Rationality problem for norm one tori for A_5 and $\mathrm{PSL}_2(\mathbb{F}_8)$ extensions

Akinari Hoshi[1] Aiichi Yamasaki[2]

[1]Niigata University

[2]Kyoto University

March 19, 2024

§1. Rationality problem for algebraic tori T (1/3)

- k: a base field which is NOT algebraically closed! (in this talk)
- T: algebraic k-torus, i.e. k-form of a split torus;
  an algebraic group over k (group k-scheme) with $T \times_k \overline{k} \simeq (\mathbb{G}_{m,\overline{k}})^n$.

Rationality problem for algebraic tori

Whether T is k-rational?, i.e. $T \approx \mathbb{P}^n$? (birationally k-equivalent)

Let $R^{(1)}_{K/k}(\mathbb{G}_m)$ be the norm one torus of K/k, i.e. the kernel of the norm map $N_{K/k} : R_{K/k}(\mathbb{G}_m) \to \mathbb{G}_m$ where $R_{K/k}$ is the Weil restriction:

$$1 \longrightarrow R^{(1)}_{K/k}(\mathbb{G}_m) \longrightarrow R_{K/k}(\mathbb{G}_m) \xrightarrow{N_{K/k}} \mathbb{G}_m \longrightarrow 1.$$

$$\dim \qquad\qquad n-1 \qquad\qquad n \qquad\qquad 1$$

- $\exists 2$ algebraic k-tori T with $\dim(T) = 1$;
  the trivial torus $\mathbb{G}_m$ and $R^{(1)}_{K/k}(\mathbb{G}_m)$ with $[K : k] = 2$, are k-rational.

Rationality problem for algebraic tori T (2/3)

- $\exists 13$ algebraic k-tori T with $\dim(T) = 2$.

Theorem (Voskresenskii 1967) 2-dim. algebraic tori T

T is k-rational.

- $\exists 73$ algebraic k-tori T with $\dim(T) = 3$.

Theorem (Kunyavskii 1990) 3-dim. algebraic tori T

(i) $\exists 58$ algebraic k-tori T which are k-rational;
(ii) $\exists 15$ algebraic k-tori T which are not retract k-rational.

- k-rational $\Rightarrow$ stably k-rational $\Rightarrow$ retract k-rational $\Rightarrow$ k-unirational.

- $\exists 710$ $\mathbb{Z}$-coujugacy subgroups $G \leq \mathrm{GL}(4, \mathbb{Z})$
  ($\exists 710$ 4-dim. algebraic k-tori T).

Rationality problem for algebraic tori T (3/3)

Theorem (Hoshi and Yamasaki 2017) 4-dim. algebraic tori T

(i) T is stably k-rational $\iff$ $\exists G$: 487 groups;
(ii) T is not stably but retract k-rational $\iff$ $\exists G$: 7 groups;
(iii) T is not retract k-rational $\iff$ $\exists G$: 216 groups.

- $\exists 6079$ $\mathbb{Z}$-coujugacy subgroups $G \leq \mathrm{GL}(5, \mathbb{Z})$ ($\exists 6079$ 5-dim. algebraic k-tori T).

Theorem (Hoshi and Yamasaki 2017) 5-dim. algebraic tori T

(i) T is stably k-rational $\iff$ $\exists G$: 3051 groups;
(ii) T is not stably but retract k-rational $\iff$ $\exists G$: 25 groups;
(iii) T is not retract k-rational $\iff$ $\exists G$: 3003 groups.

- BUT we do not know the answer for dimension 6.
- $\exists 85308$ $\mathbb{Z}$-coujugacy subgroups $G \leq \mathrm{GL}(6, \mathbb{Z})$! ($\exists 85308$ 6-dim. algebraic k-tori T!).

Algebraic k-tori T and G-lattices

- T: algebraic k-torus
  $\Longrightarrow \exists$ finite Galois extension L/k such that $T \times_k L \simeq (\mathbb{G}_{m,L})^n$.
- $G = \mathrm{Gal}(L/k)$ where L is the minimal splitting field.

Category of algebraic k-tori which split$/L$ $\overset{\text{duality}}{\longleftrightarrow}$ Category of G-lattices
(i.e. finitely generated $\mathbb{Z}$-free $\mathbb{Z}[G]$-module)

- $T \mapsto$ the character group $\widehat{T} = \mathrm{Hom}(T, \mathbb{G}_m)$: G-lattice.
- $T = \mathrm{Spec}(L[M]^G)$ which splits$/L$ with $\widehat{T} \simeq M \hookleftarrow M$: G-lattice
- Tori of dimension $n \overset{1:1}{\longleftrightarrow}$ elements of the set $H^1(\mathcal{G}, \mathrm{GL}(n, \mathbb{Z}))$
  where $\mathcal{G} = \mathrm{Gal}(\overline{k}/k)$ since $\mathrm{Aut}(\mathbb{G}_m^n) = \mathrm{GL}(n, \mathbb{Z})$.
- k-torus T of dimension n is determined uniquely by the integral representation $h : \mathcal{G} \to \mathrm{GL}(n, \mathbb{Z})$ up to conjugacy, and the group $h(\mathcal{G})$ is a finite subgroup of $\mathrm{GL}(n, \mathbb{Z})$.
- The function field of T $\overset{\text{identified}}{\longleftrightarrow}$ $L(M)^G$: invariant field.

Rationality problem for algebraic tori T

- L/k: Galois extension with $G = \mathrm{Gal}(L/k)$.
- $M = \bigoplus_{1 \leq j \leq n} \mathbb{Z} \cdot u_j$: G-lattice with a $\mathbb{Z}$-basis $\{u_1, \ldots, u_n\}$.
- G acts on $L(x_1, \ldots, x_n)$ by

$$\sigma(x_j) = \prod_{i=1}^{n} x_i^{a_{i,j}}, \quad 1 \leq j \leq n$$

  for any $\sigma \in G$, when $\sigma(u_j) = \sum_{i=1}^{n} a_{i,j} u_i$, $a_{i,j} \in \mathbb{Z}$.
- $L(M) := L(x_1, \ldots, x_n)$ with this action of G.
- $\boxed{\text{The function field of algebraic } k\text{-torus } T \overset{\text{identified}}{\longleftrightarrow} L(M)^G}$

Rationality problem for algebraic tori T (2nd form)

Whether $L(M)^G$ is k-rational?
(= purely transcendental over k?; $L(M)^G = k(\exists t_1, \ldots, \exists t_n)$?)

Some definitions

- K/k: a finite generated field extension.

Definition (stably rational)

K is called stably k-rational if $K(y_1, \ldots, y_m)$ is k-rational.

Definition (retract rational)

K is retract k-rational if $\exists k$-algebra (domain) $R \subset K$ such that
(i) K is the quotient field of R;
(ii) $\exists f \in k[x_1, \ldots, x_n]$ $\exists k$-algebra hom. $\varphi : R \to k[x_1, \ldots, x_n][1/f]$ and $\psi : k[x_1, \ldots, x_n][1/f] \to R$ satisfying $\psi \circ \varphi = 1_R$.

Definition (unirational)

K is k-unirational if $K \subset k(x_1, \ldots, x_n)$.

- k-rational $\Rightarrow$ stably k-rational $\Rightarrow$ retract k-rational $\Rightarrow$ k-unirational.
- $L(M)^G$ (resp. T) is always k-unirational.

Flabby (Flasque) resolution

- M: G-lattice, i.e. f.g. $\mathbb{Z}$-free $\mathbb{Z}[G]$-module.

Definition

(i) M is permutation $\overset{\text{def}}{\iff}$ $M \simeq \oplus_{1\le i\le m}\mathbb{Z}[G/H_i]$.

(ii) M is stably permutation $\overset{\text{def}}{\iff}$ $M \oplus \exists P \simeq P'$, P, P': permutation.

(iii) M is invertible $\overset{\text{def}}{\iff}$ $M \oplus \exists M' \simeq P$: permutation.

(iv) M is coflabby $\overset{\text{def}}{\iff}$ $H^1(H, M) = 0$ $(\forall H \le G)$.

(v) M is flabby $\overset{\text{def}}{\iff}$ $\widehat{H}^{-1}(H, M) = 0$ $(\forall H \le G)$. ($\widehat{H}$: Tate cohomology)

- "permutation"
  $\Longrightarrow$ "stably permutation"
  $\Longrightarrow$ "invertible"
  $\Longrightarrow$ "flabby and coflabby".

Commutative monoid $\mathcal{M}$

$M_1 \sim M_2 \overset{\text{def}}{\iff} M_1 \oplus P_1 \simeq M_2 \oplus P_2$ ($\exists P_1, \exists P_2$: permutation).
$\Longrightarrow$ commutative monoid $\mathcal{M}$: $[M_1] + [M_2] := [M_1 \oplus M_2]$, $0 = [P]$.

Theorem (Endo-Miyata 1974, Colliot-Thélène-Sansuc 1977)

$\exists P$: permutation, $\exists F$: flabby such that

$$0 \to M \to P \to F \to 0\text{: flabby resolution of } M.$$

- $[M]^{fl} := [F]$; flabby class of M

Theorem (Endo-Miyata 1973, Voskresenskii 1974, Saltman 1984)

(EM73) $[M]^{fl} = 0 \iff L(M)^G$ is stably k-rational.
(Vos74) $[M]^{fl} = [M']^{fl} \iff L(M)^G(x_1, \ldots, x_m) \simeq L(M')^G(y_1, \ldots, y_n)$; stably k-equivalent.
(Sal84) $[M]^{fl}$ is invertible $\iff L(M)^G$ is retract k-rational.

- $M = M_G \simeq \widehat{T} = \mathrm{Hom}(T, \mathbb{G}_m)$, $k(T) \simeq L(M)^G$, $G = \mathrm{Gal}(L/k)$

Contributions of [HY17] (Hoshi and Yamasaki, 2017, Mem. Amer. Math. Soc., v+215 pp.)

- We give a procedure to compute a flabby resolution of M, in particular $[M]^{fl} = [F]$, effectively (with smaller rank after base change) by computer software GAP.
- The function `IsFlabby` (resp. `IsCoflabby`) may determine whether M is flabby (resp. coflabby).
- The function `IsInvertibleF` may determine whether $[M]^{fl} = [F]$ is invertible ($\leftrightarrow$ whether $L(M)^G$ (resp. T) is retract k-rational).
- We provide some functions for checking a possibility of isomorphism

$$\left(\bigoplus_{i=1}^{r} a_i\, \mathbb{Z}[G/H_i]\right) \oplus a_{r+1}F \simeq \bigoplus_{i=1}^{r} b_i'\, \mathbb{Z}[G/H_i] \qquad (*)$$

  by computing some invariants (e.g. trace, $\widehat{Z}^0$, $\widehat{H}^0$) of both sides.
- [HY17, Example 10.7]. $G \simeq S_5 \leq \mathrm{GL}(5, \mathbb{Z})$ with number $(5, 946, 4)$ $\Longrightarrow \mathrm{rank}(F) = 17$ and $\mathrm{rank}(*) = 88$ holds $\Longrightarrow [F] = 0 \Longrightarrow L(M)^G$ (resp. T) is stably k-rational.

Special case: $T = R^{(1)}_{K/k}(\mathbb{G}_m)$; norm one tori (1/4)

- Rationality problem for $T = R^{(1)}_{K/k}(\mathbb{G}_m)$ is investigated by S. Endo, Colliot-Thélène and Sansuc, W. Hürlimann, L. Le Bruyn, A. Cortella and B. Kunyavskii, N. Lemire and M. Lorenz, M. Florence, etc.

Theorem (Endo-Miyata 1974), (Saltman 1984)

Let K/k be a finite Galois field extension and $G = \mathrm{Gal}(K/k)$.
(i) T is retract k-rational $\iff$ all the Sylow subgroups of G are cyclic;
(ii) T is stably k-rational $\iff$ G is a cyclic group, or a direct product of a cyclic group of order m and a group $\langle \sigma, \tau \,|\, \sigma^n = \tau^{2^d} = 1, \tau\sigma\tau^{-1} = \sigma^{-1} \rangle$, where $d, m \geq 1, n \geq 3, m, n$: odd, and $(m, n) = 1$.

Theorem (Endo 2011)

Let K/k be a finite non-Galois, separable field extension and L/k be the Galois closure of K/k. Assume that the Galois group of L/k is nilpotent. Then the norm one torus $T = R^{(1)}_{K/k}(\mathbb{G}_m)$ is not retract k-rational.

Special case: $T = R^{(1)}_{K/k}(\mathbb{G}_m)$; norm one tori (2/4)

- K/k: a finite non-Galois, separable field extension
- L/k: the Galois closure of K/k.
- $G = \mathrm{Gal}(L/k)$, $H = \mathrm{Gal}(L/K) \lneq G$.

Theorem (Endo 2011)

Assume that all the Sylow subgroups of G are cyclic.
Then T is retract k-rational.
$T = R^{(1)}_{K/k}(\mathbb{G}_m)$ is stably k-rational $\iff$ $G = D_n$, n odd $(n \geq 3)$ or $C_m \times D_n$, m, n odd $(m, n \geq 3)$, $(m, n) = 1$, $H \leq D_n$ with $|H| = 2$.

Special case: $T = R^{(1)}_{K/k}(\mathbb{G}_m)$; norm one tori (3/4)

Theorem (Endo 2011) $\dim T = n-1$

Assume that $\mathrm{Gal}(L/k) = S_n$, $n \geq 3$, and $\mathrm{Gal}(L/K) = S_{n-1}$ is the stabilizer of one of the letters in S_n.
(i) $R^{(1)}_{K/k}(\mathbb{G}_m)$ is retract k-rational $\iff$ n is a prime;
(ii) $R^{(1)}_{K/k}(\mathbb{G}_m)$ is (stably) k-rational $\iff$ $n = 3$.

Theorem (Endo 2011, Hoshi and Yamasaki 2017) $\dim T = n-1$

Assume that $\mathrm{Gal}(L/k) = A_n$, $n \geq 4$, and $\mathrm{Gal}(L/K) = A_{n-1}$ is the stabilizer of one of the letters in A_n.
(i) $R^{(1)}_{K/k}(\mathbb{G}_m)$ is retract k-rational $\iff$ n is a prime;
(ii) $R^{(1)}_{K/k}(\mathbb{G}_m)$ is stably k-rational $\iff$ $n = 5$.

Special case: $T = R^{(1)}_{K/k}(\mathbb{G}_m)$; norm one tori (4/4)

Known results on stably/retract k-rational classification for T

- $G = nTm \leq S_n$ $(n \leq 10)$ and $G \neq 9T27 \simeq PSL_2(\mathbb{F}_8)$ with $[G:H] = n$, $G = pTm \leq S_p$ and $G \neq PSL_2(\mathbb{F}_{2^e})$ $(p = 2^e + 1 \geq 17$; Fermat prime$)$ with $[G:H] = p$ (Hoshi-Yamasaki [HY21] Israel J. Math.).
- $G = nTm \leq S_n$ $(n = 12, 14, 15)$, $(n = 2^e)$ with $[G:H] = n$ (Hasegawa-Hoshi-Yamasaki [HHY20] Math. Comp.).

$Ш(T)$ and Hasse norm principle over number fields k (see next slides)

- $[K:k] = n \leq 15$ (Hoshi-Kanai-Yamasaki [HKY22], Math. Comp., [HKY23] J. Number Theory).
- T is retract k-rational (k: number field) $\Rightarrow Ш(T) = 0$ (Hasse norm principle holds for K/k).

$\text{Ш}(T)$ and HNP for K/k: Ono's theorem (1963)

- T : algebraic k-torus, i.e. $T \times_k \overline{k} \simeq (\mathbb{G}_{m,\overline{k}})^n$.
- $\text{Ш}(T) := \mathrm{Ker}\{H^1(k,T) \xrightarrow{\mathrm{res}} \bigoplus_{v \in V_k} H^1(k_v,T)\}$: Shafarevich-Tate gp.
- $T = R^{(1)}_{K/k}(\mathbb{G}_m)$ is biregularly isomorphic to the norm hyper surface $f(x_1,\ldots,x_n) = 1$ where $f \in k[x_1,\ldots,x_n]$ is the norm form of K/k.

Theorem (Ono 1963, Ann. of Math.)

Let K/k be a finite extension of number fields and $T = R^{(1)}_{K/k}(\mathbb{G}_m)$. Then

$$\text{Ш}(T) \simeq (N_{K/k}(\mathbb{A}_K^\times) \cap k^\times)/N_{K/k}(K^\times)$$

where $\mathbb{A}_K^\times$ is the idele group of K. In particular,

$$\text{Ш}(T) = 0 \iff \text{Hasse norm principle holds for } K/k.$$

Main theorems: Theorem 1 and Theorem 2 (1/3)

- K/k: a finite separable field extension.
- L/k: the Galois closure of K/k.
- $G = \mathrm{Gal}(L/k)$, $H = \mathrm{Gal}(L/K) \lneq G$.
- $T = R^{(1)}_{K/k}(\mathbb{G}_m)$; the norm one torus of K/k (with $\dim T = [K:k] - 1 = [G:H] - 1$).
- Theorem 1 gives an answer to the rationality problem (up to stable k-equivalence) for some norm one tori $R^{(1)}_{K/k}(\mathbb{G}_m)$ of K/k (when K/k is Galois, i.e. $H = 1$, this theorem is due to Endo and Miyata (1975)).

Theorem 1 (Hoshi and Yamasaki, arXiv:2309.16187)

(1) When $G \simeq A_4 \simeq \mathrm{PSL}_2(\mathbb{F}_3)$, $A_5 \simeq \mathrm{PSL}_2(\mathbb{F}_5) \simeq \mathrm{PSL}_2(\mathbb{F}_4) \simeq \mathrm{PGL}_2(\mathbb{F}_4) \simeq \mathrm{SL}_2(\mathbb{F}_4)$, $A_6 \simeq \mathrm{PSL}_2(\mathbb{F}_9)$, T is not retract k-rational except for the two cases $(G,H) \simeq (A_5, V_4)$, (A_5, A_4) with $|G| = 60$, $[G:H] = 15, 5$. For the two exceptional cases, T is stably k-rational;

Theorem 1 (Hoshi and Yamasaki, arXiv:2309.16187) (Continue)

(2) When $G \simeq S_3 \simeq \mathrm{PSL}_2(\mathbb{F}_2) \simeq \mathrm{PGL}_2(\mathbb{F}_2) \simeq \mathrm{SL}_2(\mathbb{F}_2) \simeq \mathrm{GL}_2(\mathbb{F}_2)$, $S_4 \simeq \mathrm{PGL}_2(\mathbb{F}_3)$, $S_5 \simeq \mathrm{PGL}_2(\mathbb{F}_5)$, S_6, T is not retract k-rational except for the six cases $(G,H) \simeq (S_3,\{1\})$, (S_3,C_2), (S_5,V_4) satisfying $V_4 \leq D(S_5) \simeq A_5$, (S_5,D_4), (S_5,A_4), (S_5,S_4) with $|S_3| = 6$, $[S_3 : H] = 6$, 3, $|S_5| = 120$, $[S_5 : H] = 30$, 15, 10, 5.
For the two exceptional cases $(S_3,\{1\})$, (S_3,C_2), T is stably k-rational.
For the four exceptional cases (S_5,V_4) satisfying $V_4 \leq D(S_5) \simeq A_5$, (S_5,D_4), (S_5,A_4), (S_5,S_4), T is not stably but retract k-rational;
(3) When $G \simeq \mathrm{GL}_2(\mathbb{F}_3)$, $\mathrm{GL}_2(\mathbb{F}_4) \simeq A_5 \times C_3$, $\mathrm{GL}_2(\mathbb{F}_5)$, T is not retract k-rational except for the case $(G,H) \simeq (\mathrm{GL}_2(\mathbb{F}_4), A_4)$ satisfying $A_4 \leq D(G) \simeq A_5$ with $|G| = 180$, $[G : H] = 15$.
For the exceptional case, T is stably k-rational;
(4) When $G \simeq \mathrm{SL}_2(\mathbb{F}_3)$, $\mathrm{SL}_2(\mathbb{F}_5)$, $\mathrm{SL}_2(\mathbb{F}_7)$, T is not retract k-rational;
(5) When $G \simeq \mathrm{PSL}_2(\mathbb{F}_7) \simeq \mathrm{PSL}_3(\mathbb{F}_2)$, T is not retract k-rational except for the two cases $H \simeq D_4$, S_4 with $|G| = 168$, $[G : H] = 21, 7$.
For the two exceptional cases, T is not stably but retract k-rational;

Theorem 1 (Hoshi and Yamasaki, arXiv:2309.16187) (Continue)

(6) When $G \simeq \mathrm{PSL}_2(\mathbb{F}_8) \simeq \mathrm{PGL}_2(\mathbb{F}_8) \simeq \mathrm{SL}_2(\mathbb{F}_8)$, T is not retract k-rational except for the two cases $H = \mathrm{Sy}_2(G) \simeq (C_2)^3$, $N_G(\mathrm{Sy}_2(G)) \simeq (C_2)^3 \rtimes C_7$ with $|G| = 504$, $[G : H] = 63$, 9.
For the two exceptional cases, T is stably k-rational.

In particular, for the exceptional cases in (1)–(6), e.g. $(G, H) \simeq (\mathrm{PSL}_2(\mathbb{F}_7), D_4)$, $(\mathrm{PSL}_2(\mathbb{F}_8), (C_2)^3)$, $(\mathrm{PSL}_2(\mathbb{F}_8), (C_2)^3 \rtimes C_7)$ with $[G : H] = 21$, 63, 9, T is retract k-rational.
Therefore, we get the vanishing $H^1(k, \mathrm{Pic}\,\overline{X}) \simeq H^1(G, \mathrm{Pic}\,X_L) \simeq H^1(G, F) \simeq Ш^2_\omega(G, J_{G/H}) \simeq \mathrm{Br}(X)/\mathrm{Br}(k) \simeq \mathrm{Br}_{\mathrm{nr}}(k(X)/k)/\mathrm{Br}(k) = 0$ where X is a smooth k-compactification of T. This implies that, when k is a global field, i.e. a finite extension of $\mathbb{Q}$ or $\mathbb{F}_q(t)$, $A(T) = 0$ and $Ш(T) = 0$, i.e. T has the weak approximation property, Hasse principle holds for all torsors E under T and Hasse norm principle holds for K/k.

Remark

(1) The case where $G \leq S_n$ is transitive and $[G:H]=n$ ($n \leq 15$, $n=2^e$ or $n=p$ is prime) was solved by Hasegawa, Hoshi and Yamasaki (2020) Hoshi and Yamasaki (2021) except for the stable k-rationality of T with $G \simeq 9T27$ and $G \leq S_p$ for Fermat primes $p \geq 17$. Theorem 1 (6) gives an answer for $G \simeq 9T27$ as $(G,H) \simeq (\mathrm{PSL}_2(\mathbb{F}_8), (C_2)^3 \rtimes C_7)$.
(2) $H^1(k, \mathrm{Pic}\,\overline{X})$, $A(T)$ and $\text{Ш}(T)$ were investigated by Macedo and Newton (2022) when $G \simeq A_n$, S_n and by Hoshi, Kanai and Yamasaki (2022, 2023, arXiv:2210.09119) when $[G:H] \leq 15$ and $G \simeq M_{11}$, J_1.

More precisely, for the stably k-rational cases $G \simeq S_3 \simeq \mathrm{PSL}_2(\mathbb{F}_2)$, $A_5 \simeq \mathrm{PSL}_2(\mathbb{F}_4)$, $\mathrm{PSL}_2(\mathbb{F}_8)$ as in Theorem 1 we prove the following result which implies that there exists a rational k-torus $T' = \bigoplus_i R_{k_i/k}(\mathbb{G}_m)$ (for some $k \subset k_i \subset L$) of dimension r such that $T \times T'$ is k-rational.

Main theorems: Theorem 1 and Theorem 2 (3/3)

- $G = \mathrm{Gal}(L/k)$ and $H = \mathrm{Gal}(L/K) \lneq G$.
- $J_{G/H} = (I_{G/H})^\circ = \mathrm{Hom}_{\mathbb{Z}}(I_{G/H}, \mathbb{Z}) \simeq \widehat{T} = \mathrm{Hom}(T, \mathbb{G}_m)$: Chevalley module with $0 \to \mathbb{Z} \to \mathbb{Z}[G/H] \to J_{G/H} \to 0$ which is dual to $0 \to I_{G/H} \to \mathbb{Z}[G/H] \xrightarrow{\varepsilon} \mathbb{Z} \to 0$ where ε is the augmentation map.
- $T = R^{(1)}_{K/k}(\mathbb{G}_m)$: the norm one torus of K/k (with $\dim T = [K:k] - 1 = [G:H] - 1$) whose function field over k is $k(T) \simeq L(J_{G/H})^G$.

Theorem 2 (Hoshi and Yamasaki, arXiv:2309.16187)

(1) When $(G, H) \simeq (S_3, \{1\}) \simeq (\mathrm{PSL}_2(\mathbb{F}_2), \{1\})$ with $[G:H] = 6$, there exists the flabby class $F = [J_{G/H}]^{fl}$ with $\mathrm{rank}_{\mathbb{Z}} F = 7$ such that

$$\mathbb{Z}[S_3/C_2]^{\oplus 2} \oplus \mathbb{Z}[S_3/C_3] \simeq \mathbb{Z} \oplus F$$

holds with rank $r = 2 \cdot 3 + 2 = 1 + 7 = 8$.

Theorem 2 (Hoshi and Yamasaki, arXiv:2309.16187) (Continue)

(2) When $(G,H)\simeq(S_3,C_2)\simeq(\mathrm{PSL}_2(\mathbb{F}_2),C_2)$ with $[G:H]=3$, there exists the flabby class $F=[J_{G/H}]^{fl}$ with $\mathrm{rank}_{\mathbb{Z}}\,F=4$ such that

$$\mathbb{Z}[S_3/C_2]\oplus\mathbb{Z}[S_3/C_3]\simeq\mathbb{Z}\oplus F$$

holds with rank $r=3+2=1+4=5$. In particular, we get

$$\mathbb{Z}[S_3/C_2]^{\oplus 2}\oplus\mathbb{Z}[S_3/C_3]\simeq\mathbb{Z}\oplus(\mathbb{Z}[S_3/C_2]\oplus F)$$

holds with rank $r'=2\cdot 3+2=1+7=8$.

(3) When $(G,H)\simeq(A_5,V_4)\simeq(\mathrm{PSL}_2(\mathbb{F}_4),V_4)$ with $[G:H]=15$, there exists the flabby class $F=[J_{G/H}]^{fl}$ with $\mathrm{rank}_{\mathbb{Z}}\,F=21$ such that

$$\mathbb{Z}[A_5/C_5]\oplus\mathbb{Z}[A_5/A_4]^{\oplus 2}\simeq\mathbb{Z}\oplus F$$

holds with rank $r=12+2\cdot 5=1+21=22$.

Theorem 2 (Hoshi and Yamasaki, arXiv:2309.16187) (Continue)

(4) When $(G, H) \simeq (A_5, A_4) \simeq (\mathrm{PSL}_2(\mathbb{F}_4), A_4)$ with $[G : H] = 5$, there exists the flabby class $F = [J_{G/H}]^{fl}$ with $\mathrm{rank}_{\mathbb{Z}} F = 16$ such that

$$\mathbb{Z}[A_5/C_5] \oplus \mathbb{Z}[A_5/S_3] \simeq \mathbb{Z}[A_5/D_5] \oplus F$$

holds with rank $r = 12 + 10 = 6 + 16 = 22$.

(5) When $(G, H) \simeq (\mathrm{PSL}_2(\mathbb{F}_8), (C_2)^3)$ with $[G : H] = 63$, there exists the flabby class $F = [J_{G/H}]^{fl}$ with $\mathrm{rank}_{\mathbb{Z}} F = 73$ such that

$$\mathbb{Z}[G/S_3] \oplus \mathbb{Z}[G/C_9] \oplus \mathbb{Z}[G/((C_2)^3 \rtimes C_7)]^{\oplus 2} \simeq \mathbb{Z}[G/S_3] \oplus \mathbb{Z} \oplus F$$

holds with rank $r = 84 + 56 + 2 \cdot 9 = 84 + 1 + 73 = 158$.

(6) When $(G, H) \simeq (\mathrm{PSL}_2(\mathbb{F}_8), (C_2)^3 \rtimes C_7)$ with $[G : H] = 9$, there exists the flabby class $F = [J_{G/H}]^{fl}$ with $\mathrm{rank}_{\mathbb{Z}} F = 64$ such that

$$\mathbb{Z}[G/C_3] \oplus \mathbb{Z}[G/C_9] \oplus \mathbb{Z}[G/D_7] \simeq \mathbb{Z}[G/C_3] \oplus \mathbb{Z}[G/D_9] \oplus F$$

holds with rank $r = 168 + 56 + 36 = 168 + 28 + 64 = 260$.

Theorem 2 (Hoshi and Yamasaki, arXiv:2309.16187) (Continue)

In particular, for the cases (1)–(6), $F = [J_{G/H}]^{fl}$ is stably permutation and hence $T = R^{(1)}_{K/k}(\mathbb{G}_m)$ is stably k-rational. More precisely, there exists a rational k-torus T' of dimension r such that $\widehat{T'} = \mathrm{Hom}(T', \mathbb{G}_m)$ is isomorphic to the permutation G-lattice with rank r in the left-hand side of the isomorphism, i.e. $r = 8,\ 5,\ 22,\ 22,\ 158,\ 260$, and $T \times T'$ is k-rational.

- We conjecture that $T = R^{(1)}_{K/k}(\mathbb{G}_m)$ is stably k-rational for the cases
$$(G,H) \simeq (\mathrm{PSL}_2(\mathbb{F}_{2^d}), (C_2)^d)\ (d \geq 1),$$
$$(\mathrm{PSL}_2(\mathbb{F}_{2^d}), (C_2)^d \rtimes C_{2^d-1})\ (d \geq 1)$$
(see Theorem 2 for $d = 1,\ 2,\ 3$):

Conjecture

Conjecture (Hoshi and Yamasaki, arXiv:2309.16187)

When $G \simeq \mathrm{PSL}_2(\mathbb{F}_{2^d})$ $(d \geq 1)$, T is not retract k-rational except for the case $(G, H) \simeq (S_3, \{1\})$ $(d = 1)$ and the two cases $H = \mathrm{Sy}_2(G) \simeq (C_2)^d$, $H = N_G(\mathrm{Sy}_2(G)) \simeq (C_2)^d \rtimes C_{d-1}$ with $|G| = (2^d + 1)2^d(2^d - 1)$, $[G : H] = 2^{2d} - 1 = (2^d + 1)(2^d - 1)$, $2^d + 1$ $(d \geq 1)$.
For the exceptional cases, T is stably k-rational. Moreover,
(1) for $H \simeq (C_2)^d$ $(d \geq 1)$, there exist the flabby class $F = [J_{G/H}]^{fl}$ with $\mathrm{rank}_{\mathbb{Z}}\, F = 2^{2d} + 2^d + 1$ and a permutation G-lattice P such that

$$P \oplus \mathbb{Z}[G/C_{2^d+1}] \oplus \mathbb{Z}[G/((C_2)^d \rtimes C_{2^d-1})]^{\oplus 2} \simeq P \oplus \mathbb{Z} \oplus F$$

holds with
$\mathrm{rank}_{\mathbb{Z}}\, P + 2^d(2^d - 1) + 2 \times (2^d + 1) = \mathrm{rank}_{\mathbb{Z}}\, P + 1 + (2^{2d} + 2^d + 1)$;

Conjecture (Hoshi and Yamasaki, arXiv:2309.16187) (Continue)

(2) for $H \simeq (C_2)^d \rtimes C_{2^d-1}$ $(d \geq 1)$, there exist the flabby class $F = [J_{G/H}]^{fl}$ with $\mathrm{rank}_{\mathbb{Z}} F = 2^{2d}$ and a permutation G-lattice Q such that

$$Q \oplus \mathbb{Z}[G/C_{2^d+1}] \oplus \mathbb{Z}[G/D_{2^d-1}] \simeq Q \oplus \mathbb{Z}[G/D_{2^d+1}] \oplus F$$

holds with
$\mathrm{rank}_{\mathbb{Z}} Q + 2^d(2^d-1) + 2^{d-1}(2^d+1) = \mathrm{rank}_{\mathbb{Z}} Q + 2^{d-1}(2^d-1) + 2^{2d}$
where $D_1 = C_2$ $(d = 1)$.

Note that Theorem 2 claims that
Conjecture (1) holds for $d = 1, 2, 3$ with $\mathrm{rank}_{\mathbb{Z}} P = 0, 0, 84$;
Conjecture (2) holds for $d = 1, 2, 3$ with $\mathrm{rank}_{\mathbb{Z}} Q = 0, 0, 168$.