Norm one tori and Hasse norm principle, III: Degree 16 case

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A. Hoshi, K. Kanai, A. Yamasaki,
[HKY22] Norm one tori and Hasse norm principle, Math. Comp. (2022).
[HKY23] Norm one tori and Hasse norm principle, II: Degree 12 case, JNT (2023).
[HKY1] Norm one tori and Hasse norm principle, III: Degree 16 case, arXiv:2404.01362.
[HKY2] Hasse norm principle for M_{11} and J_1 extensions, arXiv:2210.09119.

We use GAP. The related algorithms/functions are available from

`https://doi.org/10.57723/289563` (KURENAI: repository of Kyoto University),
`http://mathweb.sc.niigata-u.ac.jp/~hoshi/Algorithm/Norm1ToriHNP/`,
`https://www.math.kyoto-u.ac.jp/~yamasaki/Algorithm/Norm1ToriHNP/`.

§1 Introduction & Main theorems 1,2,3,4

- k : a global field, i.e. a number field or a finite extension of $\mathbb{F}_q(t)$.

Definition (Hasse norm principle)

Let k be a global field. K/k be a finite extension and $\mathbb{A}_K^\times$ be the idele group of K. We say that the Hasse norm principle holds for K/k if

$$\mathrm{Obs}(K/k) := (N_{K/k}(\mathbb{A}_K^\times) \cap k^\times)/N_{K/k}(K^\times) = 1$$

where $N_{K/k}$ is the norm map.

Theorem (Hasse's norm theorem 1931)

If K/k is a cyclic extension of a number field, then

$$\mathrm{Obs}(K/k) = 1.$$

Example (Hasse [Has31]): $\mathrm{Obs}(\mathbb{Q}(\sqrt{-39}, \sqrt{-3})/\mathbb{Q}) \simeq \mathbb{Z}/2\mathbb{Z}.$
$\mathrm{Obs}(\mathbb{Q}(\sqrt{2}, \sqrt{-1})/\mathbb{Q}) = 1.$

In both cases, Galois group $G \simeq V_4$ (Klein four-group).

Tate's theorem (1967)

For any Galois extension K/k, Tate gave:

Theorem (Tate 1967, *in Alg. Num. Th. ed. by Cassels and Fröhlich*)

Let K/k be a finite Galois extension with Galois group $\mathrm{Gal}(K/k) \simeq G$. Let V_k be the set of all places of k and G_v be the decomposition group of G at $v \in V_k$. Then

$$\mathrm{Obs}(K/k) \simeq \mathrm{Coker}\{\bigoplus_{v \in V_k} \widehat{H}^{-3}(G_v, \mathbb{Z}) \xrightarrow{\text{cores}} \widehat{H}^{-3}(G, \mathbb{Z})\}$$

where $\widehat{H}$ is the Tate cohomology. In particular, In particular, the Hasse norm principle holds for K/k if and only if the restriction map $H^3(G, \mathbb{Z}) \xrightarrow{\text{res}} \bigoplus_{v \in V_k} H^3(G_v, \mathbb{Z})$ is injective.

- If $G \simeq C_n$ is cyclic, then $H^3(C_n, \mathbb{Z}) \simeq H^1(C_n, \mathbb{Z}) = 0$ and hence the Hasse's original theorem follows.
- If $G \simeq V_4$, then $\mathrm{Obs}(K/k) = 0 \iff {}^{\exists}v \in V_k$ such that $G_v = V_4$ $(H^3(V_4, \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z})$ (v: should be ramified).

Known results for HNP (1/2)

The HNP for Galois extensions K/k was investigated by Gerth [Ger77], [Ger78], Gurak [Gur78a], [Gur78b], [Gur80], Morishita [Mor90], Horie [Hor93], Takeuchi [Tak94], Kagawa [Kag95], etc.

- (Gurak 1978; Endo-Miyata 1975 + Ono 1963)
  If all the Sylow subgroups of $\mathrm{Gal}(K/k)$ is cyclic, then $\mathrm{Obs}(K/k) = 0$.

However, for non-Galois extensions K/k, very little is known whether the Hasse norm principle holds:

- (Bartels 1981) $[K : k] = p$; prime $\Rightarrow$ HNP for K/k holds.
- (Bartels 1981) $[K : k] = n$ and Galois closure $\mathrm{Gal}(L/k) \simeq D_n$ $\Rightarrow$ HNP for K/k holds.
- (Voskresenskii-Kunyavskii 1984) $[K : k] = n$ and $\mathrm{Gal}(L/k) \simeq S_n$ $\Rightarrow$ HNP for K/k holds.
- (Macedo 2020) $[K : k] = n$ and $\mathrm{Gal}(L/k) \simeq A_n$ $\Rightarrow$ HNP for K/k holds if $n \geq 5$; $n = 6$ using Hoshi-Yamasaki [HY17].

Ono's theorem (1963)

- T : algebraic k-torus, i.e. $T \times_k \overline{k} \simeq (\mathbb{G}_{m,\overline{k}})^n$.
- $Ш(T) := \mathrm{Ker}\{H^1(k,T) \xrightarrow{\mathrm{res}} \bigoplus_{v \in V_k} H^1(k_v,T)\}$: Shafarevich-Tate gp.
- The norm one torus $R^{(1)}_{K/k}(\mathbb{G}_m)$ of K/k:

$$1 \longrightarrow R^{(1)}_{K/k}(\mathbb{G}_m) \longrightarrow R_{K/k}(\mathbb{G}_{m,K}) \xrightarrow{\mathrm{N}_{K/k}} \mathbb{G}_{m,k} \longrightarrow 1$$

  where $R_{K/k}$ is the Weil restriction.
- $R^{(1)}_{K/k}(\mathbb{G}_m)$ is biregularly isomorphic to the norm hyper surface $f(x_1,\ldots,x_n)=1$ where $f \in k[x_1,\ldots,x_n]$ is the norm form of K/k.

Theorem (Ono 1963, Ann. of Math.)

Let K/k be a finite extension and $T = R^{(1)}_{K/k}(\mathbb{G}_m)$. Then

$$Ш(T) \simeq \mathrm{Obs}(K/k).$$

Known results for HNP (2/2)

- $T = R^{(1)}_{K/k}(\mathbb{G}_m)$.
- $Ш(T) \simeq \mathrm{Obs}(K/k)$.

Theorem (Kunyavskii 1984)

Let $[K : k] = 4$, $G = \mathrm{Gal}(L/k) \simeq 4Tm$ $(1 \leq m \leq 5)$.
Then $Ш(T) = 0$ except for $4T2$ and $4T4$. For $4T2 \simeq V_4$, $4T4 \simeq A_4$,
(i) $Ш(T) \leq \mathbb{Z}/2\mathbb{Z}$;
(ii) $Ш(T) = 0 \Leftrightarrow {}^{\exists}v \in V_k$ such that $V_4 \leq G_v$.

Theorem (Drakokhrust-Platonov 1987)

Let $[K : k] = 6$, $G = \mathrm{Gal}(L/k) \simeq 6Tm$ $(1 \leq m \leq 16)$.
Then $Ш(T) = 0$ except for $6T4$ and $6T12$. For $6T4 \simeq A_4$, $6T12 \simeq A_5$,
(i) $Ш(T) \leq \mathbb{Z}/2\mathbb{Z}$;
(ii) $Ш(T) = 0 \Leftrightarrow {}^{\exists}v \in V_k$ such that $V_4 \leq G_v$.

Voskresenskii's theorem (1969) (1/2)

Theorem (Voskresenskii 1969)

Let k be a global field, T be an algebraic k-torus and X be a smooth k-compactification of T. Then there exists an exact sequence

$$0 \to A(T) \to H^1(k, \operatorname{Pic} \overline{X})^\vee \to Ш(T) \to 0$$

where $M^\vee = \operatorname{Hom}(M, \mathbb{Q}/\mathbb{Z})$ is the Pontryagin dual of M.

- The group $A(T) := \left(\prod_{v \in V_k} T(k_v)\right) \Big/ \overline{T(k)}$ is called the kernel of the weak approximation of T.
- T : retract rational $\iff$ $[\widehat{T}]^{fl} = [\operatorname{Pic} \overline{X}]$ is invertible
  $\implies$ $\operatorname{Pic} \overline{X}$ is flabby and coflabby
  $\implies$ $H^1(k, \operatorname{Pic} \overline{X})^\vee = 0 \quad \implies A(T) = Ш(T) = 0.$
- when $T = R^{(1)}_{K/k}(\mathbb{G}_m)$, by Ono's theorem $Ш(T) \simeq \operatorname{Obs}(K/k)$,
  T : retract k-rational $\implies \operatorname{Obs}(K/k) = 0$ (HNP for K/k holds).

Voskresenskii's theorem (1969) (2/2)

- when $T = R^{(1)}_{K/k}(\mathbb{G}_m)$, by Ono's theorem $Ш(T) \simeq \mathrm{Obs}(K/k)$, T : retract k-rational $\Longrightarrow$ $\mathrm{Obs}(K/k) = 0$ (HNP for K/k holds).
- when $T = R^{(1)}_{K/k}(\mathbb{G}_m)$, $\widehat{T} = J_{G/H}$ where $J_{G/H} = (I_{G/H})^\circ = \mathrm{Hom}(I_{G/H}, \mathbb{Z})$ is the dual lattice of $I_{G/H} = \mathrm{Ker}(\varepsilon)$ and $\varepsilon : \mathbb{Z}[G/H] \to \mathbb{Z}$ is the augmentation map.
- (Hoshi-Yamasaki, 2018, Hasegawa-Hoshi-Yamasaki, 2020) For $[K : k] = n \leq 17$ except $9T27 \simeq \mathrm{PSL}_2(\mathbb{F}_8)$, the classificasion of stably/retract rational $R^{(1)}_{K/k}(\mathbb{G}_m)$ was given.
- $H^1(k, \mathrm{Pic}\,\overline{X}) \simeq \mathrm{Br}(X)/\mathrm{Br}(k) \simeq \mathrm{Br}_{\mathrm{nr}}(k(X)/k)/\mathrm{Br}(k)$ where $\mathrm{Br}(X)$ is the étale cohomological/Azumaya Brauer group of X by Colliot-Thélène-Sansuc 1987.

Main theorems 1,2,3,4,5 (1/4)

- $\exists\ 2, 13, 73, 710, 6079$ cases of alg. k-tori T of $\dim(T) = 1, 2, 3, 4, 5$.
- X: a smooth k-compactification of T, $\overline{X} = X \times_k \overline{k}$.

Theorem 1 ([HKY22, Theorem 1.5 and Theorem 1.6])

(i) $\dim(T) = 4$. Among the 216 cases (of 710) of not retract k-rational T,

$$H^1(k, \operatorname{Pic} \overline{X}) \simeq \begin{cases} 0 & (194 \text{ of } 216), \\ \mathbb{Z}/2\mathbb{Z} & (20 \text{ of } 216), \\ (\mathbb{Z}/2\mathbb{Z})^{\oplus 2} & (2 \text{ of } 216). \end{cases}$$

(ii) $\dim(T) = 5$. Among 3003 cases (of 6079) of not retract k-rational T,

$$H^1(k, \operatorname{Pic} \overline{X}) \simeq \begin{cases} 0 & (2729 \text{ of } 3003), \\ \mathbb{Z}/2\mathbb{Z} & (263 \text{ of } 3003), \\ (\mathbb{Z}/2\mathbb{Z})^{\oplus 2} & (11 \text{ of } 3003). \end{cases}$$

- Kunyavskii (1984) showed that among the 15 cases (of 73) of not retract k-rational T of $\dim(T) = 3$, $H^1(k, \operatorname{Pic} \overline{X}) = 0$ (13 of 15), $H^1(k, \operatorname{Pic} \overline{X}) \simeq \mathbb{Z}/2\mathbb{Z}$ (2 of 15).

Main theorems 1,2,3,4,5 (2/4)

- k : a field, K/k : a separable field extension of $[K:k]=n$.
- $T = R^{(1)}_{K/k}(\mathbb{G}_m)$ with $\dim(T) = n-1$.
- X : a smooth k-compactification of T.
- L/k : Galois closure of K/k, $G := \mathrm{Gal}(L/k)$ and $H = \mathrm{Gal}(L/K)$ with $[G:H]=n \Longrightarrow G = nTm \leq S_n$: transitive.

Theorem 2 ([HKY22, Theorem 1.5], [HKY23, Theorem 1.1])

Let $2 \leq n \leq 15$ be an integer. Then $H^1(k, \mathrm{Pic}\,\overline{X}) \neq 0 \Longleftrightarrow G = nTm$ is given as in [HKY22, Table 1] $(n \neq 12)$ or [HKY23,Table 1] $(n = 12)$.

- The number of transitive subgroups nTm of S_n $(2 \leq n \leq 16)$ up to conjugacy (with $H^1(k, \mathrm{Pic}\,\overline{X}) \neq 0$) is given as follows:

n	2	3	4	5	6	7	8	9	10
# of nTm	1	2	5	5	16	7	50	34	45
(with $H^1(k, \mathrm{Pic}\,\overline{X}) \neq 0$)	0	0	2	0	2	0	15	7	3

n	11	12	13	14	15	16
# of nTm	8	301	9	63	104	1954
(with $H^1(k, \mathrm{Pic}\,\overline{X}) \neq 0$)	0	64	0	1	2	853

[HKY22, Table 1]: $H^1(k, \operatorname{Pic} \overline{X}) \simeq H^1(G, [J_{G/H}]^{fl}) \neq 0$ where $G = nTm$ with $2 \leq n \leq 15$ and $n \neq 12$

G	$H^1(k, \operatorname{Pic} \overline{X}) \simeq H^1(G, [J_{G/H}]^{fl})$
$4T2 \simeq V_4$	$\mathbb{Z}/2\mathbb{Z}$
$4T4 \simeq A_4$	$\mathbb{Z}/2\mathbb{Z}$
$6T4 \simeq A_4$	$\mathbb{Z}/2\mathbb{Z}$
$6T12 \simeq A_5$	$\mathbb{Z}/2\mathbb{Z}$
$8T2 \simeq C_4 \times C_2$	$\mathbb{Z}/2\mathbb{Z}$
$8T3 \simeq (C_2)^3$	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 3}$
$8T4 \simeq D_4$	$\mathbb{Z}/2\mathbb{Z}$
$8T9 \simeq D_4 \times C_2$	$\mathbb{Z}/2\mathbb{Z}$
$8T11 \simeq (C_4 \times C_2) \rtimes C_2$	$\mathbb{Z}/2\mathbb{Z}$
$8T13 \simeq A_4 \times C_2$	$\mathbb{Z}/2\mathbb{Z}$
$8T14 \simeq S_4$	$\mathbb{Z}/2\mathbb{Z}$
$8T15 \simeq C_8 \rtimes V_4$	$\mathbb{Z}/2\mathbb{Z}$
$8T19 \simeq (C_2)^3 \rtimes C_4$	$\mathbb{Z}/2\mathbb{Z}$
$8T21 \simeq (C_2)^3 \rtimes C_4$	$\mathbb{Z}/2\mathbb{Z}$
$8T22 \simeq (C_2)^3 \rtimes V_4$	$\mathbb{Z}/2\mathbb{Z}$
$8T31 \simeq ((C_2)^4 \rtimes C_2) \rtimes C_2$	$\mathbb{Z}/2\mathbb{Z}$
$8T32 \simeq ((C_2)^3 \rtimes V_4) \rtimes C_3$	$\mathbb{Z}/2\mathbb{Z}$
$8T37 \simeq \mathrm{PSL}_3(\mathbb{F}_2) \simeq \mathrm{PSL}_2(\mathbb{F}_7)$	$\mathbb{Z}/2\mathbb{Z}$
$8T38 \simeq (((C_2)^4 \rtimes C_2) \rtimes C_2) \rtimes C_3$	$\mathbb{Z}/2\mathbb{Z}$

[HKY22, Table 1]: $H^1(k, \mathrm{Pic}\,\overline{X}) \simeq H^1(G, [J_{G/H}]^{fl}) \neq 0$
where $G = nTm$ with $2 \leq n \leq 15$ and $n \neq 12$

G	$H^1(k, \mathrm{Pic}\,\overline{X}) \simeq H^1(G, [J_{G/H}]^{fl})$
$9T2 \simeq (C_3)^2$	$\mathbb{Z}/3\mathbb{Z}$
$9T5 \simeq (C_3)^2 \rtimes C_2$	$\mathbb{Z}/3\mathbb{Z}$
$9T7 \simeq (C_3)^2 \rtimes C_3$	$\mathbb{Z}/3\mathbb{Z}$
$9T9 \simeq (C_3)^2 \rtimes C_4$	$\mathbb{Z}/3\mathbb{Z}$
$9T11 \simeq (C_3)^2 \rtimes C_6$	$\mathbb{Z}/3\mathbb{Z}$
$9T14 \simeq (C_3)^2 \rtimes Q_8$	$\mathbb{Z}/3\mathbb{Z}$
$9T23 \simeq ((C_3)^2 \rtimes Q_8) \rtimes C_3$	$\mathbb{Z}/3\mathbb{Z}$
$10T7 \simeq A_5$	$\mathbb{Z}/2\mathbb{Z}$
$10T26 \simeq \mathrm{PSL}_2(\mathbb{F}_9) \simeq A_6$	$\mathbb{Z}/2\mathbb{Z}$
$10T32 \simeq S_6$	$\mathbb{Z}/2\mathbb{Z}$
$14T30 \simeq \mathrm{PSL}_2(\mathbb{F}_{13})$	$\mathbb{Z}/2\mathbb{Z}$
$15T9 \simeq (C_5)^2 \rtimes C_3$	$\mathbb{Z}/5\mathbb{Z}$
$15T14 \simeq (C_5)^2 \rtimes S_3$	$\mathbb{Z}/5\mathbb{Z}$

Theorem 3 ([HKY1, Theorem 1.1]) $[K:k]=16$

Assume that $G=\mathrm{Gal}(L/k)=16Tm$ $(1\leq m\leq 1954)$ is a transitive subgroup of S_{16} and $H=\mathrm{Gal}(L/K)$ with $[G:H]=16$. Then

$$H^1(k,\mathrm{Pic}\,\overline{X})=\begin{cases}\mathbb{Z}/2\mathbb{Z} & \text{if } m \text{ is given as in [HKY1, Table 1-1] (774 cases)},\\ (\mathbb{Z}/2\mathbb{Z})^{\oplus 2} & \text{if } m=7,10,11,46,58,61,73,76,82,87,89,107,113,118,\\ & 120,128,129,138,142,162,164,165,178,183,206,297,308,\\ & 319,414,731,1080 \text{ (31 cases)},\\ (\mathbb{Z}/2\mathbb{Z})^{\oplus 3} & \text{if } m=2,9,18,20,23,25,67,69,83,92,98,101,127,173,\\ & 197,202,212,241,246,270,295,301,313,358,372,440,463,\\ & 466,604,632,649,656,794,801,1082,1187,1378 \text{ (37 cases)},\\ (\mathbb{Z}/2\mathbb{Z})^{\oplus 4} & \text{if } m=64 \text{ (1 case)},\\ (\mathbb{Z}/2\mathbb{Z})^{\oplus 6} & \text{if } m=3 \text{ (1 case)},\\ \mathbb{Z}/4\mathbb{Z} & \text{if } m=4,51,63,143,185,323,375,430,769 \text{ (9 cases)},\\ 0 & \text{otherwise (1101 cases)}.\end{cases}$$

- $16T64\simeq(C_2)^4\rtimes C_3$ with $H^1(k,\mathrm{Pic}\,\overline{X})\simeq(\mathbb{Z}/2\mathbb{Z})^4$.
- $16T3\simeq(C_2)^4$ with $H^1(k,\mathrm{Pic}\,\overline{X})\simeq(\mathbb{Z}/2\mathbb{Z})^6$.
- $16T4\simeq(C_4)^2$, $16T51\simeq(C_4)^2\rtimes C_2$, $16T63\simeq(C_4)^2\rtimes C_3$, $16T143\simeq(C_4)^2\rtimes C_4$, $16T185\simeq(C_4)^2\rtimes C_6$, $16T430\simeq(C_4)^2\rtimes Q_{12}$ with $H^1(k,\mathrm{Pic}\,\overline{X})\simeq\mathbb{Z}/4\mathbb{Z}$.

Main theorems 1,2,3,4,5 (3/4)

- $G_p = \mathrm{Syl}_p(G)$ is a p-Sylow subgroup of G.

[HKY1, Lemma 4.1] ($n = p^d$: prime power, e.g. $n = 2^4 = 16$)

Let $G = nTm$ be a transitive subgroup of S_n and $G_p = \mathrm{Syl}_p(G)$.
If $n = p^d$ is a prime power, then $G_p \leq S_n$ is transitive.

[HKY1, Lemma 4.2] ($F = [J_{G/H}]^{fl} = [\mathrm{Pic}\,\overline{X}]$: flabby class)

Let $G = nTm$ be a transitive subgroup of S_n and $G_p = \mathrm{Syl}_p(G)$. Let $H \leq G$ be a subgroup with $[G : H] = n$ and $F = [J_{G/H}]^{fl}$ be a flabby class. Let $H^1(G, F)_{(p)}$ is the p-primary component of $H^1(G, F)$. Then there exists an injection $H^1(G, F)_{(p)} \hookrightarrow H^1(G_p, F|_{G_p})$ where $p \mid |G|$.
In particular, if $H^1(G_p, F|_{G_p}) = 0$ for any $p \mid n$, then $H^1(G, F) = 0$.

Main theorems 1,2,3,4,5 (4/4)

- k : a number field, K/k : a separable field extension of $[K:k]=n$.
- $T=R^{(1)}_{K/k}(\mathbb{G}_m)$, X : a smooth k-compactification of T.

Theorem 4 ([HKY22, Th 1.18], [HKY23, Th 1.3], [HKY, Th 1.4])

Let $2\leq n\leq 16$ be an integer. For $G=nTm$ with $H^1(k,\operatorname{Pic}\overline{X})\neq 0$, assume G is primitive ($\exists$ 22 cases), i.e. $H\leq G$: maximal, when $n=16$,

$$Ш(T)=0 \iff G=nTm \text{ satisfies } \boxed{\text{some conditions}} \text{ of } G_v$$

where G_v is the decomposition group of G at v.

- By Ono's theorem $Ш(T)\simeq \mathrm{Obs}(K/k)$, Theorem 4 gives a necessary and sufficient condition for HNP holds for K/k.

Theorem 5 ([HKY22, Theorem 1.17])

Assume that $G=M_n\leq S_n$ $(n=11,12,22,23,24)$ is the Mathieu group of degree n. Then $H^1(k,\operatorname{Pic}\overline{X})=0$. In particular, $Ш(T)=0$.

Examples of Theorem 4

Example ($G = 8T4 \simeq D_4$, $8T13 \simeq A_4 \times C_2$, $8T14 \simeq S_4$, $8T37 \simeq \mathrm{PSL}_2(\mathbb{F}_7)$, $10T7 \simeq A_5$, $14T30 \simeq \mathrm{PSL}_2(\mathbb{F}_{13})$)

$Ш(T) = 0 \iff {}^{\exists}v \in V_k$ such that $V_4 \leq G_v$.

Example ($G = 10T26 \simeq \mathrm{PSL}_2(\mathbb{F}_9)$)

$Ш(T) = 0 \iff {}^{\exists}v \in V_k$ such that $D_4 \leq G_v$.

Example ($G = 10T32 \simeq S_6 \leq S_{10}$)

$Ш(T) = 0 \iff {}^{\exists}v \in V_k$ such that
(i) $V_4 \leq G_v$ where $N_{\widetilde{G}}(V_4) \simeq C_8 \rtimes (C_2 \times C_2)$ for the normalizer $N_{\widetilde{G}}(V_4)$ of V_4 in $\widetilde{G}$ with the normalizer $\widetilde{G} = N_{S_{10}}(G) \simeq \mathrm{Aut}(G)$ of G in S_{10} or
(ii) $D_4 \leq G_v$ where $D_4 \leq [G, G] \simeq A_6$.

- $45/165$ subgroups $V_4 \leq G$ satisfy (i).
- $45/180$ subgroups $D_4 \leq G$ satisfy (ii).

Definition of some rationalities

- L/k : f.g. field extension. L is k-rational $\overset{\text{def}}{\iff} L \simeq k(x_1, \ldots, x_n)$.

Definition (stably rational)

L is called stably k-rational if $L(y_i, \ldots, y_m)$ is k-rational.

Definition (retract rational)

Let k be an infinite field.
L is called retract k-rational if $\exists k$-algebra $R \subset L$ such that
(i) L is the quotient field of R;
(ii) $\exists f \in k[x_1, \ldots, x_n]$, $\exists k$-algebra hom. $\varphi : R \to k[x_1, \ldots, x_n][1/f]$ and $\psi : k[x_1, \ldots, x_n][1/f] \to R$ satisfying $\psi \circ \varphi = 1_R$.

Definition (unirational)

L is called k-unirational if $L \subset k(t_1, \ldots, t_n)$.

- "rational" $\Rightarrow$ "stably rational" $\Rightarrow$ "retract rational" $\Rightarrow$ "unirational".
- algebraic k-torus T is k-unirational.

§2 Rationality problem for algebraic tori (1/3)

Problem (Rationality problem for algebraic tori)

Whether an algebraic torus T is k-rational?

- $\exists 2$ algebraic tori with $\dim(T) = 1$; the trivial torus $\mathbb{G}_m$ and $R^{(1)}_{K/k}(\mathbb{G}_m)$ with $[K : k] = 2$, which are k-rational.
- $\exists 13$ algebraic tori with $\dim(T) = 2$;

Theorem (Voskresenskii 1967)

All the algebraic tori T with $\dim(T) = 2$ are k-rational.

- $\exists 73$ algebraic tori with $\dim(T) = 3$;

Theorem (Kunyavskii 1990)

(i) $\exists 58$ algebraic tori T with $\dim(T) = 3$ which are k-rational;
(ii) $\exists 15$ algebraic tori T with $\dim(T) = 3$ which are not k-rational;
(iii) T is k-rational $\Leftrightarrow$ T is stably k-rational $\Leftrightarrow$ T is retract k-rational.

- $\exists 710$ algebraic tori with $\dim(T) = 4$;

Theorem (Hoshi-Yamasaki 2017)

(i) $\exists 487$ algebraic tori T with $\dim(T) = 4$ which are stably k-rational;
(ii) $\exists 7$ algebraic tori T with $\dim(T) = 4$ which are not stably k-rational but retract k-rational;
(iii) $\exists 216$ algebraic tori T with $\dim(T) = 4$ which are not retract k-rational.

- $\exists 6079$ algebraic tori with $\dim(T) = 5$;

Theorem (Hoshi-Yamasaki 2017)

(i) $\exists 3051$ algebraic tori T with $\dim(T) = 5$ which are stably k-rational;
(ii) $\exists 25$ algebraic tori T with $\dim(T) = 5$ which are not stably k-rational but retract k-rational;
(iii) $\exists 3003$ algebraic tori T with $\dim(T) = 5$ which are not retract k-rational.

- We do not know "k-rationality".
- Voskresenskii's conjecture: any stably k-rational torus is k-rational (Zariski problem).

Rationality problem for algebraic tori T (2/3)

- T: algebraic k-torus
  $\Longrightarrow \exists$ finite Galois extension L/k such that $T \times_k L \simeq (\mathbb{G}_{m,L})^n$.
- $G = \mathrm{Gal}(L/k)$ where L is the minimal splitting field.

Category of algebraic k-tori which split/L $\overset{\text{duality}}{\longleftrightarrow}$ Category of G-lattices
(i.e. finitely generated $\mathbb{Z}$-free $\mathbb{Z}[G]$-module)

- $T \mapsto$ the character group $X(T) = \mathrm{Hom}(T, \mathbb{G}_m)$: G-lattice.
- $T = \mathrm{Spec}(L[M]^G)$ which splits/L with $X(T) \simeq M \hookleftarrow M$: G-lattice.
- Tori of dimension $n \overset{1:1}{\longleftrightarrow}$ elements of the set $H^1(\mathcal{G}, \mathrm{GL}(n, \mathbb{Z}))$
  where $\mathcal{G} = \mathrm{Gal}(\overline{k}/k)$ since $\mathrm{Aut}(\mathbb{G}_m^n) = \mathrm{GL}(n, \mathbb{Z})$.
- k-torus T of dimension n is determined uniquely by the integral representation $h : \mathcal{G} \to \mathrm{GL}(n, \mathbb{Z})$ up to conjugacy, and the group $h(\mathcal{G})$ is a finite subgroup of $\mathrm{GL}(n, \mathbb{Z})$.
- The function field of T $\overset{\text{identified}}{\longleftrightarrow}$ $L(M)^G$: invariant field.

Rationality problem for algebraic tori T (3/3)

- L/k: Galois extension with $G = \mathrm{Gal}(L/k)$.
- $M = \bigoplus_{1 \leq j \leq n} \mathbb{Z} \cdot u_j$: G-lattice with a $\mathbb{Z}$-basis $\{u_1, \ldots, u_n\}$.
- G acts on $L(x_1, \ldots, x_n)$ by

$$\sigma(x_i) = \prod_{j=1}^{n} x_j^{a_{i,j}}, \quad 1 \leq i \leq n$$

  for any $\sigma \in G$, when $\sigma(u_i) = \sum_{j=1}^{n} a_{i,j} u_j$, $a_{i,j} \in \mathbb{Z}$.
- $L(M) := L(x_1, \ldots, x_n)$ with this action of G.
- The function field of algebraic k-torus $T \overset{\text{identified}}{\longleftrightarrow} L(M)^G$

Rationality problem for algebraic tori T (2nd form)

Whether $L(M)^G$ is k-rational?
($=$ purely transcendental over k?; $L(M)^G = k(\exists t_1, \ldots, \exists t_n)$?)

Flabby (Flasque) resolution (1/3)

- M: G-lattice, i.e. f.g. $\mathbb{Z}$-free $\mathbb{Z}[G]$-module.

Definition

(i) M is permutation $\overset{\text{def}}{\Longleftrightarrow}$ $M \simeq \oplus_{1\leq i\leq m}\mathbb{Z}[G/H_i]$;
(ii) M is stably permutation $\overset{\text{def}}{\Longleftrightarrow}$ $M \oplus \exists P \simeq P'$, P, P' : permutation;
(iii) M is invertible $\overset{\text{def}}{\Longleftrightarrow}$ $M \oplus \exists M' \simeq P$: permutation;
(iv) M is coflabby $\overset{\text{def}}{\Longleftrightarrow}$ $H^1(H, M) = 0$ $(\forall H \leq G)$;
(v) M is flabby $\overset{\text{def}}{\Longleftrightarrow}$ $\widehat{H}^{-1}(H, F) = 0$ $(\forall H \leq G)$.

- "permutation" $\Rightarrow$ "stably permutationl" $\Rightarrow$ "invertible" $\Rightarrow$ "flabby and coflabby".

Definition (Commutative monoid of G-lattices mod. permutation)

$M_1 \sim M_2 \overset{\text{def}}{\Longleftrightarrow} M_1 \oplus P_1 \simeq M_2 \oplus P_2$ ($\exists P_1, \exists P_2$: permutation)
$\Longrightarrow$ commutative monoid $\mathcal{L}$: $[M_1] + [M_2] := [M_1 \oplus M_2], 0 = [P]$.

Flabby (Flasque) resolution (2/3)

Theorem (Endo-Miyata 1975, Colliot-Thélène and Sansuc 1977)

For any G-lattice M, there exists a short exact sequence of G-lattices

$$0 \to M \to P \to F \to 0$$

where P is permutation and F is flabby.

- called a flabby resolution of the G-lattice M.
- $[M]^{fl} := [F]$: flabby class of M (well-defined).

Theorem (Endo-Miyata 1973, Voskresenskii 1974, Saltman 1984)

(EM73) $[M]^{fl} = 0 \iff L(M)^G$ is stably k-rational.
(Vos74) $[M]^{fl} = [M']^{fl} \iff L(M)^G(x_1, \ldots, x_m) \simeq L(M')^G(y_1, \ldots, y_n)$.
(Sal84) $[M]^{fl}$ is invertible $\iff L(M)^G$ is retract k-rational.

Flabby (Flasque) resolution (3/3)

Theorem (Voskresenskii 1969)

Let k be a field and $\mathcal{G} = \mathrm{Gal}(\overline{k}/k)$. Let T be an algebraic k-torus, X be a smooth k-compactification of T and $\overline{X} = X \times_k \overline{k}$. Then

$$0 \to \widehat{T} \to \widehat{Q} \to \mathrm{Pic}\,\overline{X} \to 0$$

is an exact seq. of $\mathcal{G}$-lattice where $\widehat{Q}$ is permutation and $\mathrm{Pic}\,\overline{X}$ is flabby.

- $[\widehat{T}]^{fl} = [\mathrm{Pic}\,\overline{X}]$; flabby class of $\widehat{T}$.

Theorem (Endo-Miyata 1973, Voskresenskii 1974, Saltman 1984)

(EM73) $[\mathrm{Pic}\,\overline{X}] = 0 \iff T$ is stably k-rational.
(Vos74) $[\mathrm{Pic}\,\overline{X}] = [\mathrm{Pic}\,\overline{X'}] \iff T$ and T' are stably bir. k-equivalent.
(Sal84) $[\mathrm{Pic}\,\overline{X}]$ is invertible $\iff T$ is retract k-rational.

§3 Proof of Theorem 3

- We use Drakokhrust-Platonov's method :

Definition (first obstruction to the HNP)

Let $L \supset K \supset k$ be a tower of finite extensions where L is normal over k. We call the group

$$\mathrm{Obs}_1(L/K/k) = \left(N_{K/k}(\mathbb{A}_K^\times) \cap k^\times\right) / \left((N_{L/k}(\mathbb{A}_L^\times) \cap k^\times) N_{K/k}(K^\times)\right)$$

the first obstruction to the HNP for K/k corresponding to the tower $L \supset K \supset k$.

- $\mathrm{Obs}_1(L/K/k) = \mathrm{Obs}(K/k) / (N_{L/k}(\mathbb{A}_L^\times) \cap k^\times)$.
- $\mathrm{Obs}_1(L/K/k)$ is easier than $\mathrm{Obs}(K/k)$.
- We use GAP. The related algorithms/functions we made are available from `https://doi.org/10.57723/289563`

(KURENAI: repository of Kyoto University).

Drakokhrust-Platonov's method (1/3)

Theorem (Drakokhrust-Platonov 1987)

Let $L \supset K \supset k$ be a tower of finite extensions where L is Galois over k. Let $G = \mathrm{Gal}(L/k)$ and $H = \mathrm{Gal}(L/K)$. Then

$$\mathrm{Obs}_1(L/K/k) \simeq \mathrm{Ker}\,\psi_1 \,/\, \varphi_1(\mathrm{Ker}\,\psi_2)$$

where

$$\begin{array}{ccc}
H/[H,H] & \xrightarrow{\psi_1 : H \hookrightarrow G} & G/[G,G] \\
\uparrow{\scriptstyle \varphi_1 : H_w \hookrightarrow H} & & \uparrow{\scriptstyle \varphi_2 : G_v \hookrightarrow G} \\
\displaystyle\bigoplus_{v \in V_k} \left(\bigoplus_{w \mid v} H_w/[H_w, H_w] \right) & \xrightarrow{\psi_2} & \displaystyle\bigoplus_{v \in V_k} G_v/[G_v, G_v]
\end{array}$$

and ψ_2 is defined by

$$\psi_2(h[H_w, H_w]) = x^{-1}hx[G_v, G_v]$$

for $h \in H_w = H \cap xG_vx^{-1}$ $(x \in G)$.

Drakokhrust-Platonov's method (2/3)

- ψ_2^v : the restriction of the map ψ_2 to $\bigoplus_{w|v} H_w/[H_w, H_w]$.
- $\mathrm{Obs}_1(L/K/k) = \mathrm{Ker}\,\psi_1/\varphi_1(\mathrm{Ker}\,\psi_2^{\mathrm{nr}})\varphi_1(\mathrm{Ker}\,\psi_2^r)$.

Proposition (Drakokhrust-Platonov 1987)

(i) $\mathrm{Ker}\,\psi_1 = (\overline{H} \cap [\overline{G}, \overline{G}])/[\overline{H}, \overline{H}]$;
(ii) $G_{v_1} \leq G_{v_2} \Longrightarrow \varphi_1(\mathrm{Ker}\,\psi_2^{v_1}) \subset \varphi_1(\mathrm{Ker}\,\psi_2^{v_2})$;
(iii) $\varphi_1(\mathrm{Ker}\,\psi_2^{\mathrm{nr}}) = \langle [h, x] \mid h \in H \cap xHx^{-1}, x \in G \rangle/[H, H]$;
(vi) Let $H_i \leq G_i \leq G$ $(1 \leq i \leq m)$, $H_i \leq H \cap G_i$, $k_i = L^{G_i}$ and $K_i = L^{H_i}$. If $\mathrm{Obs}(K_i/k_i) = 1$ for any $1 \leq i \leq m$ and

$$\bigoplus_{i=1}^{m} \widehat{H}^{-3}(G_i, \mathbb{Z}) \xrightarrow{\mathrm{cores}} \widehat{H}^{-3}(G, \mathbb{Z})$$

is surjective, then $\mathrm{Obs}(K/k) = \mathrm{Obs}_1(L/K/k)$. In particular,

$$[K : k] = n \text{ is square-free} \Longrightarrow \mathrm{Obs}(K/k) = \mathrm{Obs}_1(L/K/k).$$

Drakokhrust-Platonov's method (3/3)

Theorem (Drakokhrust 1989; Opolka 1980)

Let $\widetilde{L} \supset L \supset k$ be a tower of Galois extensions with $\widetilde{G} = \mathrm{Gal}(\widetilde{L}/k)$ and $\widetilde{H} = \mathrm{Gal}(\widetilde{L}/K)$ which correspond to a central extension

$$1 \to A \to \widetilde{G} \to G \to 1 \text{ with } A \cap [\widetilde{G}, \widetilde{G}] \simeq M(G) = H^2(G, \mathbb{C}^\times);$$

the Schur multiplier of G. Then

$$\mathrm{Obs}(K/k) = \mathrm{Obs}_1(\widetilde{L}/K/k).$$

In particular, if $\widetilde{G}$ is a Schur cover of G, i.e. $A \simeq M(G)$, then $\mathrm{Obs}(K/k) = \mathrm{Obs}_1(\widetilde{L}/K/k)$.

- This theorem is useful, but $\widetilde{G}$ may become large!
- We use GAP. The related algorithms/functions we made are available from `https://doi.org/10.57723/289563`

  (KURENAI: repository of Kyoto University).

Example : $G = 12T261 \simeq (S_3)^4 \rtimes V_4 \simeq S_3 \wr V_4$ (1/2)

Example ($G = 12T261 \simeq (S_3)^4 \rtimes V_4 \simeq S_3 \wr V_4$)

$\text{Ш}(T) = 0 \iff$ there exists a place v of k such that
(i) $V_4 \leq G_v$ where $V_4 \cap D(G) = 1$ for the unique characteristic subgroup $D(G) \simeq (C_3)^4 \rtimes (C_2)^3 \lhd G$,
(ii) $C_4 \times C_2 \leq G_v$ where $(C_4 \times C_2) \cap D(G1) \simeq C_2$ with $D(G) \simeq (C_3)^4 \rtimes (C_2)^3 \lhd G$,
(iii) $D_4 \leq G_v$ where $D_4 \cap (S_3)^4 \simeq C_2$ with $(S_3)^4 \lhd G$,
(iv) $Q_8 \leq G_v$, or
(v) $(C_2)^3 \rtimes C_3 \leq G_v$.

- $H^1(k, \mathrm{Pic}\,\overline{X}) \simeq \mathbb{Z}/2\mathbb{Z}$.
- $|G| = 6^4 \times 4 = 5184$.
- $H^3(G, \mathbb{Z}) \simeq M(G) \simeq (\mathbb{Z}/2\mathbb{Z})^{\oplus 4}$: Schur multiplier of G.

  $\widetilde{G} \leftarrow$ too large ! $|\widetilde{G}| = 6^4 \times 4 \times 2^4 = 82944$.

Example : $G = 12T261 \simeq (S_3)^4 \rtimes V_4 \simeq S_3 \wr V_4$ (2/2)

We can take a minimal stem ext. $\overline{G} = \widetilde{G}/A'$ (i.e. $\overline{A} \leq Z(\overline{G}) \cap [\overline{G}, \overline{G}]$) of G in the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & A = M(G) & \longrightarrow & \widetilde{G} & \xrightarrow{\pi} & G & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & \overline{A} = A/A' & \longrightarrow & \overline{G} = \widetilde{G}/A' & \xrightarrow{\overline{\pi}} & G & \longrightarrow & 1 \end{array}$$

with $\overline{A} \simeq \mathbb{Z}/2\mathbb{Z}$. There exists 15 minimal stem extensions. Then we can find exactly one (1/15) minimal stem extension which satisfies that

$$\oplus_{i=1}^{m'} \widehat{H}^{-3}(G_i, \mathbb{Z}) \xrightarrow{\text{cores}} \widehat{H}^{-3}(\overline{G}, \mathbb{Z})$$

is surjective. By Drakorust-Platonov's Proposition (vi), we have

$$\mathrm{Obs}(K/k) = \mathrm{Obs}_1(\overline{L}_j/K/k).$$

- $\mathrm{Ker}\,\psi_1 = (\mathbb{Z}/2\mathbb{Z})^{\oplus 4}$.
- $\varphi_1^{\mathrm{nr}}(\mathrm{Ker}\,\psi_2^{\mathrm{nr}}) = (\mathbb{Z}/2\mathbb{Z})^{\oplus 3}$.
- $\varphi_1^{\mathrm{r}}(\mathrm{Ker}\,\psi_2^{\mathrm{r}}) = \mathbb{Z}/2\mathbb{Z}$ $(819/891$ cases) or 0 $(72/891$ cases).

Sketch of the proof of Theorem 3 (1/2)

Step 1

• For $G = \mathrm{Gal}(L/k) = nTm \leq S_n$ and $H = \mathrm{Gal}(L/K) \leq G$ with $[G : H] = n$, determine $T = R^{(1)}_{K/k}(\mathbb{G}_m)$ satisfying $H^1(k, \mathrm{Pic}\,\overline{X}) \neq 0$. (Make Table 1)

- We shoud treat $n = (4, 6), 8, 9, 10, 12, 14, 15, 16$ because $H^1(k, \mathrm{Pic}\,\overline{X}) = 0$ when $n = p$: prime.

Step 2

• For the cases in Table1, determine $Ш(T) \simeq \mathrm{Obs}(K/k)$.

(2-1)

(a) $n = pq$ ($p \neq q$: primes) $\longrightarrow$ $\mathrm{Obs}(K/k) \simeq \mathrm{Obs}_1(L/K/k)$.

(b) Otherwise $\longrightarrow$ Find a Schur cover $\widetilde{G}$

(a minimal stem ext. $\overline{G} = \widetilde{G}/A'$ if necessary).

Then we get $\widetilde{L}/k$ s.t. $\mathrm{Obs}(K/k) \simeq \mathrm{Obs}_1(\widetilde{L}/K/k) \simeq \mathrm{Obs}_1(\overline{L}/K/k)$.

(2-2) Calculation $\mathrm{Obs}_1(\overline{L}/K/k)$ for suitable $\overline{L} \subset \widetilde{L}$.

Sketch of the proof of Theorem 3 (2/2)

(2-2) Calculation $\mathrm{Obs}_1(\overline{L}/K/k)$.

By Drakokhrust-Platonov's Thmeorem,

$$\mathrm{Obs}_1(\overline{L}/K/k) \simeq \mathrm{Ker}\,\psi_1/\varphi_1(\mathrm{Ker}\,\psi_2^{\mathrm{nr}})\varphi_1(\mathrm{Ker}\,\psi_2^{\mathrm{r}}),$$

$$\begin{array}{ccc} H/[H,H] & \xrightarrow{\psi_1} & G/[G,G] \\ \uparrow{\scriptstyle\varphi_1} & & \uparrow{\scriptstyle\varphi_2} \\ \bigoplus_{v\in V_k}\left(\bigoplus_{w|v} H_w/[H_w,H_w]\right) & \xrightarrow{\psi_2} & \bigoplus_{v\in V_k} G_v/[G_v,G_v]. \end{array}$$

We compute the following:

(i) $\mathrm{Ker}\,\psi_1 = (\overline{H}\cap[\overline{G},\overline{G}])/[\overline{H},\overline{H}]$;

(ii) $\varphi_1(\mathrm{Ker}\,\psi_2^{\mathrm{nr}}) = \langle [h,x] \mid h\in\overline{H}\cap x\overline{H}x^{-1}, x\in G\rangle/[\overline{H},\overline{H}]$;
(by Drakokhrust-Platonov's Proposition (i), (iii))

(iii) $\varphi_1(\mathrm{Ker}\,\psi_2^{\mathrm{r}})$ (in terms of G_v for ramified $v\in V_k$).

§4 Application 1: R-equivalence in algebraic k-tori (1/2)

Definition (R-equivalence, Manin 1974, *in Cubic Forms*)

- $f : Z \to X$: rational map of k-varieties covers a point $x \in X(k)$.
  $\overset{\text{def}}{\Longleftrightarrow}$ there exists a point $z \in Z(k)$ such that f is defined at z and $f(z) = x$.
- $x, y \in X(k)$ are R-equivalent.
  $\overset{\text{def}}{\Longleftrightarrow}$ there exist a fin. seq. of points $x = x_1, \ldots, x_r = y$ and rational maps $f_i : \mathbb{P}^1 \to X$ $(1 \leq i \leq r-1)$ such that f_i covers x_i, x_{i+1}.

Theorem (Colliot-Thélène and Sansuc 1977)

Let k be a field, T be an algebraic k-torus and $1 \to S \to Q \to T \to 1$ be a flabby resolution of T. Then $T(k) = H^0(k, T) \xrightarrow{\delta} H^1(k, S)$ induces

$$T(k)/R \simeq H^1(k, S).$$

Application 1: R-equivalence in algebraic k-tori (2/2)

- Let k be a local field. Using Tate-Nakayama duality, we have

$$T(k)/R \simeq H^1(k, S) \simeq H^1(k, \widehat{S}) \simeq H^1(k, \operatorname{Pic} \overline{X})$$

for norm one tori $T = R^{(1)}_{K/k}(\mathbb{G}_m)$ where $[K : k] = n \leq 15$.

Theorem ([HKY22], [HKY23])

Let $2 \leq n \leq 15$ be an integer. Let k be a local field, K/k be a separable field extension of degree n, and $T = R^{(1)}_{K/k}(\mathbb{G}_m)$ be the norm one torus of K/k. Then, $T(k)/R \simeq H^1(k, \operatorname{Pic} \overline{X}) \neq 0 \iff G$ is given as in [HKY22, Table 1] of [HKY23, Table 1].

Application 2: Tamagawa number of k-tori (1/2)

Theorem (Ono 1963)

Let k be a global field, T be an algebraic k-torus and $\tau(T)$ be the Tamagawa number of T. Then

$$\tau(T) = \frac{|H^1(k, \widehat{T})|}{|\text{Ш}(T)|}.$$

In particular, if T is retract k-rational, then $\tau(T) = |H^1(k, \widehat{T})|$.

- Let k be a number field, K/k be a field extension of degree n, $T = R^{(1)}_{K/k}(\mathbb{G}_m)$ be the norm one torus of K/k. By Ono's formula, we can calculate Tamagawa number of T explicitly.
- Example. $G = 15T9 \Rightarrow \tau(T) = \dfrac{3}{5}$ or 3 because $\text{Ш}(T) \leq \mathbb{Z}/5\mathbb{Z}$.

Application 2: Tamagawa number of k-tori (2/2)

- $\tau(T) = |H^1(k, \widehat{T})| / |Ш(T)|$.

Theorem ([HKY22, Theorem 8.2])

Let k be a global field and T be an algebraic k-torus of dimension 4 (resp. 5). Among 710 (reps. 6079) cases of algebraic k-tori T, if T is one of the 688 (resp. 5805) cases with $H^1(k, \mathrm{Pic}\,\overline{X}) = 0$, then $\tau(T) = |H^1(k, \widehat{T})|$.

Theorem ([HKY22, Theorem 8.3], [HKY23, Remark 1.4])

Let $2 \leq n \leq 15$ be an integer. Let k be a number field, K/k be a field extension of degree n, L/k be the Galois closure of K/k, and $T = R^{(1)}_{K/k}(\mathbb{G}_m)$ be the norm one torus of K/k. Then $\tau(T) = |H^1(k, \widehat{T})|$ except for the cases in [HKY22, Table 1] and [HKY23, Table 1]. For the exceptional cases, we have $\tau(T) = |H^1(G, J_{G/H})| / |Ш(T)|$.

Sporadic simple group cases: M_{11} and J_1 (1/3)

- k : a numberl field.
- K/k : a separable field extension of $[K:k]=n$ (not fixed).
- L/k : Galois closure of K/k with $G=\mathrm{Gal}(L/k)$ and $H=\mathrm{Gal}(L/K)\lneq G$ with $[G:H]=n$.
- $T=R^{(1)}_{K/k}(\mathbb{G}_m)$ with $\dim(T)=n-1$.
- X : a smooth k-compactification of T.
- $G\simeq M_{11}$ with $|G|=7920=2^4\cdot 3^2\cdot 5\cdot 11$ or $G\simeq J_1$ with $|G|=175560=2^3\cdot 3\cdot 5\cdot 7\cdot 11\cdot 19$
  $\Rightarrow M(G)\simeq H^3(G,\mathbb{Z})=0$: Schur multiplier of G.

Theorem ([HKY2, Theorem 1.6]) $G\simeq M_{11}$

Asume that $G\simeq M_{11}$ and $H=\mathrm{Gal}(L/K)\lneq G$.

$$H^1(k,\mathrm{Pic}\,\overline{X})=\begin{cases}0 & \text{if } \mathrm{Syl}_2(H)\not\simeq C_2,C_4,C_8,\\ \mathbb{Z}/2\mathbb{Z} & \text{if } \mathrm{Syl}_2(H)\simeq C_2,C_4,C_8.\end{cases}$$

Sporadic simple group cases: M_{11} and J_1 (2/3)

Theorem ([HKY2, Theorem 1.8]) $G \simeq M_{11}$

Asume that $G \simeq M_{11}$ and $H = \mathrm{Gal}(L/K) \lneq G$.
(1) If $\mathrm{Syl}_2(H) \not\simeq C_2, C_4, C_8$, then $A(T) \simeq Ш(T) \simeq H^1(k, \mathrm{Pic}\,\overline{X}) = 0$.
(2) If $\mathrm{Syl}_2(H) \simeq C_2, C_4, C_8$, then either
(a) $A(T) = 0$ and $Ш(T) \simeq H^1(k, \mathrm{Pic}\,\overline{X}) \simeq \mathbb{Z}/2\mathbb{Z}$ or
(b) $A(T) \simeq H^1(k, \mathrm{Pic}\,\overline{X}) \simeq \mathbb{Z}/2\mathbb{Z}$ and $Ш(T) = 0$,
and the condition (b) is equivalent to:
(c) there exists a place v of k such that

$$\begin{cases} V_4 \leq G_v \text{ or } Q_8 \leq G_v & \text{if } \mathrm{Syl}_2(H) \simeq C_2, \\ D_4 \leq G_v \text{ or } Q_8 \leq G_v & \text{if } \mathrm{Syl}_2(H) \simeq C_4, \\ QD_8 \leq G_v & \text{if } \mathrm{Syl}_2(H) \simeq C_8 \end{cases}$$

where G_v is the decomposition group of G at a place v of k.

- $0 \to A(T) \to H^1(k, \mathrm{Pic}\,\overline{X})^\vee \to Ш(T) \to 0$ (Voskresenskii 1969).

Sporadic simple group cases: M_{11} and J_1 (3/3)

Theorem ([HKY2, Theorem 1.7]) $G \simeq J_1$

Asume that $G \simeq J_1$ and $H = \mathrm{Gal}(L/K) \lneq G$.

$$H^1(k, \operatorname{Pic} \overline{X}) = \begin{cases} 0 & \text{if } \mathrm{Syl}_2(H) \not\simeq C_2, \\ \mathbb{Z}/2\mathbb{Z} & \text{if } \mathrm{Syl}_2(H) \simeq C_2. \end{cases}$$

Theorem ([HKY2, Theorem 1.9]) $G \simeq J_1$

Asume that $G \simeq J_1$ and $H = \mathrm{Gal}(L/K) \lneq G$.

(1) If $\mathrm{Syl}_2(H) \not\simeq C_2$, then $A(T) \simeq Ш(T) \simeq H^1(k, \operatorname{Pic} \overline{X}) = 0$.

(2) If $\mathrm{Syl}_2(H) \simeq C_2$, then either

(a) $A(T) = 0$ and $Ш(T) \simeq H^1(k, \operatorname{Pic} \overline{X}) \simeq \mathbb{Z}/2\mathbb{Z}$ or

(b) $A(T) \simeq H^1(k, \operatorname{Pic} \overline{X}) \simeq \mathbb{Z}/2\mathbb{Z}$ and $Ш(T) = 0$,

and the condition (b) is equivalent to:

(c) there exists a place v of k such that $V_4 \leq G_v$ where G_v is the decomposition group of G at a place v of k.

Proof: Macedo and Newton [MN22, Corollary 3.4]

- $L/K/k$: a tower of finite extensions with L/k Galois.
- $G = \mathrm{Gal}(L/k)$ and $H = \mathrm{Gal}(L/K) \leq G$.
- $H_p = \mathrm{Syl}_p(H)$ and $K_p = L^{H_p}$.
- X and X_p: smooth compactifications of $T = R^{(1)}_{K/k}(\mathbb{G}_m)$ and $T_p = R^{(1)}_{K_p/k}(\mathbb{G}_m)$ respectively.

Theorem (Macedo and Newton [MN22, Corollary 3.4])

We obtain a commutative diagram with exact rows as follows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A(T)_{(p)} & \longrightarrow & H^1(k, \mathrm{Pic}\, \overline{X})^\vee_{(p)} & \longrightarrow & Ш(T)_{(p)} & \longrightarrow & 0 \\ & & \simeq \downarrow & & \simeq \downarrow & & \simeq \downarrow & & \\ 0 & \longrightarrow & A(T_p)_{(p)} & \longrightarrow & H^1(k, \mathrm{Pic}\, \overline{X_p})^\vee_{(p)} & \longrightarrow & Ш(T_p)_{(p)} & \longrightarrow & 0 \end{array}$$

where (p) stands for the p-primary part and the vertical isomorphisms are induced by the natural inclusion $T \hookrightarrow T_p$.