal 1998) Assume $G \neq D_8$

Krull-Schmidt holds for G-lattices $\iff$ (i) $G = C_p$ ($p \leq 19$; prime), (ii) $G = C_n$ ($n = 1, 4, 8, 9$), (iii) $G = V_4$ or (iv) $G = D_4$.

Theorem (Endo-Hironaka 1979)

Direct sum cancellation holds, i.e. $M_1 \oplus N \simeq M_2 \oplus N \Longrightarrow M_1 \simeq M_2$, $\Longrightarrow G$ is abelian, dihedral, A_4, S_4 or A_5 (*).

- via projective class group (see Swan 1988, Corollary 1.3, Section 7).
- Except for (*) $\Longrightarrow$ Direct sum cancelation fails $\Longrightarrow$ K-S fails

Theorem ([HY17]) $G \leq \mathrm{GL}(n, \mathbb{Z})$ (up to conjugacy)

(i) $n \leq 4 \Longrightarrow$ K-S holds.
(ii) $n = 5$. K-S fails $\iff$ 11 groups G (among 6079 groups).
(iii) $n = 6$. K-S fails $\iff$ 131 groups G (among 85308 groups).

Special case: $T = R^{(1)}_{K/k}(\mathbb{G}_m)$; norm one tori (1/5)

- Rationality problem for $T = R^{(1)}_{K/k}(\mathbb{G}_m)$ is investigated by S. Endo, Colliot-Thélène and Sansuc, W. Hürlimann, L. Le Bruyn, A. Cortella and B. Kunyavskii, N. Lemire and M. Lorenz, M. Florence, etc.

Theorem (Endo-Miyata 1974), (Saltman 1984)

Let K/k be a finite Galois field extension and $G = \mathrm{Gal}(K/k)$.
(i) T is retract k-rational $\iff$ all the Sylow subgroups of G are cyclic;
(ii) T is stably k-rational $\iff$ G is a cyclic group, or a direct product of a cyclic group of order m and a group $\langle \sigma, \tau \,|\, \sigma^n = \tau^{2^d} = 1, \tau\sigma\tau^{-1} = \sigma^{-1} \rangle$, where $d, m \geq 1, n \geq 3, m, n$: odd, and $(m, n) = 1$.

Theorem (Endo 2011)

Let K/k be a finite non-Galois, separable field extension and L/k be the Galois closure of K/k. Assume that the Galois group of L/k is nilpotent. Then the norm one torus $T = R^{(1)}_{K/k}(\mathbb{G}_m)$ is not retract k-rational.

Special case: $T = R^{(1)}_{K/k}(\mathbb{G}_m)$; norm one tori (2/5)

- Let K/k be a finite non-Galois, separable field extension
- Let L/k be the Galois closure of K/k.
- Let $G = \mathrm{Gal}(L/k)$ and $H = \mathrm{Gal}(L/K) \leq G$.

Theorem (Endo 2011)

Assume that all the Sylow subgroups of G are cyclic.
Then T is retract k-rational.
$T = R^{(1)}_{K/k}(\mathbb{G}_m)$ is stably k-rational $\iff$ $G = D_n$, n odd $(n \geq 3)$ or $C_m \times D_n$, m, n odd $(m, n \geq 3)$, $(m, n) = 1$, $H \leq D_n$ with $|H| = 2$.

Special case: $T = R^{(1)}_{K/k}(\mathbb{G}_m)$; norm one tori (3/5)

Theorem (Endo 2011) $\dim T = n-1$

Assume that $\mathrm{Gal}(L/k) = S_n$, $n \geq 3$, and $\mathrm{Gal}(L/K) = S_{n-1}$ is the stabilizer of one of the letters in S_n.

(i) $R^{(1)}_{K/k}(\mathbb{G}_m)$ is retract k-rational $\iff$ n is a prime;

(ii) $R^{(1)}_{K/k}(\mathbb{G}_m)$ is (stably) k-rational $\iff$ $n = 3$.

Theorem (Endo 2011) $\dim T = n-1$

Assume that $\mathrm{Gal}(L/k) = A_n$, $n \geq 4$, and $\mathrm{Gal}(L/K) = A_{n-1}$ is the stabilizer of one of the letters in A_n.

(i) $R^{(1)}_{K/k}(\mathbb{G}_m)$ is retract k-rational $\iff$ n is a prime;

(ii) $\exists t \in \mathbb{N}$ s.t. $[R^{(1)}_{K/k}(\mathbb{G}_m)]^{(t)}$ is stably k-rational $\iff$ $n = 5$.

- $[R^{(1)}_{K/k}(\mathbb{G}_m)]^{(t)}$: the product of t copies of $R^{(1)}_{K/k}(\mathbb{G}_m)$.

Special case: $T = R^{(1)}_{K/k}(\mathbb{G}_m)$; norm one tori (4/5)

Theorem ([HY17], Rationality for $R^{(1)}_{K/k}(\mathbb{G}_m)$ (dim. 4, $[K:k]=5$))

Let K/k be a separable field extension of degree 5 and L/k be the Galois closure of K/k. Assume that $G = \mathrm{Gal}(L/k)$ is a transitive subgroup of S_5 and $H = \mathrm{Gal}(L/K)$ is the stabilizer of one of the letters in G. Then the rationality of $R^{(1)}_{K/k}(\mathbb{G}_m)$ is given by

G		$L(M) = L(x_1, x_2, x_3, x_4)^G$
$5T1$	C_5	stably k-rational
$5T2$	D_5	stably k-rational
$5T3$	F_{20}	not stably but retract k-rational
$5T4$	A_5	stably k-rational
$5T5$	S_5	not stably but retract k-rational

- This theorem is already known except for the case of A_5 (Endo).
- Stably k-rationality for the case A_5 is asked by S. Endo (2011).

Special case: $T = R^{(1)}_{K/k}(\mathbb{G}_m)$; norm one tori (5/5)

Corollary of (Endo 2011) and [HY17]

Assume that $\mathrm{Gal}(L/k) = A_n$, $n \geq 4$, and $\mathrm{Gal}(L/K) = A_{n-1}$ is the stabilizer of one of the letters in A_n. Then $R^{(1)}_{K/k}(\mathbb{G}_m)$ is stably k-rational $\iff$ $n = 5$.

More recent results on stably/retract k-rational classification for T

- $G \leq S_n$ $(n \leq 10)$ and $G \neq 9T27 \simeq PSL_2(\mathbb{F}_8)$, $G \leq S_p$ and $G \neq PSL_2(\mathbb{F}_{2^e})$ $(p = 2^e + 1 \geq 17$; Fermat prime$)$ (Hoshi-Yamasaki [HY21] Israel J. Math.)
- $G \leq S_n$ $(n = 12, 14, 15)$ $(n = 2^e)$ (Hasegawa-Hoshi-Yamasaki [HHY20] Math. Comp.)

$Ш(T)$ and Hasse norm principle over number fields k (see next slides)

- (Hoshi-Kanai-Yamasaki [HKY22] Math. Comp., [HKY23] JNT)

$\text{Ш}(T)$ and HNP for K/k: Ono's theorem (1963)

- T : algebraic k-torus, i.e. $T \times_k \overline{k} \simeq (\mathbb{G}_{m,\overline{k}})^n$.
- $\text{Ш}(T) := \mathrm{Ker}\{H^1(k,T) \xrightarrow{\mathrm{res}} \bigoplus_{v \in V_k} H^1(k_v,T)\}$: Shafarevich-Tate gp.
- $T = R^{(1)}_{K/k}(\mathbb{G}_m)$ is biregularly isomorphic to the norm hyper surface $f(x_1,\ldots,x_n) = 1$ where $f \in k[x_1,\ldots,x_n]$ is the norm form of K/k.

Theorem (Ono 1963, Ann. of Math.)

Let K/k be a finite extension of number fields and $T = R^{(1)}_{K/k}(\mathbb{G}_m)$. Then

$$\text{Ш}(T) \simeq (N_{K/k}(\mathbb{A}_K^\times) \cap k^\times)/N_{K/k}(K^\times)$$

where $\mathbb{A}_K^\times$ is the idele group of K. In particular,

$$\text{Ш}(T) = 0 \iff \text{Hasse norm principle holds for } K/k.$$

Known results for HNP (2/2)

- $T = R^{(1)}_{K/k}(\mathbb{G}_m)$.
- $\text{Ш}(T) = 0 \iff$ Hasse norm principle holds for K/k.

Theorem (Kunyavskii 1984)

Let $[K : k] = 4$, $G = \mathrm{Gal}(L/k) \simeq 4Tm$ $(1 \leq m \leq 5)$.
Then $\text{Ш}(T) = 0$ *except for* $4T2$ *and* $4T4$. *For* $4T2 \simeq V_4$, $4T4 \simeq A_4$,
(i) $\text{Ш}(T) \leq \mathbb{Z}/2\mathbb{Z}$;
(ii) $\text{Ш}(T) = 0 \Leftrightarrow {}^{\exists}v \in V_k$ *such that* $V_4 \leq G_v$.

Theorem (Drakokhrust-Platonov 1987)

Let $[K : k] = 6$, $G = \mathrm{Gal}(L/k) \simeq 6Tm$ $(1 \leq m \leq 16)$.
Then $\text{Ш}(T) = 0$ except for $6T4$ and $6T12$. For $6T4 \simeq A_4$, $6T12 \simeq A_5$,
(i) $\text{Ш}(T) \leq \mathbb{Z}/2\mathbb{Z}$;
(ii) $\text{Ш}(T) = 0 \Leftrightarrow {}^{\exists}v \in V_k$ such that $V_4 \leq G_v$.

Voskresenskii's theorem (1969) (1/2)

- Let X be a smooth k-compactification of an algebraic k-torus T

Theorem (Voskresenskii 1969)

Let k be a global field, T be an algebraic k-torus and X be a smooth k-compactification of T. Then there exists an exact sequence

$$0 \to A(T) \to H^1(k, \mathrm{Pic}\,\overline{X})^\vee \to Ш(T) \to 0$$

where $M^\vee = \mathrm{Hom}(M, \mathbb{Q}/\mathbb{Z})$ is the Pontryagin dual of M.

- The group $A(T) := \left(\prod_{v \in V_k} T(k_v)\right) \Big/ \overline{T(k)}$ is called the kernel of the weak approximation of T.
- T : retract rational $\iff$ $[\widehat{T}]^{fl} = [\mathrm{Pic}\,\overline{X}]$ is invertible
  $\Longrightarrow$ $\mathrm{Pic}\,\overline{X}$ is flabby and coflabby
  $\Longrightarrow$ $H^1(k, \mathrm{Pic}\,\overline{X})^\vee = 0 \quad \Longrightarrow \quad A(T) = Ш(T) = 0.$
- when $T = R^{(1)}_{K/k}(\mathbb{G}_m)$, by Ono's theorem,
  T : retract k-rational $\Longrightarrow Ш(T) = 0$ (HNP holds for K/k).

Voskresenskii's theorem (1969) (2/2)

- when $T = R^{(1)}_{K/k}(\mathbb{G}_m)$, $\widehat{T} = J_{G/H}$ where $J_{G/H} = (I_{G/H})^\circ = \mathrm{Hom}(I_{G/H}, \mathbb{Z})$ is the dual lattice of $I_{G/H} = \mathrm{Ker}(\varepsilon)$ and $\varepsilon : \mathbb{Z}[G/H] \to \mathbb{Z}$ is the augmentation map.
- (Hasegawa-Hoshi-Yamasaki [HHY20], Hoshi-Yamasaki [HY21]) For $[K:k] = n \leq 15$ except $9T27 \simeq \mathrm{PSL}_2(\mathbb{F}_8)$, the classificasion of stably/retract rational $R^{(1)}_{K/k}(\mathbb{G}_m)$ was given.
- when $T = R^{(1)}_{K/k}(\mathbb{G}_m)$, T : retract k-rational $\Longrightarrow H^1(k, \mathrm{Pic}\,\overline{X}) = 0$
- $H^1(k, \mathrm{Pic}\,\overline{X}) \simeq \mathrm{Br}(X)/\mathrm{Br}(k) \simeq \mathrm{Br}_{\mathrm{nr}}(k(X)/k)/\mathrm{Br}(k)$
  by Colliot-Thélène-Sansuc 1987
  where $\mathrm{Br}(X)$ is the étale cohomological/Azumaya Brauer group of X and $\mathrm{Br}_{\mathrm{nr}}(k(X)/k)$ is the unramified Brauer group of $k(X)$ over k.

Theorems 1,2,3,4 in [HKY22], [HKY23] (1/3)

- $\exists\ 2, 13, 73, 710, 6079$ cases of alg. k-tori T of $\dim(T) = 1, 2, 3, 4, 5$.

Theorem 1 ([HKY22, Theorem 1.5 and Theorem 1.6])

(i) $\dim(T) = 4$. Among the 216 cases (of 710) of not retract rational T,

$$H^1(k, \operatorname{Pic} \overline{X}) \simeq \begin{cases} 0 & (194 \text{ of } 216), \\ \mathbb{Z}/2\mathbb{Z} & (20 \text{ of } 216), \\ (\mathbb{Z}/2\mathbb{Z})^{\oplus 2} & (2 \text{ of } 216). \end{cases}$$

(ii) $\dim(T) = 5$. Among 3003 cases (of 6079) of not retract rational T,

$$H^1(k, \operatorname{Pic} \overline{X}) \simeq \begin{cases} 0 & (2729 \text{ of } 3003), \\ \mathbb{Z}/2\mathbb{Z} & (263 \text{ of } 3003), \\ (\mathbb{Z}/2\mathbb{Z})^{\oplus 2} & (11 \text{ of } 3003). \end{cases}$$

- Kunyavskii (1984) showed that among the 15 cases (of 73) of not retract rational T of $\dim(T) = 3$, $H^1(k, \operatorname{Pic} \overline{X}) = 0$ $(13$ of $15)$, $H^1(k, \operatorname{Pic} \overline{X}) \simeq \mathbb{Z}/2\mathbb{Z}$ $(2$ of $15)$.

Theorems 1,2,3,4 in [HKY22], [HKY23] (2/3)

- k : a field, K/k : a separable field extension of $[K:k]=n$.
- $T=R^{(1)}_{K/k}(\mathbb{G}_m)$ with $\dim(T)=n-1$.
- X : a smooth k-compactification of T.
- L/k : Galois closure of K/k, $G:=\mathrm{Gal}(L/k)$ and $H=\mathrm{Gal}(L/K)$ with $[G:H]=n \Longrightarrow G=nTm\leq S_n$: transitive.
- The number of transitive subgroups nTm of S_n $(2\leq n\leq 15)$ up to conjugacy is given as follows:

n	2	3	4	5	6	7	8	9	10	11	12	13	14	15
# of nTm	1	2	5	5	16	7	50	34	45	8	301	9	63	104

Theorem 2 ([HKY22, Theorem 1.5], [HKY23, Theorem 1.1])

Let $2\leq n\leq 15$ be an integer. Then $H^1(k,\mathrm{Pic}\,\overline{X})\neq 0 \Longleftrightarrow G=nTm$ is given as in [HKY22, Table 1] $(n\neq 12)$ or [HKY23,Table 1] $(n=12)$.

[HKY22, Table 1]: $H^1(k, \mathrm{Pic}\,\overline{X}) \simeq H^1(G, [J_{G/H}]^{fl}) \neq 0$ where $G = nTm$ with $2 \leq n \leq 15$ and $n \neq 12$

G	$H^1(k, \mathrm{Pic}\,\overline{X}) \simeq H^1(G, [J_{G/H}]^{fl})$
$4T2 \simeq V_4$	$\mathbb{Z}/2\mathbb{Z}$
$4T4 \simeq A_4$	$\mathbb{Z}/2\mathbb{Z}$
$6T4 \simeq A_4$	$\mathbb{Z}/2\mathbb{Z}$
$6T12 \simeq A_5$	$\mathbb{Z}/2\mathbb{Z}$
$8T2 \simeq C_4 \times C_2$	$\mathbb{Z}/2\mathbb{Z}$
$8T3 \simeq (C_2)^3$	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 3}$
$8T4 \simeq D_4$	$\mathbb{Z}/2\mathbb{Z}$
$8T9 \simeq D_4 \times C_2$	$\mathbb{Z}/2\mathbb{Z}$
$8T11 \simeq (C_4 \times C_2) \rtimes C_2$	$\mathbb{Z}/2\mathbb{Z}$
$8T13 \simeq A_4 \times C_2$	$\mathbb{Z}/2\mathbb{Z}$
$8T14 \simeq S_4$	$\mathbb{Z}/2\mathbb{Z}$
$8T15 \simeq C_8 \rtimes V_4$	$\mathbb{Z}/2\mathbb{Z}$
$8T19 \simeq (C_2)^3 \rtimes C_4$	$\mathbb{Z}/2\mathbb{Z}$
$8T21 \simeq (C_2)^3 \rtimes C_4$	$\mathbb{Z}/2\mathbb{Z}$
$8T22 \simeq (C_2)^3 \rtimes V_4$	$\mathbb{Z}/2\mathbb{Z}$
$8T31 \simeq ((C_2)^4 \rtimes C_2) \rtimes C_2$	$\mathbb{Z}/2\mathbb{Z}$
$8T32 \simeq ((C_2)^3 \rtimes V_4) \rtimes C_3$	$\mathbb{Z}/2\mathbb{Z}$
$8T37 \simeq \mathrm{PSL}_3(\mathbb{F}_2) \simeq \mathrm{PSL}_2(\mathbb{F}_7)$	$\mathbb{Z}/2\mathbb{Z}$
$8T38 \simeq (((C_2)^4 \rtimes C_2) \rtimes C_2) \rtimes C_3$	$\mathbb{Z}/2\mathbb{Z}$

[HKY22, Table 1]: $H^1(k, \mathrm{Pic}\,\overline{X}) \simeq H^1(G, [J_{G/H}]^{fl}) \neq 0$
where $G = nTm$ with $2 \leq n \leq 15$ and $n \neq 12$

G	$H^1(k, \mathrm{Pic}\,\overline{X}) \simeq H^1(G, [J_{G/H}]^{fl})$
$9T2 \simeq (C_3)^2$	$\mathbb{Z}/3\mathbb{Z}$
$9T5 \simeq (C_3)^2 \rtimes C_2$	$\mathbb{Z}/3\mathbb{Z}$
$9T7 \simeq (C_3)^2 \rtimes C_3$	$\mathbb{Z}/3\mathbb{Z}$
$9T9 \simeq (C_3)^2 \rtimes C_4$	$\mathbb{Z}/3\mathbb{Z}$
$9T11 \simeq (C_3)^2 \rtimes C_6$	$\mathbb{Z}/3\mathbb{Z}$
$9T14 \simeq (C_3)^2 \rtimes Q_8$	$\mathbb{Z}/3\mathbb{Z}$
$9T23 \simeq ((C_3)^2 \rtimes Q_8) \rtimes C_3$	$\mathbb{Z}/3\mathbb{Z}$
$10T7 \simeq A_5$	$\mathbb{Z}/2\mathbb{Z}$
$10T26 \simeq \mathrm{PSL}_2(\mathbb{F}_9) \simeq A_6$	$\mathbb{Z}/2\mathbb{Z}$
$10T32 \simeq S_6$	$\mathbb{Z}/2\mathbb{Z}$
$14T30 \simeq \mathrm{PSL}_2(\mathbb{F}_{13})$	$\mathbb{Z}/2\mathbb{Z}$
$15T9 \simeq (C_5)^2 \rtimes C_3$	$\mathbb{Z}/5\mathbb{Z}$
$15T14 \simeq (C_5)^2 \rtimes S_3$	$\mathbb{Z}/5\mathbb{Z}$

Theorems 1,2,3,4 in [HKY22], [HKY23] (3/3)

- k : a number field, K/k : a separable field extension of $[K:k]=n$.
- $T=R^{(1)}_{K/k}(\mathbb{G}_m)$, X : a smooth k-compactification of T.

Theorem 3 ([HKY22, Theorem 1.18], [HKY23, Theorem 1.3])

Let $2\leq n\leq 15$ be an integer. For the cases in [HKY22, Table 1] $(n\leq 15, n\neq 12)$ or [HKY23,Table 1] $(n=12)$,

$$\text{Ш}(T)=0 \iff G=nTm \text{ satisfies } \boxed{\text{some conditions}} \text{ of } G_v$$

where G_v is the decomposition group of G at v.

- By Ono's theorem, $\text{Ш}(T)=0 \iff$ HNP holds for K/k, Theorem 3 gives a necessary and sufficient condition for HNP for K/k.

Theorem 4 ([HKY22, Theorem 1.17])

Assume that $G=M_n\leq S_n$ $(n=11,12,22,23,24)$ is the Mathieu group of degree n. Then $H^1(k,\operatorname{Pic}\overline{X})=0$. In particular, $\text{Ш}(T)=0$.

Examples of Theorem 3

Example ($G = 8T4 \simeq D_4$, $8T13 \simeq A_4 \times C_2$, $8T14 \simeq S_4$, $8T37 \simeq \mathrm{PSL}_2(\mathbb{F}_7)$, $10T7 \simeq A_5$, $14T30 \simeq \mathrm{PSL}_2(\mathbb{F}_{13})$)

$Ш(T) = 0 \iff {}^\exists v \in V_k$ such that $V_4 \leq G_v$.

Example ($G = 10T26 \simeq \mathrm{PSL}_2(\mathbb{F}_9)$)

$Ш(T) = 0 \iff {}^\exists v \in V_k$ such that $D_4 \leq G_v$.

Example ($G = 10T32 \simeq S_6 \leq S_{10}$)

$Ш(T) = 0 \iff {}^\exists v \in V_k$ such that
(i) $V_4 \leq G_v$ where $N_{\widetilde{G}}(V_4) \simeq C_8 \rtimes (C_2 \times C_2)$ for the normalizer $N_{\widetilde{G}}(V_4)$ of V_4 in $\widetilde{G}$ with the normalizer $\widetilde{G} = N_{S_{10}}(G) \simeq \mathrm{Aut}(G)$ of G in S_{10} or
(ii) $D_4 \leq G_v$ where $D_4 \leq [G, G] \simeq A_6$.

- $45/165$ subgroups $V_4 \leq G$ satisfy (i).
- $45/180$ subgroups $D_4 \leq G$ satisfy (ii).

§2. Birational classification for algebraic tori

Problem 1: (Stably) birational classification for algebraic tori

For given two algebraic k-tori T and T',

whether T and T' are stably birationally k-equivalent?, i.e. $T \overset{\text{s.b.}}{\approx} T'$?

Theorem (Colliot-Thélène and Sansuc 1977) $\dim(T) = \dim(T') = 3$

Let L/k and L'/k be Galois extensions with $\mathrm{Gal}(L/k) \simeq \mathrm{Gal}(L'/k) \simeq V_4$. Let $T = R^{(1)}_{L/k}(\mathbb{G}_m)$ and $T' = R^{(1)}_{L'/k}(\mathbb{G}_m)$ be the corresponding norm one tori. Then $T \overset{\text{s.b.}}{\approx} T'$ (stably birationally k-equivalent) if and only if $L = L'$.

- In particular, if k is a number field, then there exist infinitely many stably birationally k-equivalent classes of (non-rational: $1\text{st}/15$) k-tori which correspond to U_1 (cf. Main theorem 1, later).

- $\overline{k}$: a fixed separable closure of k and $\mathcal{G} = \mathrm{Gal}(\overline{k}/k)$
- X: a smooth k-compactification of T, i.e. smooth projective k-variety X containing T as a dense open subvariety
- $\overline{X} = X \times_k \overline{k}$

Theorem (Voskresenskii 1969, 1970)

There exists an exact sequence of $\mathcal{G}$-lattices

$$0 \to \widehat{T} \to \widehat{Q} \to \mathrm{Pic}\,\overline{X} \to 0$$

where $\widehat{Q}$ is permutation and $\mathrm{Pic}\,\overline{X}$ is flabby.

- $M_G \simeq \widehat{T}$, $[\widehat{T}]^{fl} = [\mathrm{Pic}\,\overline{X}]$ as $\mathcal{G}$-lattices

Theorem (Voskresenskii 1970, 1973)

(i) T is stably k-rational if and only if $[\mathrm{Pic}\,\overline{X}] = 0$ as a $\mathcal{G}$-lattice.

(ii) $T \overset{\text{s.b.}}{\approx} T'$ (stably birationally k-equivalent) if and only if $[\mathrm{Pic}\,\overline{X}] = [\mathrm{Pic}\,\overline{X'}]$ as $\mathcal{G}$-lattices.

- From $\mathcal{G}$-lattice to G-lattice

Let L be the minimal splitting field of T with $G = \mathrm{Gal}(L/k) \simeq \mathcal{G}/\mathcal{H}$. We obtain a flabby resolution of $\widehat{T}$:

$$0 \to \widehat{T} \to \widehat{Q} \to \mathrm{Pic}\, X_L \to 0$$

with $[\widehat{T}]^{fl} = [\mathrm{Pic}\ X_L]$ as G-lattices.

By the inflation-restriction exact sequence
$0 \to H^1(G, \mathrm{Pic}\, X_L) \xrightarrow{\inf} H^1(k, \mathrm{Pic}\, \overline{X}) \xrightarrow{\mathrm{res}} H^1(L, \mathrm{Pic}\, \overline{X})$, we get
$\inf : H^1(G, \mathrm{Pic}\, X_L) \xrightarrow{\sim} H^1(k, \mathrm{Pic}\, \overline{X})$ because $H^1(L, \mathrm{Pic}\, \overline{X}) = 0$. We get:

Theorem (Voskresenskii 1970, 1973)

(ii)$'$ $T \overset{\text{s.b.}}{\approx} T'$ (stably birationally k-equivalent) if and only if $[\mathrm{Pic}\, X_{\widetilde{L}}] = [\mathrm{Pic}\, X'_{\widetilde{L}}]$ as $\widetilde{H}$-lattices where $\widetilde{L} = LL'$ and $\widetilde{H} = \mathrm{Gal}(\widetilde{L}/k)$.

The group $\widetilde{H}$ becomes a *subdirect product* of $G = \mathrm{Gal}(L/k)$ and $G' = \mathrm{Gal}(L'/k)$, i.e. a subgroup $\widetilde{H}$ of $G \times G'$ with surjections $\varphi_1 : \widetilde{H} \twoheadrightarrow G$ and $\varphi_2 : \widetilde{H} \twoheadrightarrow G'$.

- This observation yields a concept of "*weak stably k-equivalence*".

Definition

(i) $[M]^{fl}$ and $[M']^{fl}$ are *weak stably k-equivalent*, if there exists a subdirect product $\widetilde{H} \leq G \times G'$ of G and G' with surjections $\varphi_1 : \widetilde{H} \twoheadrightarrow G$ and $\varphi_2 : \widetilde{H} \twoheadrightarrow G'$ such that $[M]^{fl} = [M']^{fl}$ as $\widetilde{H}$-lattices where $\widetilde{H}$ acts on M (resp. M') through the surjection φ_1 (resp. φ_2).
(ii) Algebraic k-tori T and T' are *weak stably birationally k-equivalent*, denoted by $T \overset{\text{s.b.}}{\sim} T'$, if $[\widehat{T}]^{fl}$ and $[\widehat{T}']^{fl}$ are weak stably k-equivalent.

Remark

(1) $T \overset{\text{s.b.}}{\approx} T'$ (birational k-equiv.) $\Rightarrow T \overset{\text{s.b.}}{\sim} T'$ (weak birational k-equiv.).
(2) $\overset{\text{s.b.}}{\sim}$ becomes an equivalence relation and we call this equivalent class *the weak stably k-equivalent class* of $[\widehat{T}]^{fl}$ (or T) denoted by WSEC_r $(r \geq 0)$ with the stably k-rational class WSEC_0.

Rationality problem for 3-dimensional algebraic k-tori T was solved by Kunyavskii (1990). Stably/retract rationality for algebraic k-tori T of dimensions 4 and 5 are given in Hoshi and Yamasaki [HY17, Chapter 1].

Definition

(1) The 15 groups $G = N_{3,i} \leq \mathrm{GL}(3,\mathbb{Z})$ $(1 \leq i \leq 15)$ for which $k(T) \simeq L(M)^G$ is not retract k-rational are as in [HY, Table 6].
(2) The 64 groups $G = N_{31,i} \leq \mathrm{GL}(4,\mathbb{Z})$ $(1 \leq i \leq 64)$ for which $k(T) \simeq L(M)^G$ is not retract k-rational where $M \simeq M_1 \oplus M_2$ with $\mathrm{rank}\ M = 3 + 1$ are as in [HY, Table 7].
(3) The 152 groups $G = N_{4,i} \leq \mathrm{GL}(4,\mathbb{Z})$ $(1 \leq i \leq 152)$ for which $k(T) \simeq L(M)^G$ is not retract k-rational with $\mathrm{rank}\ M = 4$ are as in [HY, Table 8].
(4) The 7 groups $G = I_{4,i} \leq \mathrm{GL}(4,\mathbb{Z})$ $(1 \leq i \leq 7)$ for which $k(T) \simeq L(M)^G$ is not stably but retract k-rational with $\mathrm{rank}\ M = 4$ are as in [HY, Table 9].

Main Theorems 1, 2, 3, 4, 5, 6, 7

- Main theorem 1 $\dim(T) = 3$: up to $\overset{\text{s.b.}}{\sim}$
- Main theorem 2 $\dim(T) = 3$: up to $\overset{\text{s.b.}}{\approx}$
- Main theorem 3 $\dim(T) = 4$: up to $\overset{\text{s.b.}}{\sim}$
- Main theorem 4 $\dim(T) = 4$ $(N_{4,i})$: up to $\overset{\text{s.b.}}{\approx}$
- Main theorem 5 $\dim(T) = 4$ $(I_{4,i})$: up to $\overset{\text{s.b.}}{\approx}$
- Main theorem 6 $\dim(T) = 4$: seven $I_{4,i}$ cases
- Main theorem 7 higher dimensional cases: $\dim(T) \geq 3$

Definition

The G-lattice M_G of rank n is defined to be the G-lattice with a $\mathbb{Z}$-basis $\{u_1, \ldots, u_n\}$ on which G acts by $\sigma(u_i) = \sum_{j=1}^{n} a_{i,j} u_j$ for any $\sigma = [a_{i,j}] \in G \leq \mathrm{GL}(n, \mathbb{Z})$.

Main theorem 1 ([HY, Theorem 1.22]) $\dim(T) = 3$: up to $\overset{\text{s.b.}}{\sim}$

There exist exactly 14 weak stably birationally k-equivalent classes of algebraic k-tori T of dimension 3 which consist of the stably rational class WSEC_0 and 13 classes WSEC_r $(1 \leq r \leq 13)$ for $[\widehat{T}]^{fl}$ with $\widehat{T} = M_G$ and $G = N_{3,i}$ $(1 \leq i \leq 15)$ as in the following: (red $\leftrightarrow$ norm one tori)

r	$G = N_{3,i} : [\widehat{T}]^{fl} = [M_G]^{fl} \in \mathrm{WSEC}_r$	G
1	$N_{3,1} = U_1$ ([CTS 1977])	V_4
2	$N_{3,2} = U_2$	C_2^3
3	$N_{3,3} = W_2$	C_2^3
4	$N_{3,4} = W_1$	$C_4 \times C_2$
5	$N_{3,5} = U_3$, $N_{3,6} = U_4$	D_4
6	$N_{3,7} = U_6$	$D_4 \times C_2$
7	$N_{3,8} = U_5$	A_4
8	$N_{3,9} = U_7$	$A_4 \times C_2$
9	$N_{3,10} = W_3$	$A_4 \times C_2$
10	$N_{3,11} = U_9$, $N_{3,13} = U_{10}$	S_4
11	$N_{3,12} = U_8$	S_4
12	$N_{3,14} = U_{12}$	$S_4 \times C_2$
13	$N_{3,15} = U_{11}$	$S_4 \times C_2$

Main theorem 2 ([HY, Theorem 1.23]) $\dim(T) = 3$: up to $\overset{\text{s.b.}}{\approx}$

Let T_i and T'_j $(1 \leq i, j \leq 15)$ be algebraic k-tori of dimension 3 with the minimal splitting fields L_i and L'_j, and $\widehat{T}_i = M_G$ and $\widehat{T}'_j = M_{G'}$ which satisfy that G and G' are $\mathrm{GL}(3, \mathbb{Z})$-conjugate to $N_{3,i}$ and $N_{3,j}$ respectively. For $1 \leq i, j \leq 15$, the following conditions are equivalent:

(1) $T_i \overset{\text{s.b.}}{\approx} T'_j$ (stably birationally k-equivalent);
(2) $L_i = L'_j$, $T_i \times_k K$ and $T'_j \times_k K$ are weak stably birationally K-equivalent for any $k \subset K \subset L_i$;
(3) $L_i = L'_j$, $T_i \times_k K$ and $T'_j \times_k K$ are weak stably birationally K-equivalent for any $k \subset K \subset L_i$ corresponding to WSEC_r $(r \geq 1)$;
(4) $L_i = L'_j$, $T_i \times_k K$ and $T'_j \times_k K$ are weak stably birationally K-equivalent for any $k \subset K \subset L_i$ corresponding to WSEC_r $(r \geq 1)$ with $[K : k] = d$ where

$$d = \begin{cases} 1 & (i = 1, 3, 4, 8, 9, 10, 11, 12, 13, 14), \\ 1, 2 & (i = 2, 5, 6, 7, 15). \end{cases}$$

- $\exists G = N_{31,i} \leq \mathrm{GL}(4,\mathbb{Z})$ $(1 \leq i \leq 64)$ for which $k(T) \simeq L(M)^G$ is not retract k-rational where $M \simeq M_1 \oplus M_2$ with $\mathrm{rank}\, M = 3+1$.
- $G = N_{4,i} \leq \mathrm{GL}(4,\mathbb{Z})$ $(1 \leq i \leq 152)$ for which $k(T) \simeq L(M)^G$ is not retract k-rational with $\mathrm{rank}\, M = 4$.
- $\exists G = I_{4,i} \leq \mathrm{GL}(4,\mathbb{Z})$ $(1 \leq i \leq 7)$ for which $k(T) \simeq L(M)^G$ is not stably but retract k-rational with $\mathrm{rank}\, M = 4$.

Main theorem 3 ([HY, Theorem 1.24]) $\dim(T) = 4$: up to $\overset{\text{s.b.}}{\sim}$

There exist exactly 129 weak stably birationally k-equivalent classes of algebraic k-tori T of dimension 4 which consist of the stably rational class WSEC_0, 121 classes WSEC_r $(1 \leq r \leq 121)$ for $[\widehat{T}]^{fl}$ with $\widehat{T} = M_G$ and $G = N_{31,i}$ $(1 \leq i \leq 64)$ as in [HY, Table 3] and for $[\widehat{T}]^{fl}$ with $\widehat{T} = M_G$ and $G = N_{4,i}$ $(1 \leq i \leq 152)$ as in [HY, Table 4], and 7 classes WSEC_r $(122 \leq r \leq 128)$ for $[\widehat{T}]^{fl}$ with $\widehat{T} = M_G$ and $G = I_{4,i}$ $(1 \leq i \leq 7)$ as in [HY, Table 5].

Main theorem 4 ([HY, Theorem 1.26]) $\dim(T)=4$ $(N_{4,i})$: up to $\overset{\text{s.b.}}{\approx}$

Let T_i and T'_j $(1\leq i,j\leq 152)$ be algebraic k-tori of dimension 4 with the minimal splitting fields L_i and L'_j and the character modules $\widehat{T}_i=M_G$ and $\widehat{T}'_j=M_{G'}$ which satisfy that G and G' are $\mathrm{GL}(4,\mathbb{Z})$-conjugate to $N_{4,i}$ and $N_{4,j}$ respectively. For $1\leq i,j\leq 152$ except for the cases $i=j=137,139,145,147$, the following conditions are equivalent:

(1) $T_i\overset{\text{s.b.}}{\approx}T'_j$ (stably birationally k-equivalent);

(2) $L_i=L'_j$, $T_i\times_k K$ and $T'_j\times_k K$ are weak stably birationally K-equivalent for any $k\subset K\subset L_i$;

(3) $L_i=L'_j$, $T_i\times_k K$ and $T'_j\times_k K$ are weak stably birationally K-equivalent for any $k\subset K\subset L_i$ corresponding to WSEC_r $(r\geq 1)$;

(4) $L_i=L'_j$, $T_i\times_k K$ and $T'_j\times_k K$ are weak stably birationally K-equivalent for any $k\subset K\subset L_i$ corresponding to WSEC_r $(r\geq 1)$ with $[K:k]=d$ where d is given as in [HY, Theorem 1.26].

For the exceptional cases $i=j=137,139,145,147$ $(G\simeq Q_8\times C_3$, $(Q_8\times C_3)\rtimes C_2$, $\mathrm{SL}(2,\mathbb{F}_3)\rtimes C_4$, $(\mathrm{GL}(2,\mathbb{F}_3)\rtimes C_2)\rtimes C_2\simeq(\mathrm{SL}(2,\mathbb{F}_3)\rtimes C_4)\rtimes C_2)$, we have the

Main theorem 4 ([HY, Theorem 1.26]) $\dim(T) = 4$ $(N_{4,i})$: up to $\overset{\text{s.b.}}{\approx}$

For the exceptional cases $i = j = 137, 139, 145, 147$ ($G \simeq Q_8 \times C_3$, $(Q_8 \times C_3) \rtimes C_2$, $\mathrm{SL}(2, \mathbb{F}_3) \rtimes C_4$, $(\mathrm{GL}(2, \mathbb{F}_3) \rtimes C_2) \rtimes C_2 \simeq (\mathrm{SL}(2, \mathbb{F}_3) \rtimes C_4) \rtimes C_2$), we have the implications $(1) \Rightarrow (2) \Leftrightarrow (3) \Leftrightarrow (4)$, there exists $\tau \in \mathrm{Aut}(G)$ such that $G' = G^\tau$ and $X = Y \lhd Z$ with $Z/Y \simeq C_2, C_2^2, C_2, C_2$ respectively where

$$\mathrm{Inn}(G) \leq X \leq Y \leq Z \leq \mathrm{Aut}(G),$$

$$X = \mathrm{Aut}_{\mathrm{GL}(4,\mathbb{Z})}(G) = \{\sigma \in \mathrm{Aut}(G) \mid G \text{ and } G^\sigma \text{ are conjugate in } \mathrm{GL}(4, \mathbb{Z})\},$$
$$Y = \{\sigma \in \mathrm{Aut}(G) \mid [M_G]^{fl} = [M_{G^\sigma}]^{fl} \text{ as } \widetilde{H}\text{-lattices where } \widetilde{H} = \{(g, g^\sigma) \mid g \in G\} \simeq G\},$$
$$Z = \{\sigma \in \mathrm{Aut}(G) \mid [M_H]^{fl} \sim [M_{H^\sigma}]^{fl} \text{ for any } H \leq G\}.$$

Moreover, we have $(1) \Leftrightarrow M_G \simeq M_{G^\tau}$ as $\widetilde{H}$-lattices $\Leftrightarrow M_G \otimes_{\mathbb{Z}} \mathbb{F}_p \simeq M_{G^\tau} \otimes_{\mathbb{Z}} \mathbb{F}_p$ as $\mathbb{F}_p[\widetilde{H}]$-lattices for $p = 2$ $(i = j = 137)$, for $p = 2$ and 3 $(i = j = 139)$, for $p = 3$ $(i = j = 145, 147)$.

Main theorem 5 ([HY, Theorem 1.29]) $\dim(T)=4$ $(I_{4,i})$: up to $\overset{\text{s.b.}}{\approx}$

Let T_i and T'_j $(1\leq i,j\leq 7)$ be algebraic k-tori of dimension 4 with the minimal splitting fields L_i and L'_j and the character modules $\widehat{T}_i=M_G$ and $\widehat{T}'_j=M_{G'}$ which satisfy that G and G' are $\mathrm{GL}(4,\mathbb{Z})$-conjugate to $I_{4,i}$ and $I_{4,j}$ respectively. For $1\leq i,j\leq 7$ except for the case $i=j=7$, the following conditions are equivalent:

(1) $T_i \overset{\text{s.b.}}{\approx} T'_j$ (stably birationally k-equivalent);

(2) $L_i=L'_j$, $T_i\times_k K$ and $T'_j\times_k K$ are weak stably birationally K-equivalent for any $k\subset K\subset L_i$;

(3) $L_i=L'_j$, $T_i\times_k K$ and $T'_j\times_k K$ are weak stably birationally K-equivalent for any $k\subset K\subset L_i$ corresponding to WSEC_r $(r\geq 1)$;

(4) $L_i=L'_j$, $T_i\times_k K$ and $T'_j\times_k K$ are weak stably birationally K-equivalent for any $k\subset K\subset L_i$ corresponding to WSEC_r $(r\geq 1)$ with $[K:k]=d$ where $d=1$ $(i=1,2,4,5,7)$, $d=1,2$ $(i=3,6)$.

For the exceptional case $i=j=7$ $(G\simeq C_3\rtimes C_8)$, we have the implications $(1)\Rightarrow(2)\Leftrightarrow(3)\Leftrightarrow(4)$, there exists $\tau\in\mathrm{Aut}(G)$ such that $G'=G^\tau$ and $X=Y\lhd Z$ with $Z/Y\simeq C_2$ where

Main theorem 5 ([HY, Theorem 1.29]) $\dim(T) = 4$ $(I_{4,i})$: up to $\overset{\text{s.b.}}{\approx}$

For the exceptional case $i = j = 7$ $(G \simeq C_3 \rtimes C_8)$, we have the implications $(1) \Rightarrow (2) \Leftrightarrow (3) \Leftrightarrow (4)$, there exists $\tau \in \mathrm{Aut}(G)$ such that $G' = G^\tau$ and $X = Y \lhd Z$ with $Z/Y \simeq C_2$ where

$$\mathrm{Inn}(G) \simeq S_3 \leq X \leq Y \leq Z \leq \mathrm{Aut}(G) \simeq S_3 \times C_2^2,$$

$$X = \mathrm{Aut}_{\mathrm{GL}(4,\mathbb{Z})}(G) = \{\sigma \in \mathrm{Aut}(G) \mid G \text{ and } G^\sigma \text{ are conjugate in } \mathrm{GL}(4,\mathbb{Z})\} \simeq D_6,$$
$$Y = \{\sigma \in \mathrm{Aut}(G) \mid [M_G]^{fl} = [M_{G^\sigma}]^{fl} \text{ as } \widetilde{H}\text{-lattices where } \widetilde{H} = \{(g, g^\sigma) \mid g \in G\} \simeq G\},$$
$$Z = \{\sigma \in \mathrm{Aut}(G) \mid [M_H]^{fl} \sim [M_{H^\sigma}]^{fl} \text{ for any } H \leq G\} \simeq S_3 \times C_2^2.$$

Moreover, we have $(1) \Leftrightarrow M_G \simeq M_{G^\tau}$ as $\widetilde{H}$-lattices $\Leftrightarrow M_G \otimes_{\mathbb{Z}} \mathbb{F}_3 \simeq M_{G^\tau} \otimes_{\mathbb{Z}} \mathbb{F}_3$ as $\mathbb{F}_3[\widetilde{H}]$-lattices.

Main theorem 6 ([HY, Theorem 1.31]) $\dim(T)=4$: seven $I_{4,i}$ cases

Let T_i $(1 \leq i, j \leq 7)$ be an algebraic k-torus of dimension 4 with the character module $\widehat{T}_i = M_G$ which satisfies that G is $\mathrm{GL}(4,\mathbb{Z})$-conjugate to $I_{4,i}$. Let T_i^σ be the algebraic k-torus with $\widehat{T}_i^\sigma = M_{G^\sigma}$ $(\sigma \in \mathrm{Aut}(G))$. Then T_i and T_i^σ are not stably k-rational but we have:

(1) $T_1 \times_k T_2$ is stably k-rational;

(2) $T_3 \times_k T_3^\sigma$ stably k-rational for $\sigma \in \mathrm{Aut}(G)$ with $1 \neq \overline{\sigma} \in \mathrm{Aut}(G)/\mathrm{Inn}(G) \simeq C_2$;

(3) $T_4 \times_k T_5$ is stably k-rational;

(4) $T_6 \times_k T_6^\sigma$ is stably k-rational for $\sigma \in \mathrm{Aut}(G)$ with $1 \neq \overline{\sigma} \in \mathrm{Aut}(G)/\mathrm{Inn}(G) \simeq C_2$;

(5) $T_7 \times_k T_7^\sigma$ is stably k-rational for $\sigma \in \mathrm{Aut}(G)$ with $1 \neq \overline{\sigma} \in \mathrm{Aut}(G)/X \simeq C_2$ where

$$X = \mathrm{Aut}_{\mathrm{GL}(4,\mathbb{Z})}(G) = \{\sigma \in \mathrm{Aut}(G) \mid G \text{ and } G^\sigma \text{ are conjugate in } \mathrm{GL}(4,\mathbb{Z})\} \simeq D_6.$$

Higher dimensional cases: $\dim(T) \geq 3$

The following theorem can answer Problem 1 for algebraic k-tori T and T' of dimensions $m \geq 3$ and $n \geq 3$ respectively with $[\widehat{T}]^{fl}, [\widehat{T}']^{fl} \in \mathrm{WSEC}_r$ $(1 \leq r \leq 128)$ via Main theorem 2, Main theorem 4, and Main theorem 5.

Main theorem 7 ([HY, Theorem 1.32]) higher dimensional cases

Let T be an algebraic k-torus of dimension $m \geq 3$ with the minimal splitting field L, $\widehat{T} = M_G$, $G \leq \mathrm{GL}(m, \mathbb{Z})$ and $[\widehat{T}]^{fl} \in \mathrm{WSEC}_r$ $(1 \leq r \leq 128)$. Then there exists an algebraic k-torus T'' of dimension 3 or 4 with the minimal splitting field L'', $\widehat{T}'' = M_{G''}$, and $G'' = N_{3,i}$ $(1 \leq i \leq 15)$, $G'' = N_{4,i}$ $(1 \leq i \leq 152)$ or $G'' = I_{4,i}$ $(1 \leq i \leq 7)$ such that T'' and T are stably birationally k-equivalent and $L'' \subset L$, i.e. $[M_{G''}]^{fl} = [M_G]^{fl}$ as G-lattices and G acts on $[M_{G''}]^{fl}$ through $G'' \simeq G/N$ for the corresponding normal subgroup $N \lhd G$.