On the simplest cubic fields and related Thue equations

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- On correspondence between solutions of a family of cubic Thue equations and isomorphism classes of the simplest cubic fields, J. Number Theory **131** (2011) 2135–2150.

§0 fun

Diophantine Approximation

Approximating $\alpha \in \mathbb{R}$ by $\dfrac{p}{q} \in \mathbb{Q}$.

Example

- $\alpha = \sqrt{2}$.
  $1, \frac{14}{10}, \frac{141}{100}, \frac{1414}{1000}, \frac{14142}{10000}, \frac{141421}{100000}, \frac{1414213}{1000000}, \frac{14142135}{10000000},$
  $\frac{141421356}{100000000}, \frac{1414213562}{1000000000}, \frac{14142135623}{10000000000}, \frac{141421356237}{100000000000}, \ldots$
- $\alpha = \pi$.
  $3, \frac{31}{10}, \frac{314}{100}, \frac{3141}{1000}, \frac{31415}{10000}, \frac{314159}{100000}, \frac{3141592}{1000000}, \frac{31415926}{10000000},$
  $\frac{314159265}{100000000}, \frac{3141592653}{1000000000}, \frac{31415926535}{10000000000}, \frac{314159265358}{100000000000}, \ldots$

Diophantine approximation

- $\alpha = \sqrt{2} \approx 1.4142135623730950488.$

  $1, \frac{14}{10}, \frac{141}{100}, \frac{1414}{1000}, \frac{14142}{10000}, \frac{141421}{100000}, \frac{1414213}{1000000}, \frac{14142135}{10000000},$

  $\frac{141421356}{100000000}, \frac{1414213562}{1000000000}, \frac{14142135623}{10000000000}, \frac{141421356237}{100000000000}, \ldots$
- $\frac{3}{2} = 1.5, \frac{7}{5} = 1.4, \frac{17}{12} \approx 1.4166, \frac{41}{29} \approx 1.4139$
- $\frac{99}{70} \approx 1.4128, \frac{239}{169} \approx 1.414201, \frac{577}{408} \approx 1.414215$
- $\frac{1393}{985} \approx 1.4142131, \frac{3363}{2378} \approx 1.4142136$
- $\frac{8119}{5741} \approx 1.41421355, \frac{19601}{13860} \approx 1.414213564$
- $\frac{47321}{33461} \approx 1.4142135620, \frac{114243}{80782} \approx 1.4142135624$
- $\frac{275807}{195025} \approx 1.41421356236, \frac{665857}{470832} \approx 1.414213562374$
- $\frac{1607521}{1136689} \approx 1.414213562372, \frac{3880899}{2744210} \approx 1.4142135623731$
- $\frac{9369319}{6625109} \approx 1.41421356237308$
- $\frac{22619537}{15994428} \approx 1.414213562373096$
- $\frac{54608393}{38613965} \approx 1.4142135623730948$

Diophantine approximation

- $\alpha = \pi \approx 3.14159265358979323846\,26.$

  $3,\ \frac{31}{10},\ \frac{314}{100},\ \frac{3141}{1000},\ \frac{31415}{10000},\ \frac{314159}{100000},\ \frac{3141592}{1000000},\ \frac{31415926}{10000000},$
  $\frac{314159265}{100000000},\ \frac{3141592653}{1000000000},\ \frac{31415926535}{10000000000},\ \frac{314159265358}{100000000000},\ \ldots$
- $\frac{22}{7} \approx 3.142,\ \frac{333}{106} \approx 3.14150,\ \frac{355}{113} \approx 3.1415929$
- $\frac{103993}{33102} \approx 3.1415926530,\ \frac{104348}{33215} \approx 3.1415926539$
- $\frac{208341}{66317} \approx 3.1415926534,\ \frac{312689}{99532} \approx 3.1415926536$
- $\frac{833719}{265381} \approx 3.141592653581,\ \frac{1146408}{364913} \approx 3.141592653591$
- $\frac{4272943}{1360120} \approx 3.1415926535893$
- $\frac{5419351}{1725033} \approx 3.1415926535898$
- $\frac{80143857}{25510582} \approx 3.141592653589792$
- $\frac{165707065}{52746197} \approx 3.141592653589793\,4$
- $\frac{245850922}{78256779} \approx 3.141592653589793\,1$
- $\frac{411557987}{131002976} \approx 3.141592653589793\,25$
- $\frac{1068966896}{340262731} \approx 3.141592653589793\,235$

Dirichlet's theorem

Theorem (Dirichlet 1842)

Let α, $Q \in \mathbb{R}$ $(Q > 1)$.

$\exists p, q \in \mathbb{Z}$ s.t. $\gcd(p, q) = 1$, $1 \leq q \leq Q$ and $|q\alpha - p| < \dfrac{1}{Q}$.

- $\left|\alpha - \dfrac{p}{q}\right| \leq \dfrac{1}{qQ} < \dfrac{1}{q^2}$.

Corollary (Dirichlet 1842)

If α is irrational, then $\exists\infty\ \dfrac{p}{q} \in \mathbb{Q}$ s.t. $\left|\alpha - \dfrac{p}{q}\right| < \dfrac{1}{q^2}$.

How to construct such a "good" $\dfrac{p}{q}$?

- Continued fractions.

Go back to Theorem of continued fractions

Definition (Continued fractions)

A finite or infinite expression of the form

$$[a_0; a_1, a_2, \cdots] = a_0 + \cfrac{1}{a_1 + \frac{1}{a_2 + \cdots}}$$

where $a_i > 0$, is called a continued fraction.

- $\sqrt{2} = [1; 2, 2, 2, 2, \ldots]$, $\sqrt{3} = [1; 1, 2, 1, 2, 1, 2, \ldots]$, $\pi = [3; 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, 2, \ldots]$.

Definition (Convergents)

Let $\alpha = [a_0; a_1, a_2, \ldots]$ be a continued fraction.
The i-th convergent of α is $C_i = [a_0; a_1, \ldots, a_i]$.
$[a_0; a_1, \ldots]$ is called convergent if $\lim_{i \to \infty} C_i = \exists \alpha \in \mathbb{R}$.

- For any $a_i \in \mathbb{Z}_{>0}$, $[a_0; a_1, a_2, \ldots]$ is convergent.

Let $\alpha = [a_0; a_1, \ldots]$ be a simple c.f. (i.e. $a_i \in \mathbb{Z}_{>0}$).
Let $C_i = [a_0; a_1, \ldots, a_i]$ be the i-th convergent to α.

Lemma

C_i is given by $C_i = p_i/q_i$ for $i \geq 0$ where

$$\begin{cases} p_{-1} = 1, q_{-1} = 0, \\ p_0 = a_0, q_0 = 1, \\ p_i = a_i p_{i-1} + p_{i-2}, q_i = a_i q_{i-1} + q_{i-2} \ (i \geq 1). \end{cases}$$

Lemma

(1) $p_i q_{i-1} - p_{i-1} q_i = (-1)^i$ for $i \geq 0$;
(2) $(p_i, q_i) = 1$;
(3) $1 = q_0 \leq q_1 < q_2 < q_3 < \cdots$;
(4) $C_0 < C_2 < C_4 < \cdots < \alpha < \cdots < C_5 < C_3 < C_1$;
(5) $|C_i - C_{i+1}| = 1/(q_i q_{i+1})$ for $i \geq 0$.

Let $\alpha = [a_0; a_1, \ldots]$ be a simple c.f. (i.e. $a_i \in \mathbb{Z}_{>0}$).
Let $C_i = [a_0; a_1, \ldots, a_i]$ be the i-th convergent to α.

Theorem

(1) Any convergent $C_i = p_i/q_i$ satisfies

$$\left|\alpha - \frac{p_i}{q_i}\right| < \frac{1}{q_i q_{i+1}} < \frac{1}{q_i^2};$$

(2) Conversely, if $p/q \in \mathbb{Q}$ satisty

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{2q^2},$$

then $\frac{p}{q}$ is one of the convergents of α (i.e. $\frac{p}{q} = C_i = \frac{p_i}{q_i}$).

Jump to Dirichlet's Theorem

§1 Introduction: Brief history of Thue equations

Definition (Thue equations)

Let $F(x,y) \in \mathbb{Z}[x,y]$ be an irreducible binary form of degree $n \geq 3$. A Diophantine equation

$$F(x,y) = a_n x^n + a_{n-1}x^{n-1}y + \cdots + a_1 xy^{n-1} + a_0 y^n = \lambda$$

where $\lambda \in \mathbb{Z}$ $(\lambda \neq 0)$, is called a Thue equation.

Theorem (Thue 1909)

A Thue equation has finitely many solutions $(x,y) \in \mathbb{Z}^2$.

Example

The equation $F(x,y) = x^3 - 3xy^2 - y^3 = 1$ has only 6 integer solutions:
$(x,y) = (0,-1), (-1,1), (1,0), (2,1), (1,-3), (-3,2)$.

Let α be an algebraic number of degree $n \geq 2$,
i.e. $f(\alpha) = 0$ with $f(x) = a_n x^n + \cdots + a_1 x + a_0$, $a_i \in \mathbb{Z}$.

Theorem (Liouville 1844)

$$\exists c_1 = c_1(\alpha) > 0 \quad \text{s.t.} \quad \left|\alpha - \frac{x}{y}\right| > \frac{c_1}{y^n} \quad \text{for } \forall x, y (y > 0)$$

This theorem is effective, i.e. can compute c_1 explicitly.
Indeed, we may take

$$c_1(\alpha) = \min(1, 1/C), C = \max_{[\alpha-1,\alpha+1]} |f'(x)|.$$

Example

- $\alpha = \sqrt{2}$, $c_1 = \frac{1}{2(\alpha+1)} \approx 0.20717$
- $\alpha = \sqrt[3]{2}$, $c_1 = \frac{1}{3(\alpha+1)^2} \approx 0.0652668$

Let α be an algebraic number of degree $n \geq 2$,
i.e. $f(\alpha) = 0$ with $f(x) = a_n x^n + \cdots + a_1 x + a_0$, $a_i \in \mathbb{Z}$.

Theorem (Thue 1909)

$$\exists c_2 = c_2(\alpha) > 0 \quad \text{s.t.} \quad \left|\alpha - \frac{x}{y}\right| > \frac{c_2}{y^{\frac{n}{2}+1}} \quad \text{for } \forall x, y \ (y > 0).$$

- This theorem is not effective (cannot compute c_2).

Corollary (Thue 1909)

$\left|\alpha - \frac{x}{y}\right| < \frac{1}{y^{\frac{n}{2}+1+\varepsilon}}$ has only finitely many solutions $\frac{x}{y} \in \mathbb{Q}$.

Through many mathematicians' efforts, Roth proved:

Theorem (Roth 1955)

$\left|\alpha - \frac{x}{y}\right| < \frac{1}{y^{2+\varepsilon}}$ has only finitely many solutions $\frac{x}{y} \in \mathbb{Q}$.

Proof of Thue's theorem : $F(x,y)=\lambda$ has only finitely many solutions

By contradiction. Suppose not.

Let $F(x,1)=f(x)=a_n(x-\alpha_1)\cdots(x-\alpha_n)$.
Define $\gamma=\min_{i\neq j}|\alpha_i-\alpha_j|$.

Claim 1 $\exists i\ \exists\infty(x,y)$ with $F(x,y)=\lambda$ and $|\alpha_i-\frac{x}{y}|<\frac{\gamma}{2}$.
$\because$ If not, $\exists\infty(x,y)$ s.t. $|\lambda|=|F(x,y)|=|y^nf(\frac{x}{y})|=$
$|a_ny^n|\prod_{i=1}^n|\frac{x}{y}-\alpha_i|\geq|a_ny^n|\left(\frac{\gamma}{2}\right)^n\to\infty\ (|y|\to\infty)$.

Fix $\alpha=\alpha_1$ with $\exists\infty(x,y)$ s.t. $|\alpha-\frac{x}{y}|<\frac{\gamma}{2}$ (WLOG).

Claim 2 For $i=2,\ldots,n$, $|\frac{x}{y}-\alpha_i|>\frac{\gamma}{2}$.
$\because\ \gamma\leq|\alpha-\alpha_i|\leq|\alpha-\frac{x}{y}|+|\frac{x}{y}-\alpha_i|<\frac{\gamma}{2}+|\frac{x}{y}-\alpha_i|$.

$$\frac{\exists c_2}{|y|^{\frac{n}{2}+1}}<\left|\alpha-\frac{x}{y}\right|=\left|\frac{F(x,y)}{a_ny^n\prod_{i=2}^n(\frac{x}{y}-\alpha_i)}\right|<\frac{|\lambda|}{|a_n||y^n|(\frac{\gamma}{2})^{n-1}}.$$

($\uparrow$ by Thue's Theorem) ($\uparrow$ by Claim 2)

Hence $|y|<\left(\frac{\lambda}{c_2|a_n|(\frac{\gamma}{2})^{n-1}}\right)^{1/(n-\frac{n}{2}-1)}$. Contradiction. $\square$

Effective methods: Baker's theory

Theorem (Baker 1967)

Let $n \geq 2$. Let $\alpha_1, \ldots, \alpha_n$ be non-zero algebraic numbers s.t. $\log \alpha_1, \ldots, \log \alpha_n$ are linearly independent over $\mathbb{Q}$.
Suppose $\kappa > 2n+1$ and $d \in \mathbb{Z}_{>0}$.
Then $\exists C = C(n, \alpha_1, \ldots, \alpha_n, \kappa, d) > 0$; effectively computable number s.t.
$\forall$ algebraic numbers $\beta_1, \ldots, \beta_n$ not all zero with $\deg \beta_i \leq d$,

$$|\beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n| > Ce^{-(\log H)^\kappa}$$

where $H = \max\{H(\beta_1), \ldots, H(\beta_n)\}$ and $H(\beta_i)$; hight of β_i.

Theorem (Baker 1968) Suppose $\kappa > n+1 \geq 4$.
All solutions $(x,y) \in \mathbb{Z}^2$ of Thue eq. $F(x,y) = \lambda$ satisfy

$$\max\{|x|, |y|\} < Ce^{(\log \lambda)^\kappa}$$

where $C = C(n, \kappa, a_0, \ldots, a_n)$ is effectively computable.

Thomas' theorem for a family of Thue equations

$$F_m^{(3)}(X,Y) = X^3 - mX^2Y - (m+3)XY^2 - Y^3 = 1$$

By using Baker's theory, Thomas proved

Theorem (Thomas 1990)

If $-1 \leq m \leq 10^3$ or $1.365 \times 10^7 \leq m$, then all solutions of $F_m^{(3)}(x,y) = 1$ are given by trivial solutions $(x,y) = (0,-1), (-1,1), (1,0)$ for $\forall m$ and additionally

$$\begin{aligned}
(x,y) &= (-1,-1), (-1,2), (2,-1) && \text{for} \quad m = -1,\\
(x,y) &= (5,4), (4,-9), (-9,5) && \text{for} \quad m = -1,\\
(x,y) &= (2,1), (1,-3), (-3,2) && \text{for} \quad m = 0,\\
(x,y) &= (-7,-2), (-2,9), (9,-7) && \text{for} \quad m = 2.
\end{aligned}$$

Theorem (Mignotte 1993)

For the remaining case, $\exists$ only trivial solutions.

Theorem C

§2 Results

We consider Thomas' family of cubic Thue equations

$$\boxed{F_m^{(3)}(X,Y) := X^3 - mX^2Y - (m+3)XY^2 - Y^3 = \lambda}$$

for $m \in \mathbb{Z}$ and general $\lambda \in \mathbb{Z}$ $(\lambda \neq 0)$.

- We may assume that $-1 \leq m$ and $0 < \lambda$ because

$$F_{-m-3}^{(3)}(X,Y) = F_m^{(3)}(-Y,-X),$$
$$-F_m^{(3)}(X,Y) = F_m^{(3)}(-X,-Y).$$

- $\lambda = a^3$ for some $a \in \mathbb{Z}$, $F_m^{(3)}(x,y) = a^3$ has three trivial solutions $(a,0), (0,-a), (-a,a)$, i.e. $xy(x+y) = 0$.
- If $(x,y) \in \mathbb{Z}^2$ is solution, then $(y,-x-y)$, $(-x-y,x)$ are also solutions because $F_m^{(3)}(x,y)$ is invariant under the action $x \longmapsto y \longmapsto -x-y \longmapsto x$ of order three.

Mignotte-Pethö-Lemmermeyer (1996)

$$F_m^{(3)}(X,Y) = X^3 - mX^2Y - (m+3)XY^2 - Y^3 = \lambda$$

By using Baker's theory, they proved:

Theorem Mignotte-Pethö-Lemmermeyer (1996)

Let $m \geq 1649$ and $\lambda > 1$. If $F_m^{(3)}(x,y) = \lambda$, then

$$\log|y| < c_1 \log^2(m+3) + c_2 \log(m+1) \log\lambda$$

where

$$c_1 = 700 + 476.4\left(1 - \frac{1432.1}{m+1}\right)^{-1}\left(1.501 - \frac{1902}{m+1}\right) < 1956.4,$$

$$c_2 = 29.82 + \left(1 - \frac{1432.1}{m+1}\right)^{-1}\frac{1432}{(m+1)\log(m+1)} < 30.71.$$

Example (much smaller than previous bounds)

- If $m = 1649$ and $\lambda = 10^9$, then $|y| < 10^{48698}$.

Mignotte-Pethö-Lemmermeyer (1996)

$$F_m^{(3)}(X,Y) = X^3 - mX^2Y - (m+3)XY^2 - Y^3 = \lambda$$

By using Baker's theory, they proved:

Theorem Mignotte-Pethö-Lemmermeyer (1996)

For $-1 \le m$ and $1 < \lambda \le 2m+3$, all solutions to $F_m^{(3)}(x,y) = \lambda$ are given by trivial solutions for $\lambda = a^3$ and

$$(x,y) \in \{(-1,2),(2,-1),(-1,-1),\\ (-1,m+2),(m+2,-m-1),(-m-1,-1)\}$$

for $\lambda = 2m+3$,
except for $m=1$ in which case $\exists$extra solutions:

$$(x,y) \in \{(1,-4),(-4,3),(3,1),(3,-11),(-11,8),(8,3)\}$$

for $\lambda = 5$ $(= 2m+3)$.

Lettl-Pethö-Voutier (1999)

Let θ_2 be a root of $f_m(X) := F_m(X,1)$ with $-\frac{1}{2} < \theta_2 < 0$.
By using hypergeometric method, they proved:

Theorem Lettl-Pethö-Voutier (1999)

Let $m \geq 1$ and assume that $(x, y) \in \mathbb{Z}^2$ is a primitive solution to $|F_m^{(3)}(x, y)| \leq \lambda(m)$ with $-\frac{y}{2} < x \leq y$ and $\frac{8\lambda(m)}{2m+3} \leq y$ where $\lambda(m) : \mathbb{Z} \to \mathbb{N}$. Then

(i) x/y is a convergent to θ_2, and we have either $y = 1$ or

$$\left|\frac{x}{y} - \theta_2\right| < \frac{\lambda(m)}{y^3(m+1)} \quad \text{and} \quad y \geq m+2.$$

(ii)Define

$$\kappa = \frac{\log(\sqrt{m^2+3m+9}) + 0.83}{\log(m+\frac{3}{2}) - 1.3}.$$

If $m \geq 30$, then $y^{2-\kappa} < 17.78 \cdot 2.59^{\kappa}\lambda(m)$.

Example (comparing with MPL (1996))

- For $m = 1649$, $|y| < 635\lambda(m)^{1.54}$ instead of $|y| < 10^{46649}\lambda(m)^{288}$.

$$f_m^{(3)}(X) := F_m^{(3)}(X,1), \quad L_m^{(3)} := \mathrm{Spl}_{\mathbb{Q}} f_m^{(3)}(X)$$

- The splitting fields $L_m^{(3)}$ are totally real cyclic cubic fields which are called Shanks' simplest cubic fields.
- $L_m^{(3)} = L_{-m-3}^{(3)}$ for $m \in \mathbb{Z}$. $\mathrm{disc}_X f_m^{(3)} = (m^2+3m+9)^2$.

Theorem C (Correspondence)

For a given $m \in \mathbb{Z}$,
$\exists (x,y) \in \mathbb{Z}^2$ with $xy(x+y) \neq 0$ s.t. $F_m^{(3)}(x,y) = \lambda$
for some $\lambda \in \mathbb{N}$ with $\lambda \,|\, m^2+3m+9$
$\iff \exists n \in \mathbb{Z} \setminus \{m, -m-3\}$ s.t. $L_m^{(3)} = L_n^{(3)}$.
Moreover integers n, m and $(x,y) \in \mathbb{Z}^2$ satisfy

$$N = m + \frac{(m^2+3m+9)xy(x+y)}{F_m^{(3)}(x,y)}$$

where N is either n or $-n-3$.

Thomas' theorem

For a fixed $m \in \mathbb{Z}$, we obtain the correspondence

$$\boxed{\exists n \in \mathbb{Z} \setminus \{m, -m-3\} \text{ s.t. } L_m^{(3)} = L_n^{(3)}} \qquad \text{(I)}$$

$$1:3 \ \Updownarrow \ \text{Theorem C}$$

$$\boxed{\begin{array}{l} \exists (x,y) \in \mathbb{Z}^2 \text{ with } xy(x+y) \neq 0 \\ \qquad \text{s.t. } F_m^{(3)}(x,y) = \lambda \,|\, m^2+3m+9 \end{array}} \qquad \text{(II)}$$

- $\mathrm{disc}(F_m^{(3)}(X,Y)) = (m^2+3m+9)^2$.

Corollary

For a fixed $m \geq -1$,

$$\#\{n \geq -1 \,|\, \text{(I)}\} = \frac{\#\{(x,y) \,|\, \text{(II)} \ \& \ \gcd(x,y)=1\}}{3}.$$

R. Okazaki's theorems O_1, O_2

Okazaki announced the following theorems in 2002.
He use his result on gaps between sol's (2002) which is based on Baker's theory: Laurent-Mignotte-Nesterenko (1995).

Theorem O_1 (Okazaki 2002+α)
For $-1 \leq m < n \in \mathbb{Z}$, if $L_m^{(3)} = L_n^{(3)}$ then $m \leq 35731$.

Theorem O_2 (Okazaki unpublished)
For $-1 \leq m < n \in \mathbb{Z}$, if $L_m^{(3)} = L_n^{(3)}$ then $m, n \in \{-1, 0, 1, 2, 3, 5, 12, 54, 66, 1259, 2389\}$.

In particular, we get

$$L_{-1}^{(3)} = L_5^{(3)} = L_{12}^{(3)} = L_{1259}^{(3)},$$
$$L_0^{(3)} = L_3^{(3)} = L_{54}^{(3)}, \qquad L_1^{(3)} = L_{66}^{(3)}, \qquad L_2^{(3)} = L_{2389}^{(3)}.$$

A brief sketch of the proof (slide) is available at
`http://www1.doshisha.ac.jp/~rokazaki/papers.html`

Thomas' $4 \times 3 = 12$ non-trivial solutions for $\lambda = 1$

$$\begin{aligned}
(x,y) &= (-1,-1),(-1,2),(2,-1) && \text{for} \quad m=-1,\\
(x,y) &= (5,4),(4,-9),(-9,5) && \text{for} \quad m=-1,\\
(x,y) &= (2,1),(1,-3),(-3,2) && \text{for} \quad m=0,\\
(x,y) &= (-7,-2),(-2,9),(9,-7) && \text{for} \quad m=2
\end{aligned}$$

correspond to

$$\boxed{L_{-1}^{(3)} = L_{12}^{(3)}, \quad L_{-1}^{(3)} = L_{1259}^{(3)}, \quad L_{0}^{(3)} = L_{54}^{(3)}, \quad L_{2}^{(3)} = L_{2389}^{(3)}.}$$

$$\boxed{\begin{array}{l} L_{-1}^{(3)} = L_{5}^{(3)}, \quad L_{0}^{(3)} = L_{3}^{(3)}, \quad L_{1}^{(3)} = L_{66}^{(3)}, \quad L_{3}^{(3)} = L_{54}^{(3)}, \\ L_{5}^{(3)} = L_{12}^{(3)}, \quad L_{5}^{(3)} = L_{1259}^{(3)}, \quad L_{12}^{(3)} = L_{1259}^{(3)} \end{array}}$$

correspond to $7 \times 3 = \exists 21$ (non-trivial) solutions for $\lambda > 1$.

$L_m^{(3)} = L_n^{(3)}$ (33 solutions), $L_n^{(3)} = L_m^{(3)}$ (33 solutions)

Conclusion: in total $\exists 66$ solutions.

Theorem S: Solutions

$$F_m^{(3)}(X,Y) = X^3 - mX^2Y - (m+3)XY^2 - Y^3 = \lambda$$

By Theorem C and Theorems O_1, O_2, we get:

Theorem S

For $m \geq -1$,
all integer solutions $(x,y) \in \mathbb{Z}^2$ with $xy(x+y) \neq 0$
to $F_m^{(3)}(x,y) = \lambda$ with $\lambda \in \mathbb{N}$ and $\lambda \,|\, m^2+3m+9$
are given in Table 1. (66 solutions)

Corollary

For $m \geq -1$, all solutions to $F_m^{(3)}(x,y) = m^2+3m+9$ are given in Table 1 and $(x,y) = (3,0), (0,-3), (-3,3)$.

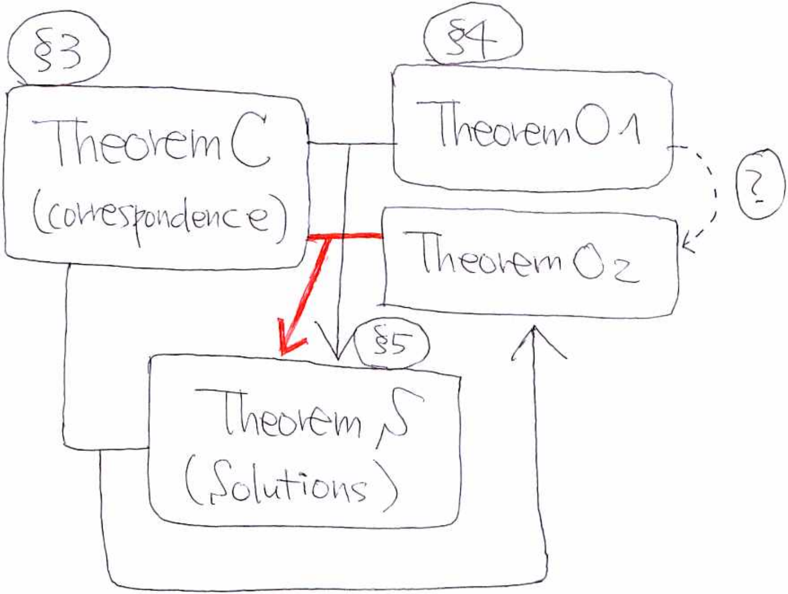
- $\because$ the elliptic curve $y^2 + 3y = x^3 - 9$ over $\mathbb{Q}$ has only two integral points $(x,y) = (3,-6), (3,3)$.

Table 1

m	n	$-n-3$	$2m+3$	λ	m^2+3m+9	(x,y)
-1	-15	12	1	1	7	$(-1,2)$, $(2,-1)$, $(-1,-1)$
-1	1259	-1262	1	1	7	$(4,-9)$, $(-9,5)$, $(5,4)$
-1	5	-8	1	7	7	$(1,-3)$, $(-3,2)$, $(2,1)$
0	54	-57	3	1	9	$(1,-3)$, $(-3,2)$, $(2,1)$
0	-6	3	3	3	9	$(-1,2)$, $(2,-1)$, $(-1,-1)$
1	-69	66	5	13	13	$(-2,7)$, $(7,-5)$, $(-5,-2)$
2	-2392	2389	7	1	19	$(-2,9)$, $(9,-7)$, $(-7,-2)$
3	-3	0	9	9	27	$(-1,2)$, $(2,-1)$, $(-1,-1)$
3	-57	54	9	9	27	$(-1,5)$, $(5,-4)$, $(-4,-1)$
5	1259	-1262	13	49	49	$(3,-22)$, $(-22,19)$, $(19,3)$
5	-15	12	13	49	49	$(-1,5)$, $(5,-4)$, $(-4,-1)$
5	-1	-2	13	49	49	$(-1,-2)$, $(-2,3)$, $(3,-1)$
12	-2	-1	27	27	$3^3\cdot 7$	$(-1,2)$, $(2,-1)$, $(-1,-1)$
12	-1262	1259	27	27	$3^3\cdot 7$	$(-1,14)$, $(14,-13)$, $(-13,-1)$
12	-8	5	27	$3^3\cdot 7$	$3^3\cdot 7$	$(-1,5)$, $(5,-4)$, $(-4,-1)$
54	0	-3	111	7^3	$3^2\cdot 7^3$	$(-1,-2)$, $(-2,3)$, $(3,-1)$
54	-6	3	111	$3\cdot 7^3$	$3^2\cdot 7^3$	$(-1,5)$, $(5,-4)$, $(-4,-1)$
66	-4	1	135	$3^3\cdot 13^2$	$3^3\cdot 13^2$	$(-2,7)$, $(7,-5)$, $(-5,-2)$
1259	-1	-2	2521	61^3	$7\cdot 61^3$	$(-4,-5)$, $(-5,9)$, $(9,-4)$
1259	-15	12	2521	61^3	$7\cdot 61^3$	$(-1,14)$, $(14,-13)$, $(-13,-1)$
1259	5	-8	2521	$7\cdot 61^3$	$7\cdot 61^3$	$(-3,-19)$, $(-19,22)$, $(22,-3)$
2389	-5	2	4781	67^3	$19\cdot 67^3$	$(-2,9)$, $(9,-7)$, $(-7,-2)$

Table 1

m	n	$-n-3$	$2m+3$	λ	m^2+3m+9	(x,y)
-1	-15	12	1	1	7	$(-1,2)$, $(2,-1)$, $(-1,-1)$
-1	1259	-1262	1	1	7	$(4,-9)$, $(-9,5)$, $(5,4)$
-1	5	-8	1	7	7	$(1,-3)$, $(-3,2)$, $(2,1)$
0	54	-57	3	1	9	$(1,-3)$, $(-3,2)$, $(2,1)$
0	-6	3	3	3	9	$(-1,2)$, $(2,-1)$, $(-1,-1)$
1	-69	66	5	13	13	$(-2,7)$, $(7,-5)$, $(-5,-2)$
2	-2392	2389	7	1	19	$(-2,9)$, $(9,-7)$, $(-7,-2)$
3	-3	0	9	9	27	$(-1,2)$, $(2,-1)$, $(-1,-1)$
3	-57	54	9	9	27	$(-1,5)$, $(5,-4)$, $(-4,-1)$
5	1259	-1262	13	49	49	$(3,-22)$, $(-22,19)$, $(19,3)$
5	-15	12	13	49	49	$(-1,5)$, $(5,-4)$, $(-4,-1)$
5	-1	-2	13	49	49	$(-1,-2)$, $(-2,3)$, $(3,-1)$
12	-2	-1	27	27	$3^3\cdot 7$	$(-1,2)$, $(2,-1)$, $(-1,-1)$
12	-1262	1259	27	27	$3^3\cdot 7$	$(-1,14)$, $(14,-13)$, $(-13,-1)$
12	-8	5	27	$3^3\cdot 7$	$3^3\cdot 7$	$(-1,5)$, $(5,-4)$, $(-4,-1)$
54	0	-3	111	7^3	$3^2\cdot 7^3$	$(-1,-2)$, $(-2,3)$, $(3,-1)$
54	-6	3	111	$3\cdot 7^3$	$3^2\cdot 7^3$	$(-1,5)$, $(5,-4)$, $(-4,-1)$
66	-4	1	135	$3^3\cdot 13^2$	$3^3\cdot 13^2$	$(-2,7)$, $(7,-5)$, $(-5,-2)$
1259	-1	-2	2521	61^3	$7\cdot 61^3$	$(-4,-5)$, $(-5,9)$, $(9,-4)$
1259	-15	12	2521	61^3	$7\cdot 61^3$	$(-1,14)$, $(14,-13)$, $(-13,-1)$
1259	5	-8	2521	$7\cdot 61^3$	$7\cdot 61^3$	$(-3,-19)$, $(-19,22)$, $(22,-3)$
2389	-5	2	4781	67^3	$19\cdot 67^3$	$(-2,9)$, $(9,-7)$, $(-7,-2)$



§3
Theorem C
(correspondence)
§4
Theorem O1
?
Theorem O2
§5
Theorem S
(Solutions)

§3 Theorem C: Correspondence

$$F_m^{(3)}(X,Y) = X^3 - mX^2Y - (m+3)XY^2 - Y^3,$$
$$f_m^{(3)}(X) := F_m^{(3)}(X,1), \quad L_m^{(3)} := \mathsf{Spl}_{\mathbb{Q}} f_m^{(3)}(X)$$

- $f_m^{(3)}(X)$ is a generic polynomial for C_3 over any field k.

Definition (generic polynomial) $\mathbf{t} = (t_1, \ldots, t_n)$.

A polynomial $f(\mathbf{t}, X) \in k(\mathbf{t})[X]$ is called generic over k if
(1) Gal$(f/k(\mathbf{t}) \simeq G$;
(2) $\forall M/K \supset k$ with Gal$(M/K) \simeq G$,
$\exists \mathbf{a} \in K^n$ s.t. $M = \mathsf{Spl}_K f(\mathbf{a}, X)$.

- related to Inverse Galois Problem.
- When $L_m^{(3)} = L_n^{(3)}$?

§3 Theorem C: Correspondence

$$F_m^{(3)}(X,Y) = X^3 - mX^2Y - (m+3)XY^2 - Y^3,$$
$$f_m^{(3)}(X) := F_m^{(3)}(X,1), \quad L_m^{(3)} := \mathsf{Spl}_{\mathbb{Q}} f_m^{(3)}(X)$$

Theorem (Morton 1994, Chapman 1996, Hoshi-Miyake 2009)

Assume that char $k \neq 2$.

For $m, n \in k$, $L_m^{(3)} = L_n^{(3)}$

$$\iff \exists u \in k \text{ s.t. either } n = m + \frac{(m^2+3m+9)u(u+1)}{f_m(u)}$$
$$\text{or } \ n = -m - 3 - \frac{(m^2+3m+9)u(u+1)}{f_m(u)}.$$

- using multi-resolvent polynomial.

Proof of Theorem C

By above Theorem, $L_n^{(3)} = L_m^{(3)} \iff \exists u \in \mathbb{Q}$ s.t. either $n = m + \frac{(m^2+3m+9)u(u+1)}{f_m^{(3)}(u)}$ or $n = -m-3-\frac{(m^2+3m+9)u(u+1)}{f_m^{(3)}(u)}$.

Write $u = x/y$ with $\gcd(x,y) = 1$ then we have $n = m + \frac{(m^2+3m+9)xy(x+y)}{F_m^{(3)}(x,y)}$ or $n = -m-3-\frac{(m^2+3m+9)xy(x+y)}{F_m^{(3)}(x,y)}$.

- If $\exists x, y \in \mathbb{Z}$ s.t. $F_m^{(3)}(x,y) =: \lambda \mid m^2+3m+9$ and $xy(x+y) \neq 0$, then $\exists n \in \mathbb{Z} \setminus \{m, -m-3\}$ s.t. $L_n^{(3)} = L_m^{(3)}$.
- We should show that the converse also holds, namely $L_n^{(3)} = L_m^{(3)}, (n \neq m, -m-3) \implies \lambda \mid m^2+3m+9$.

Define $h(u) := (m^2+3m+9)u(u+1)$. Then

$$\frac{h(u)}{f_m^{(3)}(u)} \in \mathbb{Z} \text{ where } u = x/y \text{ with } \gcd(x,y) = 1.$$

Take the resultant $R := \mathrm{Res}_u(h(u), f_m^{(3)}(u))$

$$R = \begin{vmatrix} m^2+3m+9 & m^2+3m+9 & 0 & 0 & 0 \\ 0 & m^2+3m+9 & m^2+3m+9 & 0 & 0 \\ 0 & 0 & m^2+3m+9 & m^2+3m+9 & 0 \\ 1 & m+3 & m & -1 & 0 \\ 0 & 1 & m+3 & m & -1 \end{vmatrix}$$
$$= -(m^2+3m+9)^3.$$

We also see

$$R = \begin{vmatrix} m^2+3m+9 & m^2+3m+9 & 0 & 0 & h(u)u^2 \\ 0 & m^2+3m+9 & m^2+3m+9 & 0 & h(u)u \\ 0 & 0 & m^2+3m+9 & m^2+3m+9 & h(u) \\ 1 & m+3 & m & -1 & f_m^{(3)}(u)u \\ 0 & 1 & m+3 & m & f_m^{(3)}(u) \end{vmatrix}$$

$= -(m^2+3m+9)^2\Big(h(u)p(u) + f_m^{(3)}(u)q(u)\Big)$ where
$p(u) = 2u^2-(2m+1)u-m-5$, $q(u) = -(m^2+3m+9)(2u+1)$.
Hence we have

$$h(u)p(u) + f_m^{(3)}(u)q(u) = (m^2+3m+9).$$

Take the homogeneous form

$$H(x,y) := y^3 \cdot h(x/y), \quad P(x,y) := y^2 \cdot p(x/y),$$
$$Q(x,y) := y^2 \cdot p(x/y), \quad F_m^{(3)}(x,y) = y^3 \cdot f_m^{(3)}(x/y).$$

Then

$$H(x,y)P(x,y) + F_m^{(3)}(x,y)Q(x,y) = (m^2+3m+9)y^5.$$

Hence

$$\frac{H(x,y)P(x,y)}{F_m^{(3)}(x,y)} + Q(x,y) = \frac{(m^2+3m+9)y^5}{F_m^{(3)}(x,y)} \in \mathbb{Z}.$$

We also see

$$\frac{H(x,y)P(x,y)^{\sigma^2}}{F_m^{(3)}(x,y)} + Q(x,y)^{\sigma^2} = \frac{(m^2+3m+9)x^5}{F_m^{(3)}(x,y)} \in \mathbb{Z}.$$

where $\sigma^2 \ : \ y \longmapsto x \longmapsto -x-y$

because $F_m^{(3)}(x,y)$ and $H(x,y)$ are σ-invariants.

Since $\gcd(x,y)=1$, $\lambda = F_m^{(3)}(x,y) \,|\, m^2+3m+9$. ▸ Theorem C

□

§4 Theorem O_1: Okazaki's Theorem

For $m \in \mathbb{Z}$, we take

$$F_m^{(3)}(X,Y) = (X - \theta_1^{(m)}Y)(X - \theta_2^{(m)}Y)(X - \theta_3^{(m)}Y),$$

and $L_m = \mathbb{Q}(\theta_1^{(m)})$. We see
$-2 < \theta_3^{(m)} < -1, \quad -\frac{1}{2} < \theta_2^{(m)} < 0, \quad 1 < \theta_1^{(m)}$.
Take the exterior product

$$\boldsymbol{\delta} = {}^t(\delta_1, \delta_2, \delta_3) := \mathbf{1} \times \boldsymbol{\theta} = {}^t(\theta_2 - \theta_3, \theta_3 - \theta_1, \theta_1 - \theta_2)$$

where $\mathbf{1} = {}^t(1,1,1)$, $\boldsymbol{\theta} = {}^t(\theta_1, \theta_2, \theta_3) \in \mathbb{R}^3$.

The norm $\mathrm{N}(\boldsymbol{\delta}) = \delta_1\delta_2\delta_3 = -\sqrt{D}$ where
$D = ((\theta_1 - \theta_2)(\theta_1 - \theta_3)(\theta_2 - \theta_3))^2$.

The canonical lattice

$$\mathcal{L}^\natural = \boldsymbol{\delta}(\mathbb{Z}\mathbf{1} + \mathbb{Z}\boldsymbol{\theta})$$

of F is orthogonal to $\mathbf{1}$, where the product of vectors is the component-wise product. We consider the curve $\mathcal{H}$

$$\mathcal{H} : z_1 + z_2 + z_3 = 0, \quad z_1 z_2 z_3 = \sqrt{D}.$$

on the plane $\mathit{\Pi} = \{{}^t(z_1, z_2, z_3) \in \mathbb{R}^3 \mid z_1 + z_2 + z_3 = 0\}$.
For (x, y) with $F_m^{(3)}(x, y) = 1$, we see $x\mathbf{1} - y\boldsymbol{\theta} \in (\mathcal{O}_{L_m}^\times)^3$ because $\mathrm{N}(x\mathbf{1} - y\boldsymbol{\theta}) = 1$. Then we get a bijection

$$(x, y) \longleftrightarrow \boldsymbol{z} = \boldsymbol{\delta}(-x\mathbf{1} + y\boldsymbol{\theta}) \in \mathcal{L}^\natural \cap \mathcal{H}$$

via $\mathrm{N}(\boldsymbol{z}) = \mathrm{N}(\boldsymbol{\delta})\mathrm{N}(-x\mathbf{1} + y\boldsymbol{\theta}) = (-\sqrt{D})(-1) = \sqrt{D}$. Let

$$\log : (\mathbb{R}^\times)^3 \ni {}^t(z_1, z_2, z_3) \mapsto {}^t(\log|z_1|, \log|z_2|, \log|z_3|) \in \mathbb{R}^3$$

be the logarithmic map. By Dirichlet's unit theorem, the set

$$\mathcal{E}(L_m) := \{\log \boldsymbol{\varepsilon} \mid \boldsymbol{\varepsilon} = {}^t(\varepsilon, \varepsilon^\sigma, \varepsilon^{\sigma^2}), \varepsilon \in \mathcal{O}_{L_m}^\times\}$$

is a lattice of rank 2 on the plane
$\mathit{\Pi}_{\log} := \{{}^t(u_1, u_2, u_3) \in \mathbb{R}^3 \mid u_1 + u_2 + u_3 = 0\}$.

We use the modified logarithmic map

$$\phi : (\mathbb{R}^\times)^3 \ni \boldsymbol{z} \mapsto \boldsymbol{u} = {}^t(u_1, u_2, u_3) = \log(D^{-1/6}\boldsymbol{z}) \in \mathbb{R}^3.$$

For (x, y) with $F_m^{(3)}(x, y) = 1$ and
$\boldsymbol{z} = \boldsymbol{\delta}(-x\mathbf{1} + y\boldsymbol{\theta}) \in \mathcal{L}^\natural \cap \mathcal{H}$,
$\boldsymbol{u} = \phi(\boldsymbol{z}) = \phi(\boldsymbol{\delta}(-x\mathbf{1} + y\boldsymbol{\theta})) \in \phi(\boldsymbol{\delta}) + \mathcal{E}(L_m) \subset \mathit{\Pi}_{\log}$; the displaced lattice, since $-x\mathbf{1} + y\boldsymbol{\theta} \in (\mathcal{O}_{L_m}^\times)^3$. We can show

- $3\,\phi(\boldsymbol{\delta}) \in \mathcal{E}(L_m)$.

We now assume that $L_m = L_n$ for $-1 \leq m < n$ and take a common trivial solution $(x, y) = (1, 0)$. Then

$$\boldsymbol{u}^{(m)}, \boldsymbol{u}^{(n)} \in \mathcal{M} = \mathbb{Z}\,\phi(\boldsymbol{\delta}^{(m)}) + \mathbb{Z}\,\phi(\boldsymbol{\delta}^{(n)}) + \mathcal{E}(L_m) \subset \mathit{\Pi}_{\log}$$

where $\mathcal{M}$ is a lattice with discriminant
$d(\mathcal{M}) = d(\mathcal{E}(L_m))$, $\frac{1}{3}d(\mathcal{E}(L_m))$ or $\frac{1}{9}d(\mathcal{E}(L_m))$. We may get:

- $d(\mathcal{M}) = d(\mathcal{E}(L_m))$ or $\frac{1}{3}d(\mathcal{E}(L_m))$.

We adopt local coordinates for $\mathcal{C} := \phi(\mathcal{H}) \subset \mathit{\Pi}_{\log}$ by

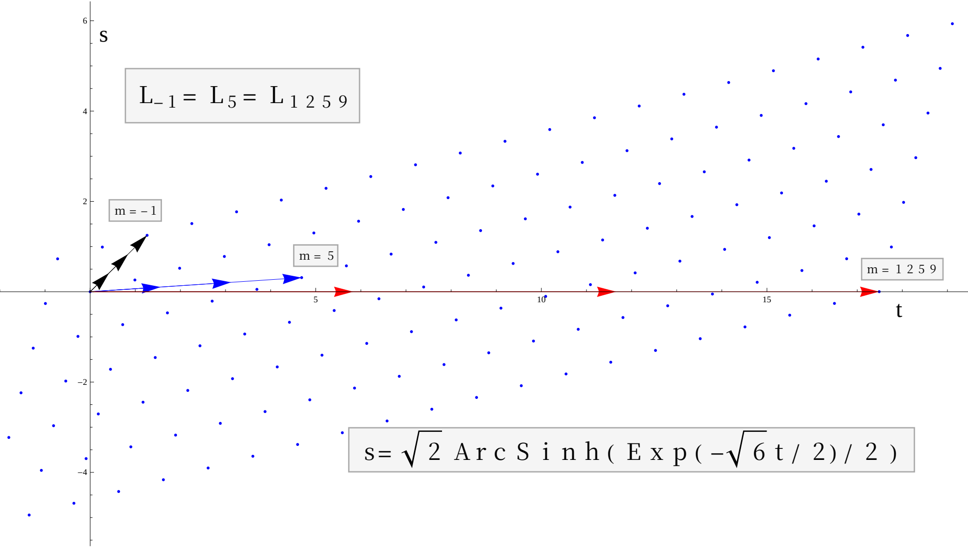
$$s = s(\boldsymbol{u}) := \frac{u_2 - u_3}{\sqrt{2}}, \quad t = t(\boldsymbol{u}) := -\frac{\sqrt{6}u_1}{2}.$$

Then

$$s = \sqrt{2}\ \mathrm{arcsinh}\left(\exp\left(-\sqrt{6}t/2\right)/2\right), \qquad 0 \le s \le \sqrt{3}t.$$

Example

m	-1	0	1	2	3	4	5
s	0.4163	0.3016	0.2263	0.1773	0.1444	0.1212	0.1042
t	0.4206	0.6893	0.9267	1.1269	1.2952	1.4385	1.5624



s
t
6
4
2
−2
−4
5
10
15
$L_{-1} = L_5 = L_{1259}$
m = −1
m = 5
m = 1259
$s = \sqrt{2}\,\mathrm{ArcSinh}(\mathrm{Exp}(-\sqrt{6}\,t/2)/2)$

Using a result of Laurent-Mignotte-Nesterenko (1995) in Baker's theory, Okazaki gave:

Theorem (Okazaki 2002)

Assume distinct points $\boldsymbol{u} = \boldsymbol{u}^{(m)}$ and $\boldsymbol{u}' = \boldsymbol{u}^{(n)}$ of $\mathcal{M}$ on $\mathcal{C}$. Assume $t' = t(\boldsymbol{u}') \geq t = t(\boldsymbol{u})$. Then

$$\frac{\sqrt{2}\, d(\mathcal{M}) \exp(\sqrt{6}t/2)}{1 + \exp(-2(t'-t)/\sqrt{6}\log 2)} \leq t'.$$

Theorem (Okazaki 2002)

For $\boldsymbol{z}' \in \mathcal{L}^{\natural} \cap \mathcal{H}$ and $t' = t(\boldsymbol{z}')$, we have

$$\frac{t'}{d(\mathbb{Z}\phi(\boldsymbol{\delta}) + \mathcal{E}(L_m))} \leq 5.04 \times 10^4.$$

We can show

$$0.14\exp(\sqrt{6}t/2) - t < t' - t.$$

Then it follows

$$\begin{aligned}&\frac{\sqrt{2}\exp(\sqrt{6}t/2)}{1+\exp(-2(0.14\exp(\sqrt{6}t/2)-t)/\sqrt{6}\log 2)}\\ &< \frac{\sqrt{2}\exp(\sqrt{6}t/2)}{1+\exp(-2(t'-t)/\sqrt{6}\log 2)} \le \frac{t'}{d(\mathcal{M})} \le 5.04\times 10^4.\end{aligned}$$

We get $t \le 8.56$ and hence $m \le 35731$. $\square$

§5 Theorem S: Solutions

Theorem C + Theorem O_1 $\Rightarrow$ Theorem S

▸ Theorem S

Let θ_2 be a root of $f_m^{(3)}(X)$ with $-\frac{1}{2} < \theta_2 < 0$. For $m \geq 18$,
$$-\frac{1}{m} + \frac{2}{m^2} - \frac{3}{m^3} - \frac{3}{m^4} + \frac{17}{m^5} - \frac{28}{m^6}$$
$$< \theta_2 < -\frac{1}{m} + \frac{2}{m^2} - \frac{3}{m^3} - \frac{3}{m^4} + \frac{17}{m^5} - \frac{27}{m^6}.$$
The continued fraction expansion of the root θ_2 of $f_m^{(3)}(X)$ is given as follows:
When m is even,

$$\theta_2 = \begin{cases} [-1; 1, m+1, \frac{m}{2}, 1, 3, \frac{m-14}{14}, 1, 6, [\frac{m}{6}], \ldots], \text{ if } m = 14k \geq 28, \\ [-1; 1, m+1, \frac{m}{2}, 1, 3, \frac{m-2}{14}, [\frac{49m+72}{6}], \ldots], \text{ if } m = 14k+2 \geq 156, \\ [-1; 1, m+1, \frac{m}{2}, 1, 3, \frac{m-4}{14}, 6, 1, [\frac{m-4}{6}], \ldots], \text{ if } m = 14k+4 \geq 18, \\ [-1; 1, m+1, \frac{m}{2}, 1, 3, \frac{m-6}{14}, 3, 2, [\frac{m-2}{6}], \ldots], \text{ if } m = 14k+6 \geq 20, \\ [-1; 1, m+1, \frac{m}{2}, 1, 3, \frac{m-8}{14}, 2, 3, [\frac{m-2}{6}], \ldots], \text{ if } m = 14k+8 \geq 22, \\ [-1; 1, m+1, \frac{m}{2}, 1, 3, \frac{m-10}{14}, 1, 1, 2, 1, [\frac{m-4}{6}], \ldots], \text{ if } m = 14k+10 \geq 24, \\ [-1; 1, m+1, \frac{m}{2}, 1, 3, \frac{m-12}{14}, 1, 2, 1, 1, [\frac{m-2}{6}], \ldots], \text{ if } m = 14k+12 \geq 68. \end{cases}$$

When m is odd,

$$\theta_2 = \begin{cases} [-1; 1, m+1, \frac{m+1}{2}, 3, 1, \frac{m-15}{14}, 2, 3, [\frac{m-1}{6}], \ldots], \text{ if } m = 14k+1 \geq 29, \\ [-1; 1, m+1, \frac{m+1}{2}, 3, 1, \frac{m-17}{14}, 1, 1, 2, 1, [\frac{m-3}{6}], \ldots], \text{ if } m = 14k+3 \geq 31, \\ [-1; 1, m+1, \frac{m+1}{2}, 3, 1, \frac{m-19}{14}, 1, 2, 1, 1, [\frac{m-3}{6}], \ldots], \text{ if } m = 14k+5 \geq 33, \\ [-1; 1, m+1, \frac{m+1}{2}, 3, 1, \frac{m-21}{14}, 1, 6, [\frac{m-1}{6}], \ldots], \text{ if } m = 14k+7 \geq 35, \\ [-1; 1, m+1, \frac{m+1}{2}, 3, 1, \frac{m-9}{14}, 2, 3, [\frac{49m+73}{6}], \ldots], \text{ if } m = 14k+9 \geq 65, \\ [-1; 1, m+1, \frac{m+1}{2}, 3, 1, \frac{m-11}{14}, 6, 1, [\frac{m-5}{6}], \ldots], \text{ if } m = 14k+11 \geq 25, \\ [-1; 1, m+1, \frac{m+1}{2}, 3, 1, \frac{m-13}{14}, 3, 2, [\frac{m-3}{6}], \ldots], \text{ if } m = 14k+13 \geq 27. \end{cases}$$

It is enough to find all non-trivial solutions $(x, y) \in \mathbb{Z}^2$ to $F_m^{(3)}(x, y) = \lambda \mid m^2 + 3m + 9$ for $-1 \leq m \leq 35731$.
Indeed if there exists a non-trivial solution $(x, y) \in \mathbb{Z}^2$ to $F_n^{(3)}(x, y) = \lambda \mid n^2 + 3n + 9$ for $n \geq 35732$ then there exists $-1 \leq m \leq 35731$ such that $L_m = L_n$.

(i) $-1 \leq m \leq 2407$. For small m, we can use MAGMA (Bilu-Hanrot).
(ii) $2408 \leq m \leq 35731$ and $2(2m + 3 + \frac{27}{2m+3}) \leq y$. We consider $|F_m^{(3)}(x, y)| \leq m^2 + 3m + 9$. Applying Lettel-Pethö-Voutier Theorem Theorem

$$\lambda(m) = m^2 + 3m + 9, \quad \frac{8\lambda(m)}{2m+3} = 2\left(2m + 3 + \frac{27}{2m+3}\right),$$

x/y is a convergent to θ_2. But this is impossible.
(iii) $2408 \leq m \leq 35731$ and $y < 2(2m + 3 + \frac{27}{2m+3})$. The bound is small enough to reach using a computer.

Degree 6 case

$$\begin{aligned}F_m^{(6)}(x,y) = x^6 - 2mx^5y - (5m+15)x^4y^2 \\ -20x^3y^3 + 5mx^2y^4 + (2m+6)xy^5 + y^6 = \lambda\end{aligned}$$

- $f_m^{(6)}(X) := F_m^{(6)}(X,1)$.
- $f_m^{(6)}(X)$ is irreducible/$\mathbb{Q}$ for $m \in \mathbb{Z} \setminus \{-8,-3,0,5\}$.
- $L_m^{(6)} :=$ Spl$_{\mathbb{Q}} f_m^{(6)}(X)$, then $L_m^{(6)} = L_{-m-3}^{(6)}$; the simplest sextic fields.
- $L_m^{(3)} \subset L_m^{(6)}$ for $\forall m \in \mathbb{Z}$.

Theorem

For a given $m \in \mathbb{Z}$, $\exists n \in \mathbb{Z}\setminus\{m,-m-3\}$ s.t. $L_m^{(6)} = L_n^{(6)}$ $\iff$ $\exists (x,y) \in \mathbb{Z}^2$ with $xy(x+y)(x-y)(x+2y)(2x+y) \neq 0$ s.t $F_m^{(6)}(x,y) = \lambda$ for some $\lambda \in \mathbb{N}$ with $\lambda \,|\, 27(m^2+3m+9)$.

Moreover integers n, m and $(x, y) \in \mathbb{Z}^2$ satisfy

$$N = m + \frac{(m^2+3m+9)xy(x+y)(x-y)(x+2y)(2x+y)}{F_m^{(6)}(x,y)}$$

where N is either n or $-n-3$.

By Okazaki's theorem and the fact $L_m^{(3)} \subset L_m^{(6)}$, we get:

Theorem

For $m, n \in \mathbb{Z}$, $L_m^{(6)} = L_n^{(6)} \iff m = n$ or $m = -n-3$.

Theorem

For $m \in \mathbb{Z}$, $F_m^{(6)}(x,y) = \lambda$ with $\lambda \mid 27(m^2+3m+9)$ has only trivial solutions, i.e. $xy(x+y)(x-y)(x+2y)(2x+y) = 0$.