Rationality problem for norm one tori for dihedral extensions

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§1. Rationality problem for algebraic tori T (1/3)

- k: a base field which is NOT algebraically closed! (in this talk)
- T: algebraic k-torus, i.e. k-form of a split torus;
  an algebraic group over k (group k-scheme) with $T \times_k \overline{k} \simeq (\mathbb{G}_{m,\overline{k}})^n$.

Rationality problem for algebraic tori

Whether T is k-rational?, i.e. $T \approx \mathbb{P}^n$? (birationally k-equivalent)

Let $R^{(1)}_{K/k}(\mathbb{G}_m)$ be the norm one torus of K/k, i.e. the kernel of the norm map $N_{K/k} : R_{K/k}(\mathbb{G}_m) \to \mathbb{G}_m$ where $R_{K/k}$ is the Weil restriction:

$$1 \longrightarrow R^{(1)}_{K/k}(\mathbb{G}_m) \longrightarrow R_{K/k}(\mathbb{G}_m) \xrightarrow{N_{K/k}} \mathbb{G}_m \longrightarrow 1.$$

$$\dim \qquad n-1 \qquad\qquad n \qquad\qquad 1$$

- $\exists 2$ algebraic k-tori T with $\dim(T) = 1$;
  the trivial torus $\mathbb{G}_m$ and $R^{(1)}_{K/k}(\mathbb{G}_m)$ with $[K:k] = 2$, are k-rational.

Rationality problem for algebraic tori T (2/3)

- $\exists 13$ algebraic k-tori T with $\dim(T) = 2$.

Theorem (Voskresenskii 1967) 2-dim. algebraic tori T

T is k-rational.

- $\exists 73$ algebraic k-tori T with $\dim(T) = 3$.

Theorem (Kunyavskii 1990) 3-dim. algebraic tori T

(i) $\exists 58$ algebraic k-tori T which are k-rational;
(ii) $\exists 15$ algebraic k-tori T which are not retract k-rational.

- k-rational $\Rightarrow$ stably k-rational $\Rightarrow$ retract k-rational $\Rightarrow$ k-unirational.

- $\exists 710$ $\mathbb{Z}$-coujugacy subgroups $G \leq \mathrm{GL}(4, \mathbb{Z})$ ($\exists 710$ 4-dim. algebraic k-tori T).

Rationality problem for algebraic tori T (3/3)

Theorem (Hoshi and Yamasaki 2017) 4-dim. algebraic tori T

(i) T is stably k-rational $\iff$ $\exists G$: 487 groups;
(ii) T is not stably but retract k-rational $\iff$ $\exists G$: 7 groups;
(iii) T is not retract k-rational $\iff$ $\exists G$: 216 groups.

- $\exists 6079$ $\mathbb{Z}$-coujugacy subgroups $G \leq \mathrm{GL}(5, \mathbb{Z})$ ($\exists 6079$ 5-dim. algebraic k-tori T).

Theorem (Hoshi and Yamasaki 2017) 5-dim. algebraic tori T

(i) T is stably k-rational $\iff$ $\exists G$: 3051 groups;
(ii) T is not stably but retract k-rational $\iff$ $\exists G$: 25 groups;
(iii) T is not retract k-rational $\iff$ $\exists G$: 3003 groups.

- BUT we do not know the answer for dimension 6.
- $\exists 85308$ $\mathbb{Z}$-coujugacy subgroups $G \leq \mathrm{GL}(6, \mathbb{Z})$! ($\exists 85308$ 6-dim. algebraic k-tori T!).

Algebraic k-tori T and G-lattices

- T: algebraic k-torus
  $\Longrightarrow \exists$ finite Galois extension L/k such that $T \times_k L \simeq (\mathbb{G}_{m,L})^n$.
- $G = \mathrm{Gal}(L/k)$ where L is the minimal splitting field.

Category of algebraic k-tori which split$/L$ $\stackrel{\text{duality}}{\longleftrightarrow}$ Category of G-lattices
(i.e. finitely generated $\mathbb{Z}$-free $\mathbb{Z}[G]$-module)

- $T \mapsto$ the character group $\widehat{T} = \mathrm{Hom}(T, \mathbb{G}_m)$: G-lattice.
- $T = \mathrm{Spec}(L[M]^G)$ which splits$/L$ with $\widehat{T} \simeq M \hookleftarrow M$: G-lattice
- Tori of dimension $n \stackrel{1:1}{\longleftrightarrow}$ elements of the set $H^1(\mathcal{G}, \mathrm{GL}(n,\mathbb{Z}))$
  where $\mathcal{G} = \mathrm{Gal}(\overline{k}/k)$ since $\mathrm{Aut}(\mathbb{G}_m^n) = \mathrm{GL}(n,\mathbb{Z})$.
- k-torus T of dimension n is determined uniquely by the integral representation $h : \mathcal{G} \to \mathrm{GL}(n,\mathbb{Z})$ up to conjugacy, and the group $h(\mathcal{G})$ is a finite subgroup of $\mathrm{GL}(n,\mathbb{Z})$.
- The function field of T $\stackrel{\text{identified}}{\longleftrightarrow}$ $L(M)^G$: invariant field.

Rationality problem for algebraic tori T

- L/k: Galois extension with $G = \mathrm{Gal}(L/k)$.
- $M = \bigoplus_{1 \leq j \leq n} \mathbb{Z} \cdot u_j$: G-lattice with a $\mathbb{Z}$-basis $\{u_1, \ldots, u_n\}$.
- G acts on $L(x_1, \ldots, x_n)$ by

$$\sigma(x_j) = \prod_{i=1}^{n} x_i^{a_{i,j}}, \quad 1 \leq j \leq n$$

  for any $\sigma \in G$, when $\sigma(u_j) = \sum_{i=1}^{n} a_{i,j} u_i$, $a_{i,j} \in \mathbb{Z}$.
- $L(M) := L(x_1, \ldots, x_n)$ with this action of G.
- $\boxed{\text{The function field of algebraic } k\text{-torus } T \overset{\text{identified}}{\longleftrightarrow} L(M)^G}$

Rationality problem for algebraic tori T (2nd form)

Whether $L(M)^G$ is k-rational?
(= purely transcendental over k?; $L(M)^G = k(\exists t_1, \ldots, \exists t_n)$?)

Some definitions

- K/k: a finite generated field extension.

Definition (stably rational)

K is called stably k-rational if $K(y_1, \ldots, y_m)$ is k-rational.

Definition (retract rational)

K is retract k-rational if $\exists k$-algebra (domain) $R \subset K$ such that
(i) K is the quotient field of R;
(ii) $\exists f \in k[x_1, \ldots, x_n]$ $\exists k$-algebra hom. $\varphi : R \to k[x_1, \ldots, x_n][1/f]$ and $\psi : k[x_1, \ldots, x_n][1/f] \to R$ satisfying $\psi \circ \varphi = 1_R$.

Definition (unirational)

K is k-unirational if $K \subset k(x_1, \ldots, x_n)$.

- k-rational $\Rightarrow$ stably k-rational $\Rightarrow$ retract k-rational $\Rightarrow$ k-unirational.
- $L(M)^G$ (resp. T) is always k-unirational.

Flabby (Flasque) resolution

- M: G-lattice, i.e. f.g. $\mathbb{Z}$-free $\mathbb{Z}[G]$-module.

Definition

(i) M is permutation $\stackrel{\mathrm{def}}{\iff}$ $M \simeq \oplus_{1\leq i\leq m}\mathbb{Z}[G/H_i]$.

(ii) M is stably permutation $\stackrel{\mathrm{def}}{\iff}$ $M \oplus \exists P \simeq P'$, P, P': permutation.

(iii) M is invertible $\stackrel{\mathrm{def}}{\iff}$ $M \oplus \exists M' \simeq P$: permutation.

(iv) M is coflabby $\stackrel{\mathrm{def}}{\iff}$ $H^1(H, M) = 0$ $(\forall H \leq G)$.

(v) M is flabby $\stackrel{\mathrm{def}}{\iff}$ $\widehat{H}^{-1}(H, M) = 0$ $(\forall H \leq G)$. ($\widehat{H}$: Tate cohomology)

- "permutation"
  $\Longrightarrow$ "stably permutation"
  $\Longrightarrow$ "invertible"
  $\Longrightarrow$ "flabby and coflabby".

Commutative monoid $\mathcal{M}$

$M_1 \sim M_2 \overset{\text{def}}{\iff} M_1 \oplus P_1 \simeq M_2 \oplus P_2$ ($\exists P_1, \exists P_2$: permutation).
$\Longrightarrow$ commutative monoid $\mathcal{M}$: $[M_1] + [M_2] := [M_1 \oplus M_2]$, $0 = [P]$.

Theorem (Endo-Miyata 1974, Colliot-Thélène-Sansuc 1977)

$\exists P$: permutation, $\exists F$: flabby such that

$$0 \to M \to P \to F \to 0\text{: flabby resolution of } M.$$

- $[M]^{fl} := [F]$; flabby class of M

Theorem (Endo-Miyata 1973, Voskresenskii 1974, Saltman 1984)

(EM73) $[M]^{fl} = 0 \iff L(M)^G$ is stably k-rational.
(Vos74) $[M]^{fl} = [M']^{fl} \iff L(M)^G(x_1, \ldots, x_m) \simeq L(M')^G(y_1, \ldots, y_n)$;
stably k-equivalent.
(Sal84) $[M]^{fl}$ is invertible $\iff L(M)^G$ is retract k-rational.

- $M = M_G \simeq \widehat{T} = \mathrm{Hom}(T, \mathbb{G}_m)$, $k(T) \simeq L(M)^G$, $G = \mathrm{Gal}(L/k)$

Contributions of [HY17] (Hoshi and Yamasaki, 2017, Mem. Amer. Math. Soc., v+215 pp.)

- We give a procedure to compute a flabby resolution of M, in particular $[M]^{fl} = [F]$, effectively (with smaller rank after base change) by computer software GAP.
- The function `IsFlabby` (resp. `IsCoflabby`) may determine whether M is flabby (resp. coflabby).
- The function `IsInvertibleF` may determine whether $[M]^{fl} = [F]$ is invertible ($\leftrightarrow$ whether $L(M)^G$ (resp. T) is retract k-rational).
- We provide some functions for checking a possibility of isomorphism

$$\left(\bigoplus_{i=1}^{r} a_i\,\mathbb{Z}[G/H_i]\right) \oplus a_{r+1}F \simeq \bigoplus_{i=1}^{r} b_i'\,\mathbb{Z}[G/H_i] \qquad (*)$$

  by computing some invariants (e.g. trace, $\widehat{Z}^0$, $\widehat{H}^0$) of both sides.
- [HY17, Example 10.7]. $G \simeq S_5 \leq \mathrm{GL}(5,\mathbb{Z})$ with number $(5,946,4)$ $\Longrightarrow \mathrm{rank}(F) = 17$ and $\mathrm{rank}(*) = 88$ holds $\Longrightarrow [F] = 0 \Longrightarrow L(M)^G$ (resp. T) is stably k-rational.

Special case: $T = R^{(1)}_{K/k}(\mathbb{G}_m)$; norm one tori (1/2)

- Rationality problem for $T = R^{(1)}_{K/k}(\mathbb{G}_m)$ is investigated by S. Endo, Colliot-Thélène and Sansuc, W. Hürlimann, L. Le Bruyn, A. Cortella and B. Kunyavskii, N. Lemire and M. Lorenz, M. Florence, etc.

Theorem (Endo-Miyata 1974), (Saltman 1984)

Let K/k be a finite Galois field extension and $G = \mathrm{Gal}(K/k)$.
(i) T is retract k-rational $\iff$ all the Sylow subgroups of G are cyclic;
(ii) T is stably k-rational $\iff$ G is a cyclic group, or a direct product of a cyclic group of order m and a group $\langle \sigma, \tau \,|\, \sigma^n = \tau^{2^d} = 1, \tau\sigma\tau^{-1} = \sigma^{-1} \rangle$, where $d, m \geq 1, n \geq 3, m, n$: odd, and $(m, n) = 1$.

Theorem (Endo 2011)

Let K/k be a finite non-Galois, separable field extension and L/k be the Galois closure of K/k. Assume that the Galois group of L/k is nilpotent. Then the norm one torus $T = R^{(1)}_{K/k}(\mathbb{G}_m)$ is not retract k-rational.

Special case: $T = R^{(1)}_{K/k}(\mathbb{G}_m)$; norm one tori (2/2)

- K/k: a finite non-Galois, separable field extension
- L/k: the Galois closure of K/k.
- $G = \mathrm{Gal}(L/k)$, $H = \mathrm{Gal}(L/K) \lneq G$.

Theorem (Endo 2011)

Assume that all the Sylow subgroups of G are cyclic.
Then T is retract k-rational. And we have:
$T = R^{(1)}_{K/k}(\mathbb{G}_m)$ is stably k-rational $\iff$ $G = D_n$, n odd $(n \geq 3)$ or $C_m \times D_n$, m, n odd $(m, n \geq 3)$, $(m, n) = 1$, $H \leq D_n$ with $|H| = 2$.

- For $G = D_n$, we have:
  K/k: Galois $(|H| = 1)$
  $\Rightarrow T$ is stably k-rational (not retract k-rational) if n is odd (is even).
  K/k: non-Galois $(|H| = 2$ with $H \neq Z(D_n))$
  $\Rightarrow T$ is stably k-rational (not retract k-rational) if n is odd $(n = 2^r)$.

Main theorem

Main theorem (Hoshi and Yamasaki 2024 J. Algebra)

Let $D_n = \langle x, y \mid x^n = y^2 = 1, y^{-1}xy = x^{-1}\rangle$ be the dihedral group of order $2n$ $(n \geq 4 : \text{even})$ and $Z(D_n) = \langle x^{\frac{n}{2}}\rangle$ be the center of D_n.
Assume that $G = \mathrm{Gal}(L/k) \simeq D_n$ and $H = \mathrm{Gal}(L/K) \leq G$.
Then $|H| = 2$ with $H \neq Z(D_n)$ and we have:
(i) if $n \equiv 0 \pmod 4$, then $R^{(1)}_{K/k}(\mathbb{G}_m)$ is not retract k-rational;
(ii) if $n \equiv 2 \pmod 4$, then $R^{(1)}_{K/k}(\mathbb{G}_m)$ is stably k-rational. More precisely, $R^{(1)}_{K/k}(\mathbb{G}_m) \times R_{k_1/k}(\mathbb{G}_{m,k_1}) \times R_{k_2/k}(\mathbb{G}_{m,k_2}) \times R_{k_3/k}(\mathbb{G}_{m,k_3})$ of dimension $(n-1) + \frac{n}{2} + \frac{n}{2} + 2 = 2n+1$ and $R_{k_4/k}(\mathbb{G}_{m,k_4}) \times R_{k_5/k}(\mathbb{G}_{m,k_5}) \times \mathbb{G}_{m,k}$ of dimension $n + n + 1 = 2n+1$ are birationally k-equivalent where $R_{k_i/k}(\mathbb{G}_{m,k_i})$ are k-rational with
$k_1 = L^{\langle x^{\frac{n}{2}}, x^2y\rangle}$, $k_2 = L^{\langle x^{\frac{n}{2}}, xy\rangle}$, $k_3 = L^{\langle x\rangle}$, $k_4 = L^{\langle x^{\frac{n}{2}}\rangle}$, $k_5 = L^{\langle xy\rangle}$,
$\langle x^{\frac{n}{2}}, x^2y\rangle \simeq \langle x^{\frac{n}{2}}, xy\rangle \simeq C_2 \times C_2$, $\langle x\rangle \simeq C_n$, $\langle x^{\frac{n}{2}}\rangle \simeq \langle xy\rangle \simeq C_2$ and
$[k_1 : k] = [k_2 : k] = \frac{n}{2}$, $[k_3 : k] = 2$, $[k_4 : k] = [k_5 : k] = n$.

Proof of Main theorem (1/7)

- $G \simeq D_n = \langle x, y \mid x^n = y^2 = 1, y^{-1}xy = x^{-1} \rangle$ and $n = 2m$ $(m \geq 2)$.
- We may assume that $H \simeq C_2 = \langle xy \rangle$,
  $x = (1\ 2\ \cdots\ n)$, $y = (1\ n)(2\ n-1)\cdots(m\ m+1)$,
  $xy = (2\ n)(3\ n-1)\cdots(m\ m+2)$ with $m = n/2$.

(i) Case $n \equiv 0 \pmod 4$. Write $n = 2m$ ($m \geq 2$: even).
We consider $[J_{G/H}]^{fl}|_{G'}$ where $G' = \langle x^2, y \rangle \simeq D_m$ with $H' = H \cap G' = 1$.
Then $G' \leq S_n$: transitive and $[J_{G/H}]^{fl}|_{G'} = [J_{G'}]^{fl}$ is not invertible.
$\Rightarrow [J_{G/H}]^{fl}$ is not invertible.
$\Rightarrow R^{(1)}_{K/k}(\mathbb{G}_m)$ is not retract k-rational.

Proof of Main theorem (2/7)

(ii) Case $n \equiv 2 \pmod 4$. Write $n = 2m$ ($m \geq 3$: odd).
We take $G = \langle x, y \mid x^n = y^2 = 1, y^{-1}xy = x^{-1} \rangle \simeq D_n$ and $H = \langle xy \rangle \simeq C_2$. We have the exact sequence

$$0 \to I_{G/H} \to \mathbb{Z}[G/H] \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

where ε is the augmentation map. We will construct an exact sequence of G-lattices

$$0 \to C \to P \to I_{G/H} \to 0$$

with P permutation and C stably permutation, i.e. $[C] = 0$. In particular, C and the dual $(C)^\circ = \mathrm{Hom}_{\mathbb{Z}}(C, \mathbb{Z})$ are invertible. Then we get a flabby resolution

$$0 \to J_{G/H} \to (P)^\circ \to (C)^\circ \to 0$$

of $J_{G/H}$. This implies that $[J_{G/H}]^{fl} = [(C)^\circ] = 0$ and hence $R^{(1)}_{K/k}(\mathbb{G}_m)$ is stably k-rational.

Proof of Main theorem (3/7)

Let $e_1, \ldots, e_n$ be the standard $\mathbb{Z}$-basis of $\mathbb{Z}[G/H]$ via the left regular representation of G/H:

$$x : e_1 \mapsto \cdots \mapsto e_n \mapsto e_1,$$
$$y : e_i \leftrightarrow e_{n+1-i} \ (1 \leq i \leq m).$$

Define $f_i := e_i - e_{i+1}$ $(1 \leq i \leq n-1)$ and $f_n := e_n - e_1$. Then $f_1, \ldots, f_{n-1}$ becomes a $\mathbb{Z}$-basis of $I_{G/H}$ with $\sum_{i=1}^{n} f_i = 0$, i.e. $f_n = -\sum_{i=1}^{n-1} f_i$:

$$x : f_1 \mapsto \cdots \mapsto f_{n-1} \mapsto -\sum_{i=1}^{n-1} f_i,$$
$$y : f_i \leftrightarrow -f_{n-i} \ (1 \leq i \leq m).$$

Proof of Main theorem (4/7)

We consider a permutation G-lattice $P \simeq \mathbb{Z}[G/Z(G)] \oplus \mathbb{Z}[G/H] \simeq \mathbb{Z}[G/\langle x^m \rangle] \oplus \mathbb{Z}[G/\langle xy \rangle] \simeq \mathbb{Z}[D_m] \oplus \mathbb{Z}[D_n/C_2]$ $(n = 2m)$ where $H = \mathrm{Stab}_1(G)$ with $\mathbb{Z}$-basis $\alpha_1, \ldots, \alpha_m$, $\beta_1, \ldots, \beta_m$, $\gamma_1, \ldots, \gamma_n$ on which G acts by

$$x : \alpha_1 \mapsto \cdots \mapsto \alpha_m \mapsto \alpha_1, \beta_1 \mapsto \cdots \mapsto \beta_m \mapsto \beta_1, \gamma_1 \mapsto \cdots \mapsto \gamma_n \mapsto \gamma_1,$$
$$y : \alpha_i \leftrightarrow \beta_{m-i} \ (1 \leq i \leq m-1), \alpha_m \leftrightarrow \beta_m, \gamma_j \leftrightarrow \gamma_{n+1-j} \ (1 \leq j \leq n).$$

Note that $\mathrm{rank}_{\mathbb{Z}} P = 2m + n = 2n$.
We define a G-homomorphism

$$\varphi : P \to I_{G/H}, \alpha_i \mapsto f_i + f_{m+i},\ \beta_i \mapsto -(f_i + f_{m+i}) \ (1 \leq i \leq m),$$
$$\gamma_j \mapsto x^{j-1} \left(\sum_{l=1}^{m-1} f_l + f_{m+1} \right) \ (1 \leq j \leq n).$$

Our claim is that φ is surjective.

Proof of Main theorem (5/7)

We get an exact sequence of G-lattices

$$0 \to C \to P \xrightarrow{\varphi} I_{G/H} \to 0$$

where $C = \mathrm{Ker}(\varphi)$ with $\mathrm{rank}_{\mathbb{Z}} C = n + 1 = 2m + 1$. We find a $\mathbb{Z}$-basis $a_i = \alpha_i + \beta_i$, $b_i = x^{i-1}(\alpha_1 + \beta_m - \gamma_1 - \gamma_{m+1})$ $(1 \leq i \leq m)$, $c_1 = \sum_{i=1}^m \alpha_i$ of C and the action of G on $C = \langle a_1, \ldots, a_m, b_1, \ldots, b_m, c_1 \rangle_{\mathbb{Z}}$ is given by

$$\begin{aligned}
&x : a_1 \mapsto \cdots \mapsto a_m \mapsto a_1, b_1 \mapsto \cdots \mapsto b_m \mapsto b_1, c_1 \mapsto c_1,\\
&y : a_i \leftrightarrow a_{m-i}, b_i \leftrightarrow b_{m+1-i} \ (1 \leq i \leq \tfrac{m-1}{2}),\\
&\quad a_m \mapsto a_m, b_{\frac{m+1}{2}} \mapsto b_{\frac{m+1}{2}}, c_1 \mapsto \sum_{i=1}^m a_i - c_1.
\end{aligned}$$

Take an element $c_2 = \sum_{i=1}^m \beta_i \in P$. Then we have

$$\begin{aligned}
&x : c_1 \mapsto c_1, c_2 \mapsto c_2,\\
&y : c_1 \mapsto c_2, c_2 \mapsto c_1.
\end{aligned}$$

Proof of Main theorem (6/7)

Then we consider the trivial G-lattice $\langle z_0 \rangle_{\mathbb{Z}} \simeq \mathbb{Z}$ and extend the map φ from P to $P \oplus \langle z_0 \rangle_{\mathbb{Z}}$ by $\widetilde{\varphi}(z_0) = 0$:

$$0 \to C \oplus \langle z_0 \rangle_{\mathbb{Z}} \to P \oplus \langle z_0 \rangle_{\mathbb{Z}} \xrightarrow{\widetilde{\varphi}} I_{G/H} \to 0$$

where $C \oplus \langle z_0 \rangle_{\mathbb{Z}} = \mathrm{Ker}(\widetilde{\varphi})$ with
$\mathrm{rank}_{\mathbb{Z}}(C \oplus \langle z_0 \rangle_{\mathbb{Z}}) = (n+1) + 1 = 2m + 2$.
Because $\gcd\{2, m\} = 1$, there exist $u, v \in \mathbb{Z}$ such that $2u + mv = 1$.
Then we can get a $\mathbb{Z}$-basis $a'_i := a_i + vz_0$, b_i $(1 \leq i \leq m)$, $c'_l := c_l - uz_0$ $(1 \leq l \leq 2)$ of $C \oplus \langle z_0 \rangle_{\mathbb{Z}} \simeq \mathbb{Z}[D_m/\langle x^2 y \rangle] \oplus \mathbb{Z}[D_m/\langle xy \rangle] \oplus \mathbb{Z}[D_m/\langle x \rangle]$ with $\mathbb{Z}$-rank $(2m+1) + 1 = m + m + 2$ on which $D_m = G/\langle x^m \rangle$ acts by

$$\begin{aligned} x &: a'_1 \mapsto \cdots \mapsto a'_m \mapsto a'_1, b_1 \mapsto \cdots \mapsto b_m \mapsto b_1, c'_1 \mapsto c'_1, c'_2 \mapsto c'_2, \\ y &: a'_i \leftrightarrow a'_{m-i}, b_i \leftrightarrow b_{m+1-i} \ (1 \leq i \leq \tfrac{m-1}{2}), \\ &\quad a'_m \mapsto a'_m, b_{\frac{m+1}{2}} \mapsto b_{\frac{m+1}{2}}, c'_1 \leftrightarrow c'_2. \end{aligned}$$

Proof of Main theorem (7/7)

Indeed, we can confirm that

$$\det \begin{array}{c} \begin{array}{cc} \overbrace{\hphantom{1 \quad \cdots \quad 1}}^{m} & \overbrace{\hphantom{-1 \quad -u}}^{2} \end{array} \\ \left(\begin{array}{ccc|cc} 1 & & & 0 & v \\ & \ddots & & \vdots & \vdots \\ & & 1 & 0 & v \\ \hline 0 & \cdots & 0 & 1 & -u \\ 1 & \cdots & 1 & -1 & -u \end{array}\right) \end{array} = -2u - mv = -1.$$

Then we find that $C \oplus \langle z_0 \rangle_{\mathbb{Z}} \simeq \mathbb{Z}[D_m/\langle x^2 y \rangle] \oplus \mathbb{Z}[D_m/\langle xy \rangle] \oplus \mathbb{Z}[D_m/\langle x \rangle]$ is a permutation G-lattice (D_m-lattice) and hence $[C] = 0$.
The last statement follows from the exact sequence

$$0 \to J_{G/H} \to (P)^\circ \oplus \mathbb{Z} \to (C)^\circ \oplus \mathbb{Z} \to 0$$

with $(P)^\circ \oplus \mathbb{Z} \simeq \mathbb{Z}[G/\langle x^m \rangle] \oplus \mathbb{Z}[G/\langle xy \rangle] \oplus \mathbb{Z}$,
$(C^\circ) \oplus \mathbb{Z} \simeq \mathbb{Z}[D_m/\langle x^2 y \rangle] \oplus \mathbb{Z}[D_m/\langle xy \rangle] \oplus \mathbb{Z}[D_m/\langle x \rangle]$.
This implies that $L(J_{G/H} \oplus (C)^\circ \oplus \mathbb{Z})^G \simeq L((P)^\circ \oplus \mathbb{Z})^G$. $\square$

Corollary of Main theorem

As a consequence of Main theorem, we get:

Corollary of Main theorem

Let $C = \mathrm{Ker}(\varphi)$ be a D_m-lattice with $\mathrm{rank}_{\mathbb{Z}} C = 2m+1$ $(m \geq 3 : \text{odd})$ given as in (ii) Case $n \equiv 2 \pmod 4$ in Main theorem. Then we have $\widehat{H}^0(D_m, C) \simeq \mathbb{Z}/2\mathbb{Z}$.
In particular, the D_m-lattice C is not permutation but stably permutation which satisfies $C \oplus \mathbb{Z} \simeq \mathbb{Z}[D_m/\langle x^2y\rangle] \oplus \mathbb{Z}[D_m/\langle xy\rangle] \oplus \mathbb{Z}[D_m/\langle x\rangle]$ with $\mathbb{Z}$-rank $(2m+1)+1 = m+m+2$.

- We can also give a refinement of the proof of Endo-Miyata (1974) and Endo (2011) of the stably k-rational case using a similar technique:
- For $G = D_n$, we have:
  K/k: Galois $(|H| = 1)$
  $\Rightarrow T$ is stably k-rational (not retract k-rational) if n is odd (is even).
  K/k: non-Galois ($|H| = 2$ with $H \neq Z(D_n)$)
  $\Rightarrow T$ is stably k-rational (not retract k-rational) if n is odd $(n = 2^r)$.