Rationality problem for fields of invariants, norm one tori and Hasse norm principle

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Table of contents

$\mathrm{Br}_{\mathrm{nr}}(X/\mathbb{C}) \simeq H^3(X,\mathbb{Z})_{\mathrm{tors}}$; Artin-Mumford invariant $(X : RC)$

$H^3_{\mathrm{nr}}(X,\mathbb{Q}/\mathbb{Z}) \simeq \mathrm{Hdg}^4(X,\mathbb{Z})/\mathrm{Hdg}^4(X,\mathbb{Z})_{\mathrm{alg}} \leftrightarrow$ integral Hodge conjecture

cf. Colliot-Thélène and Voisin, Duke Math. J. **161** (2012) 735–801.

§0. Introduction

Inverse Galois problem (IGP)

Does every finite group occur as a quotient group of the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$?

- Related to rationality problem (Emmy Noether's strategy: 1913)

A finite group $G \curvearrowright k(x_g \mid g \in G)$: rational function field over k by permutation $h(x_g) = x_{hg}$ (for any $g, h \in G$)

$k(x_g \mid g \in G)^G$ is rational over k, i.e. $k(x_g \mid g \in G)^G \simeq k(t_1, \ldots, t_n)$ (Noether's problem has an affirmative answer)

$\Longrightarrow k(x_g \mid g \in G)^G$ is retract rational over k (weaker concept)

$\Longleftrightarrow$ $\exists$ generic extension (polynomial) for (G, k) (Saltman's sense)

$\overset{k\text{:Hilbertian}}{\Longrightarrow}$ IGP for (k, G) has an affirmative answer

Rationality problem for quasi-monomial actions

Definition (quasi-monomial action)

Let K/k be a finite field extension and $G \leq \mathrm{Aut}_k(K(x_1,\ldots,x_n))$; finite where $K(x_1,\ldots,x_n)$ is the rational function field of n variables over K.
The action of G on $K(x_1,\ldots,x_n)$ is called quasi-monomial if
(i) $\sigma(K) \subset K$ for any $\sigma \in G$;
(ii) $K^G = k$;
(iii) for any $\sigma \in G$, $$\sigma(x_j) = c_j(\sigma)\prod_{i=1}^{n} x_i^{a_{ij}}$$
where $c_j(\sigma) \in K^\times$, $1 \leq j \leq n$, $[a_{i,j}]_{1\leq i,j\leq n} \in GL_n(\mathbb{Z})$.

Rationality problem

Under what situation the fixed field $K(x_1,\ldots,x_n)^G$ is rational over k, i.e. $K(x_1,\ldots,x_n)^G \simeq k(t_1,\ldots,t_n)$ (=purely transcendental over k), if G acts on $K(x_1,\ldots,x_n)$ by quasi-monomial k-automorphisms.

Rationality problem for quasi-monomial actions

Definition (quasi-monomial action)

Let K/k be a finite field extension and $G \leq \mathrm{Aut}_k(K(x_1,\ldots,x_n))$; finite where $K(x_1,\ldots,x_n)$ is the rational function field of n variables over K. The action of G on $K(x_1,\ldots,x_n)$ is called quasi-monomial if
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where $c_j(\sigma) \in K^\times$, $1 \leq j \leq n$, $[a_{i,j}]_{1\leq i,j\leq n} \in GL_n(\mathbb{Z})$.

- When $G \curvearrowright K$; trivial (i.e. $K = k$), called (just) monomial action.
- When $G \curvearrowright K$; trivial and permutation $\leftrightarrow$ $\boxed{\text{Noether's problem}}$.
- When $c_j(\sigma) = 1$ $(\forall\sigma \in G, \forall j)$, called purely (quasi-)monomial.
- $G = \mathrm{Gal}(K/k)$ and purely $\leftrightarrow$ $\boxed{\text{Rationality problem for algebraic tori}}$.

Exercises (1/2): Noether's problem

- $S_n \curvearrowright \mathbb{Q}(x_1, \ldots, x_n)$; permutation
  $\boxed{\text{Q.}}$ Is $\mathbb{Q}(x_1, \ldots, x_n)^{S_n}$ rational over $\mathbb{Q}$? $\boxed{\text{Ans.}}$ Yes!

  $\mathbb{Q}(x_1, \ldots, x_n)^{S_n} = \mathbb{Q}(s_1, \ldots, s_n)$; s_i, ith elementary symmetric

  $\Longrightarrow$ IGP for $(\mathbb{Q}, S_n)$ has affirmative solution.

- $A_n \curvearrowright \mathbb{Q}(x_1, \ldots, x_n)$; permutation
  $\boxed{\text{Q.}}$ Is $\mathbb{Q}(x_1, \ldots, x_n)^{A_n}$ rational over $\mathbb{Q}$? $\boxed{\text{Ans.}}$ Yes? ?? ??

  $\mathbb{Q}(x_1, \ldots, x_n)^{A_n} = \mathbb{Q}(s_1, \ldots, s_n, \Delta)$; but ...

  $\boxed{\text{Open problem}}$ Is $\mathbb{Q}(x_1, \ldots, x_n)^{A_n}$ rational over $\mathbb{Q}$? $(n \geq 6)$

- $\mathbb{Q}(x_1, \ldots, x_5)^{A_5}$ is rational over $\mathbb{Q}$ (Maeda, 1989).

Exercises (2/2): Noether's problem

- $\mathbb{Q}(x_1, x_2, x_3)^{A_3} = \mathbb{Q}(x_1, x_2, x_3)^{C_3} = \mathbb{Q}(t_1, t_2, t_3)$, $\boxed{\text{Q.}}$ t_1, t_2, t_3?
  $(C_3 : x_1 \mapsto x_2 \mapsto x_3 \mapsto x_1)$
- $\boxed{\text{Ans.}}$ $\mathbb{Q}(x_1, x_2, x_3)^{C_3} = \mathbb{Q}(t_1, t_2, t_3)$ where

$$t_1 = x_1 + x_2 + x_3,$$

$$t_2 = \frac{x_1x_2^2 + x_2x_3^2 + x_3x_1^2 - 3x_1x_2x_3}{x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_2x_3 - x_3x_1},$$

$$t_3 = \frac{x_1^2x_2 + x_2^2x_3 + x_3^2x_1 - 3x_1x_2x_3}{x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_2x_3 - x_3x_1}.$$

- $\mathbb{Q}(x_1, x_2, \ldots, x_8)^{C_8} = \mathbb{Q}(t_1, t_2, \ldots, t_8)$, $\boxed{\text{Q.}}$ $t_1, t_2, \ldots, t_8$?
  $(C_8 : x_1 \mapsto x_2 \mapsto x_3 \mapsto \cdots \mapsto x_8 \mapsto x_1)$
- $\boxed{\text{Ans.}}$ None: $\mathbb{Q}(x_1, x_2, \ldots, x_8)^{C_8}$ is not rational over $\mathbb{Q}$!

Today's talk (1/2)

Definition (quasi-monomial action)

Let K/k be a finite field extension and $G \leq \mathrm{Aut}_k(K(x_1,\ldots,x_n))$; finite where $K(x_1,\ldots,x_n)$ is the rational function field of n variables over K. The action of G on $K(x_1,\ldots,x_n)$ is called quasi-monomial if
(i) $\sigma(K) \subset K$ for any $\sigma \in G$;
(ii) $K^G = k$;
(iii) for any $\sigma \in G$, $$\sigma(x_j) = c_j(\sigma) \prod_{i=1}^{n} x_i^{a_{ij}}$$
where $c_j(\sigma) \in K^\times$, $1 \leq j \leq n$, $[a_{i,j}]_{1\leq i,j\leq n} \in GL_n(\mathbb{Z})$.

§1. $G \curvearrowright K$; trivial: monomial action & Noether's problem
§2. $G \curvearrowright K$; trivial and permutation: Noether's problem over $\mathbb{C}$
§3. (general) quasi-monomial actions (1-dim. and 2-dim. cases)
§4. $G = \mathrm{Gal}(K/k)$ and purely: rationality problem for algebraic tori

Today's talk (2/2)

§1. $G \curvearrowright K$; trivial: monomial action & Noether's problem

Hoshi-Kitayama-Yamasaki, J. Algebra **341** (2011) 45–108.

§2. $G \curvearrowright K$; trivial and permutation: Noether's problem over $\mathbb{C}$

Hoshi-Kang-Kunyavskii, Asian J. Math. **17** (2013) 689–714.

Chu-Hoshi-Hu-Kang, J. Algebra **442** (2015) 233–259.

Hoshi, J. Algebra **445** (2016) 394–432.

Hoshi-Kang-Yamasaki, J. Algebra **458** (2016) 120–133.

Hoshi-Kang-Yamasaki, J. Algebra **544** (2020) 262–301.

Hoshi-Kang-Yamasaki, Mem. AMS **283** (2023) no. 1403, 137 pp.

§3. (general) quasi-monomial actions (1-dim. and 2-dim. cases)

Hoshi-Kang-Kitayama, J. Algebra **403** (2014) 363–400.

§4. $G = \mathrm{Gal}(K/k)$ and purely: rationality problem for algebraic tori

Hoshi-Yamasaki, Mem. AMS **248** (2017) no. 1176, 215 pp.

Various rationalities: definitions

$k \subset L$; f.g. field extension, L is rational over $k \overset{\text{def}}{\iff} L \simeq k(x_1, \ldots, x_n)$.

Definition (stably rational)

L is called stably rational over $k \overset{\text{def}}{\iff} L(y_1, \ldots, y_m)$ is rational over k.

Definition (retract rational)

L is retract rational over $k \overset{\text{def}}{\iff} \exists k$-algebra $R \subset L$ such that
(i) L is the quotient field of R;
(ii) $\exists f \in k[x_1, \ldots, x_n]$ $\exists k$-algebra hom. $\varphi : R \to k[x_1, \ldots, x_n][1/f]$ and $\psi : k[x_1, \ldots, x_n][1/f] \to R$ satisfying $\psi \circ \varphi = 1_R$.

Definition (unirational)

L is unirational over $k \overset{\text{def}}{\iff} L \subset k(t_1, \ldots, t_n)$.

- Assume $L_1(x_1, \ldots, x_n) \simeq L_2(y_1, \ldots, y_m)$; stably isomorphic. If L_1 is retract rational over k, then L_2 is retract rational over k.
- "rational" $\Longrightarrow$ "stably rational" $\Longrightarrow$ "retract rational" $\Longrightarrow$ "unirational"

"rational" $\Longrightarrow$ "stably rational" $\Longrightarrow$ "retract rational" $\Longrightarrow$ "unirational"

- The direction of the implication cannot be reversed.
- (Lüroth's problem) "unirational" $\Longrightarrow$ "rational" ? YES if trdeg$= 1$
- (Castelnuovo, 1894)
  L is unirational over $\mathbb{C}$ and $\mathrm{trdeg}_{\mathbb{C}} L = 2 \Longrightarrow L$ is rational over $\mathbb{C}$.
- (Zariski, 1958) Let k be an alg. closed field and $k \subset L \subset k(x,y)$. If $k(x,y)$ is separable algebraic over L, then L is rational over k.
- (Zariski cancellation problem) $V_1 \times \mathbb{P}^n \approx V_2 \times \mathbb{P}^n \Longrightarrow V_1 \approx V_2$?
  In particular, "stably rational" $\Longrightarrow$ "rational"?
- (Beauville, Colliot-Thélène, Sansuc, Swinnerton-Dyer, 1985, Ann. Math.)
  $L = \mathbb{Q}(x,y,t)$ with $x^2 + 3y^2 = t^3 - 2$ (Châtelet surface)
  $\Longrightarrow L$ is not rational but stably rational over $\mathbb{Q}$.
  Indeed, $L(y_1, y_2, y_3)$ is rational over $\mathbb{Q}$.
- $L(y_1, y_2)$ is rational over $\mathbb{Q}$ (Shepherd-Barron, 2002, Fano Conf.).
- $\mathbb{Q}(x_1, \ldots, x_{47})^{C_{47}}$ is not stably but retract rational over $\mathbb{Q}$.
- $\mathbb{Q}(x_1, \ldots, x_8)^{C_8}$ is not retract but unirational over $\mathbb{Q}$.

Châtelet surface as an invariant field

- (Beauville, Colliot-Thélène, Sansuc, Swinnerton-Dyer, 1985, Ann. Math.) $L = \mathbb{Q}(x, y, t)$ with $x^2 + 3y^2 = t^3 - 2$ (Châtelet surface) $\Longrightarrow L$ is not rational but stably rational over $\mathbb{Q}$.
- $L = \mathbb{Q}(x, y, t) = \mathbb{Q}(\sqrt{-3})(X, Y)^{\langle\sigma\rangle}$ where

$$\sigma : \sqrt{-3} \mapsto -\sqrt{-3}, X \mapsto X, Y \mapsto \frac{X^3 - 2}{Y}.$$

Indeed, we have

$$\begin{aligned} x &= \frac{1}{2}\left(Y + \frac{X^3 - 2}{Y}\right), \\ y &= \frac{1}{2\sqrt{-3}}\left(Y - \frac{X^3 - 2}{Y}\right), \\ t &= X. \end{aligned}$$

Retract rationality and generic extension

Theorem (Saltman, 1982, DeMeyer)

Let k be an infinite field and G be a finite group.
The following are equivalent:
(i) $k(x_g \mid g \in G)^G$ is retract rational over k.
(ii) There is a generic G-Galois extension over k;
(iii) There exists a generic G-polynomial over k.

- related to Inverse Galois Problem (IGP). (i) $\Longrightarrow$ IGP(G/k): true

Definition (generic polynomial)

A polynomial $f(t_1, \ldots, t_n; X) \in k(t_1, \ldots, t_n)[X]$ is generic for G over k if
(1) $\mathrm{Gal}(f/k(t_1, \ldots, t_n)) \simeq G$;
(2) $\forall L/M \supset k$ with $\mathrm{Gal}(L/M) \simeq G$,
$\exists a_1, \ldots, a_n \in M$ such that $L = \mathrm{Spl}(f(a_1, \ldots, a_n; X)/M)$.

- By Hilbert's irreducibility theorem, $\exists L/\mathbb{Q}$ such that $\mathrm{Gal}(L/\mathbb{Q}) \simeq G$.

§1. Monomial action & Noether's problem

Definition (monomial action) $G \curvearrowright K$; trivial, $k = K^G = K$

An action of G on $k(x_1, \ldots, x_n)$ is monomial $\overset{\text{def}}{\Longleftrightarrow}$

$$\sigma(x_j) = c_j(\sigma) \prod_{i=1}^{n} x_i^{a_{i,j}}, \ 1 \leq j \leq n, \forall \sigma \in G$$

where $[a_{i,j}]_{1 \leq i,j \leq n} \in \mathrm{GL}_n(\mathbb{Z})$, $c_j(\sigma) \in k^\times := k \setminus \{0\}$.

If $c_j(\sigma) = 1$ for any $1 \leq j \leq n$ then σ is called purely monomial.

- Application to Noether's problem (permutation action)

Noether's problem (1/3) [$G = A$; abelian case]

- k; field, G; finite group
- $G \curvearrowright k$; trivial, $G \curvearrowright k(x_g \mid g \in G)$; permutation.
- $k(G) := k(x_g \mid g \in G)^G$; invariant field

Noether's problem (Emmy Noether, 1913)

Is $k(G)$ rational over k?, i.e. $k(G) \simeq k(t_1, \ldots, t_n)$?

- Is the quotient variety $\mathbb{P}^n/G$ rational over k?

- Assume $G = A$; abelian group.
- (Fisher, 1915) $\mathbb{C}(A)$ is rational over $\mathbb{C}$.
- (Masuda, 1955, 1968) $\mathbb{Q}(C_p)$ is rational over $\mathbb{Q}$ for $p \leq 11$.
- (Swan, 1969, Invent. Math.)
  $\mathbb{Q}(C_{47}), \mathbb{Q}(C_{113}), \mathbb{Q}(C_{233})$ are not rational over $\mathbb{Q}$.
- S. Endo and T. Miyata (1973), V.E. Voskresenskii (1973), ...
  e.g. $\mathbb{Q}(C_8)$ is not rational over $\mathbb{Q}$.
- (Lenstra, 1974, Invent. Math.)
  $k(A)$ is rational over k $\iff$ $\boxed{\text{some condition}}$;

Noether's problem (2/3) [$G = A$; abelian case]

- (Endo-Miyata, 1973) $\mathbb{Q}(C_{p^r})$ is rational over $\mathbb{Q}$
  $\iff \exists \alpha \in \mathbb{Z}[\zeta_{\varphi(p^r)}]$ such that $N_{\mathbb{Q}(\zeta_{\varphi(p^r)})/\mathbb{Q}}(\alpha) = \pm p$
- $h(\mathbb{Q}(\zeta_m)) = 1$ if $m < 23$
  $\Longrightarrow \mathbb{Q}(C_p)$ is rational over $\mathbb{Q}$ for $p \leq 43$ and $p = 61, 67, 71$.
- (Endo-Miyata, 1973) For $p = 47, 79, 113, 137, 167, \ldots,$
  $\mathbb{Q}(C_p)$ is not rational over $\mathbb{Q}$.
- However, for $p = 59, 83, 89, 97, 107, 163, \ldots$, unknown.
  Under the GRH, $\mathbb{Q}(C_p)$ is not rational for the above primes.
  But it was unknown for $p = 251, 347, 587, 2459, \ldots$
- For $p \leq 20000$, see speaker's paper (using PARI/GP):
  Proc. Japan Acad. Ser. A 91 (2015) 39-44.

Theorem (Plans, 2017, Proc. AMS)

$\mathbb{Q}(C_p)$ is rational over $\mathbb{Q} \iff p \leq 43$ or $p = 61, 67, 71$.

- Using lower bound of height, $\mathbb{Q}(C_p)$ is rational $\Rightarrow p < 173$.

Noether's problem (3/3) [G; non-abelian case]

Noether's problem (Emmy Noether, 1913)

Is $k(G)$ rational over k?, i.e. $k(G) \simeq k(t_1, \ldots, t_n)$?

- Assume G; non-abelian group.
- (Maeda, 1989) $k(A_5)$ is rational over k;
- (Rikuna, 2003; Plans, 2007)
  $k(GL_2(\mathbb{F}_3))$ and $k(SL_2(\mathbb{F}_3))$ is rational over k;
- (Serre, 2003)
  if 2-Sylow subgroup of $G \simeq C_{8m}$, then $\mathbb{Q}(G)$ is not rational over $\mathbb{Q}$;
  if 2-Sylow subgroup of $G \simeq Q_{16}$, then $\mathbb{Q}(G)$ is not rational over $\mathbb{Q}$;
  e.g. $G = Q_{16}, SL_2(\mathbb{F}_7), SL_2(\mathbb{F}_9)$,
  $SL_2(\mathbb{F}_q)$ with $q \equiv 7$ or $9 \pmod{16}$.

From Noether's problem to monomial actions (1/2)

- $k(G) := k(x_g \mid g \in G)^G$; invariant field

Noether's problem (Emmy Noether, 1913)

Is $k(G)$ rational over k?, i.e. $k(G) \simeq k(t_1, \ldots, t_n)$?

By Hilbert 90, we have:

No-name lemma (e.g. Miyata, 1971, Remark 3)

Let G act faithfully on k-vector space V, $W \subset V$ faithful $k[G]$-submodule. Then $K(V)^G = K(W)^G(t_1, \ldots, t_m)$.

Rationality problem: linear action

Let G act on finite-dimensional k-vector space V and $\rho : G \to GL(V)$ be a representation. Whether $k(V)^G$ is rational over k?

- the quotient variety V/G is rational over k?

From Noether's problem to monomial actions (2/2)

- For $\rho : G \to GL(V)$; monomial representation, i.e. matrix rep. has exactly one non-zero entry in each row and each column, G acts on $k(\mathbb{P}(V)) = k(\frac{w_1}{w_n}, \ldots, \frac{w_{n-1}}{w_n})$ by monomial action

By Hilbert 90, we have:

Lemma (e.g. Miyata, 1971, Lemma)

$k(V)^G = k(\mathbb{P}(V))^G(t)$.

- $V/G \approx \mathbb{P}(V)/G \times \mathbb{P}^1$ (birational)
- $k(\mathbb{P}(V))^G$ (monomial action) is rational over k
  $\Longrightarrow k(V)^G$ (linear action) is rational over k
  $\Longrightarrow k(G)$ (permutation action) is rational over k
  (Noether's problem has an affirmative answer)

Example: Noether's problem for $GL_2(\mathbb{F}_3)$ and $SL_2(\mathbb{F}_3)$

- $G = GL_2(\mathbb{F}_3) = \langle A, B, C, D\rangle \subset GL_4(\mathbb{Q})$, $|G| = 48$,
- $H = SL_2(\mathbb{F}_3) = \langle A, B, C\rangle \subset GL_4(\mathbb{Q})$, $|H| = 24$, where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, C = \begin{bmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

- G and H act on $k(V) = k(w_1, w_2, w_3, w_4)$ by

$$A : w_1 \mapsto -w_2 \mapsto -w_1 \mapsto w_2 \mapsto w_1,\ w_3 \mapsto -w_4 \mapsto -w_3 \mapsto w_4 \mapsto w_3,$$
$$B : w_1 \mapsto -w_3 \mapsto -w_1 \mapsto w_3 \mapsto w_1,\ w_2 \mapsto w_4 \mapsto -w_2 \mapsto -w_4 \mapsto w_2,$$
$$C : w_1 \mapsto -w_2 \mapsto w_3 \mapsto w_1,\ w_4 \mapsto w_4, \quad D : w_1 \mapsto w_1,\ w_2 \mapsto -w_2,\ w_3 \leftrightarrow w_4.$$

- $k(\mathbb{P}(V)) = k(x, y, z)$, $x = w_1/w_4$, $y = w_2/w_4$, $z = w_3/w_4$.
- G and H act on $k(x, y, z)$ as $G/Z(G) \simeq S_4$ and $H/Z(H) \simeq A_4$:

$$A : x \mapsto \frac{y}{z},\ y \mapsto \frac{-x}{z},\ z \mapsto \frac{-1}{z},\ B : x \mapsto \frac{-z}{y},\ y \mapsto \frac{-1}{y},\ z \mapsto \frac{x}{y},$$
$$C : x \mapsto y \mapsto z \mapsto x,\ D : x \mapsto \frac{x}{z},\ y \mapsto \frac{-y}{z},\ z \mapsto \frac{1}{z}.$$

- $k(\mathbb{P}(V))^G$: rational $\Longrightarrow k(V)^G$: rational $\Longrightarrow k(G)$: rational.

Monomial action (1/3) [3-dim. case]

Theorem (Hajja,1987) 2-dim. monomial action

$k(x_1, x_2)^G$ is rational over k.

Theorem (Hajja-Kang 1994, Hoshi-Rikuna 2008) 3-dim. purely monomial

$k(x_1, x_2, x_3)^G$ is rational over k.

Theorem (Prokhorov, 2010) 3-dim. monomial action over $k = \mathbb{C}$

$\mathbb{C}(x_1, x_2, x_3)^G$ is rational over $\mathbb{C}$.

However,
$\mathbb{Q}(x_1, x_2, x_3)^{\langle\sigma\rangle}$, $\sigma : x_1 \mapsto x_2 \mapsto x_3 \mapsto \frac{-1}{x_1 x_2 x_3}$ is not rational over $\mathbb{Q}$ (Hajja,1983).

Monomial action (2/3) [3-dim. case]

Theorem (Saltman, 2000) char $k \neq 2$

If $[k(\sqrt{a_1}, \sqrt{a_2}, \sqrt{a_3}) : k] = 8$, then $k(x_1, x_2, x_3)^{\langle\sigma\rangle}$,

$$\sigma : x_1 \mapsto \frac{a_1}{x_1}, x_2 \mapsto \frac{a_2}{x_2}, x_3 \mapsto \frac{a_3}{x_3}$$

is not retract rational over k (hence not rational over k).

Theorem (Kang, 2004)

$k(x_1, x_2, x_3)^{\langle\sigma\rangle}$, σ: $x_1 \mapsto x_2 \mapsto x_3 \mapsto \dfrac{c}{x_1 x_2 x_3} \mapsto x_1$, is rational over k
$\iff$ at least one of the following conditions is satisfied:
(i) char $k = 2$; (ii) $c \in k^2$; (iii) $-4c \in k^4$; (iv) $-1 \in k^2$.
If $k(x, y, z)^{\langle\sigma\rangle}$ is not rational over k, then it is not retract rational over k.

Recall that

- "rational" $\Longrightarrow$ "stably rational" $\Longrightarrow$ "retract rational" $\Longrightarrow$ "unirational"

Monomial action (3/3) [3-dim. case] (char $k \neq 2$)

Theorem (Yamasaki, 2012) 3-dim. monomial

$\exists$ 8 cases $G \leq GL_3(\mathbb{Z})$ s.t $k(x_1, x_2, x_3)^G$ is not retract rational over k.
Moreover, the necessary and sufficient conditions are given.

- Two of 8 cases are Saltman's and Kang's cases.
- $\exists G \leq GL_3(\mathbb{Z})$; 73 finite subgroups (up to conjugacy)

Theorem (Hoshi-Kitayama-Yamasaki, 2011) 3-dim. monomial

$k(x_1, x_2, x_3)^G$ is rational over k except for the 8 cases and $G = A_4$.
For $G = A_4$, if $[k(\sqrt{a}, \sqrt{-1}) : k] \leq 2$, then it is rational over k.

Corollary

$\exists L = k(\sqrt{a})$ such that $L(x_1, x_2, x_3)^G$ is rational over L.

- However, $\exists 4$-dim. $\mathbb{C}(x_1, x_2, x_3, x_4)^{C_2 \times C_2}$ is not retract rational.

§2. Noether's problem over $\mathbb{C}$ (1/3)

Let G be a p-group. $\mathbb{C}(G) := \mathbb{C}(x_g \mid g \in G)^G$.

- (Fisher, 1915) $\mathbb{C}(A)$ is rational over $\mathbb{C}$ if A; finite abelian group.
- (Saltman, 1984, Invent. Math.)
  For $\forall p$; prime, $\exists$ meta-abelian p-group G of order p^9
  such that $\mathbb{C}(G)$ is not retract rational over $\mathbb{C}$.
- (Bogomolov, 1988)
  For $\forall p$; prime, $\exists$ p-group G of order p^6
  such that $\mathbb{C}(G)$ is not retract rational over $\mathbb{C}$.

Indeed they showed $\mathrm{Br}_{\mathrm{nr}}(\mathbb{C}(G)/\mathbb{C}) \neq 0$; unramified Brauer group

- rational $\Longrightarrow$ stably rational $\Longrightarrow$retract rational $\Longrightarrow$ $\mathrm{Br}_{\mathrm{nr}}(\mathbb{C}(G)) = 0$.

not rational $\Leftarrow$ not stably rational $\Leftarrow$ not retract rational $\Leftarrow$ $\mathrm{Br}_{\mathrm{nr}}(\mathbb{C}(G)) \neq 0$.

- $k(G)$; retract rational $\Longrightarrow$ IGP for (k, G) has an affirmative answer.

Unramified Brauer group

Definition (Unramified Brauer group) Saltman (1984)

Let $k \subset K$ be an extension of fields.
$\mathrm{Br}_{\mathrm{nr}}(K/k) = \cap_R \mathrm{Image}\{\mathrm{Br}(R) \to \mathrm{Br}(K)\}$ where $\mathrm{Br}(R) \to \mathrm{Br}(K)$ is the natural map of Brauer groups and R runs over all the DVR such that $k \subset R \subset K$ and $K = \mathrm{Quot}(R)$.

- If K is retract rational over k, then $\mathrm{Br}(k) \xrightarrow{\sim} \mathrm{Br}_{\mathrm{nr}}(K/k)$.
  In particular, if K is retract rational over $\mathbb{C}$, then $\mathrm{Br}_{\mathrm{nr}}(K/\mathbb{C}) = 0$.
- For a smooth projective variety X over $\mathbb{C}$ with function field K, $\mathrm{Br}_{\mathrm{nr}}(K/\mathbb{C}) \simeq H^3(X, \mathbb{Z})_{\mathrm{tors}}$ which is given by Artin-Mumford (1972).

Theorem (Bogomolov 1988, Saltman 1990) $\mathrm{Br}_{\mathrm{nr}}(\mathbb{C}(G)/\mathbb{C}) \simeq B_0(G)$

Let G be a finite group. Then $\mathrm{Br}_{\mathrm{nr}}(\mathbb{C}(G)/\mathbb{C})$ is isomorphic to

$$B_0(G) = \bigcap_{A} \mathrm{Ker}\{\mathrm{res} : H^2(G, \mathbb{Q}/\mathbb{Z}) \to H^2(A, \mathbb{Q}/\mathbb{Z})\}$$

where A runs over all the bicyclic subgroups of G
(bicyclic $=$ cyclic or direct product of two cyclic groups).

- $\mathbb{C}(G)$: "retract rational" $\Longrightarrow B_0(G) = 0$.
  $B_0(G) \neq 0 \Longrightarrow \mathbb{C}(G)$: not (retract) rational over k.
- $B_0(G) \leq H^2(G, \mu) \simeq H_2(G, \mathbb{Z})$; Schur multiplier.
- $B_0(G)$ is called Bogomolov multiplier.

Noether's problem over $\mathbb{C}$ (2/3)

- (Chu-Kang, 2001) G is p-group ($|G| \leq p^4$) $\Longrightarrow \mathbb{C}(G)$ is rational.

Theorem (Moravec, 2012, Amer. J. Math.)

Assume $|G| = 3^5 = 243$. $B_0(G) \neq 0 \iff G = G(243, i)$, $28 \leq i \leq 30$.
In particular, $\exists 3$ groups G such that $\mathbb{C}(G)$ is not retract rational over $\mathbb{C}$.

- $\exists G$: 67 groups such that $|G| = 243$.

Theorem (Hoshi-Kang-Kunyavskii, 2013, Asian J. Math.)

Assume $|G| = p^5$ where p is odd prime.
$B_0(G) \neq 0 \iff G$ belongs to the isoclinism family Φ_{10}.
In particular, $\exists$ gcd$(4, p-1)+$ gcd$(3, p-1) + 1$ (resp. $\exists 3$) groups G of order p^5 $(p \geq 5)$ (resp. $p = 3$) s.t. $\mathbb{C}(G)$ is not retract rational over $\mathbb{C}$.

- $\exists 2p + 61+$ gcd$(4, p-1) + 2$ gcd$(3, p-1)$ groups such that $|G| = p^5 (p \geq 5)$. $(\exists \Phi_1, \ldots, \Phi_{10})$

From the proof (1/3)

Definition (isoclinic)

p-groups G_1 and G_2 are isoclinic $\overset{\text{def}}{\Longleftrightarrow}$
isom. $\theta : G_1/Z(G_1) \overset{\sim}{\to} G_2/Z(G_2)$, $\phi : [G_1, G_1] \overset{\sim}{\to} [G_2, G_2]$ such that

$$\begin{tikzcd}
G_1/Z(G_1) \times G_1/Z(G_1) \arrow[r, "(\theta,\theta)", "\simeq"'] \arrow[d, "{[\ ,\ ]}"'] \arrow[dr, phantom, "\circlearrowleft"] & G_2/Z(G_2) \times G_2/Z(G_2) \arrow[d, "{[\ ,\ ]}"] \\
{[G_1, G_1]} \arrow[r, "\phi", "\simeq"'] & {[G_2, G_2]}
\end{tikzcd}$$

Invariants

- lower central series
- $\#$ of conj. classes with precisely p^i members
- $\#$ of irr. complex rep. of G of degree p^i

From the proof (2/3)

- $|G| = p^4 (p > 2)$. $\exists 15$ groups (Φ_1, Φ_2, Φ_3)
- $|G| = 2^4 = 16$. $\exists 14$ groups (Φ_1, Φ_2, Φ_3)
- $|G| = p^5 (p > 3)$. $\exists 2p + 61 + (4, p-1) + 2 \times (3, p-1)$ groups $(\Phi_1, \dots, \Phi_{10})$

	Φ_1	Φ_2	Φ_3	Φ_4	Φ_5	Φ_6	Φ_7	Φ_8
$\#$	7	15	13	$p+8$	2	$p+7$	5	1
$(p=3)$						7		

	Φ_9	Φ_{10}
$\#$	$2 + (3, p-1)$	$1 + (4, p-1) + (3, p-1)$
$(p=3)$		3

From the proof (3/3)

Hoshi-Kang-Kunyavskii [HKK, Question 1.11] (2013) (arXiv:1202.5812)

Let G_1 and G_2 be isoclinic p-groups.
Is it true that the fields $k(G_1)$ and $k(G_2)$ are stably isomorphic, or, at least, that $B_0(G_1)$ is isomorphic to $B_0(G_2)$?

Theorem (Moravec, 2013) (arXiv:1203.2422)

G_1 and G_2 are isoclinic $\Longrightarrow B_0(G_1) \simeq B_0(G_2)$.

Theorem (Bogomolov-Böhning, 2013) (arXiv: 1204.4747)

G_1 and G_2 are isoclinic $\Longrightarrow \mathbb{C}(G_1)$ and $\mathbb{C}(G_2)$ are stably isomorphic.

Proof (Φ_{10}): $B_0(G) \neq 0$

Lemma 1. $N \lhd G$.

(i) $\mathrm{tr}\colon H^1(N,\mathbb{Q}/\mathbb{Z})^G \to H^2(G/N,\mathbb{Q}/\mathbb{Z})$ is not surjective where tr is the transgression map.
(ii) $AN/N \leq G/N$ is cyclic ($\forall A \leq G$; bicyclic).
$\Longrightarrow B_0(G) \neq 0$.

Proof. Consider the Hochschild-Serre 5-term exact sequence

$$0 \to H^1(G/N,\mathbb{Q}/\mathbb{Z}) \to H^1(G,\mathbb{Q}/\mathbb{Z}) \to H^1(N,\mathbb{Q}/\mathbb{Z})^G$$
$$\xrightarrow{\mathrm{tr}} H^2(G/N,\mathbb{Q}/\mathbb{Z}) \xrightarrow{\psi} H^2(G,\mathbb{Q}/\mathbb{Z})$$

where ψ is an inflation map.

(i) $\Longrightarrow$ ψ is not zero-map $\Longrightarrow \mathrm{Image}(\psi) \neq 0$.

We will show that $\mathrm{Image}(\psi) \subset B_0(G)$ by (ii).

It suffices to show that $H^2(G/N,\mathbb{Q}/\mathbb{Z}) \xrightarrow{\psi} H^2(G,\mathbb{Q}/\mathbb{Z}) \xrightarrow{\mathrm{res}} H^2(A,\mathbb{Q}/\mathbb{Z})$ is zero-map ($\forall A \leq G$: bicyclic).

Consider the following commutative diagram:

$$\begin{tikzcd}
H^2(G/N, \mathbb{Q}/\mathbb{Z}) \arrow[r, "\psi"] \arrow[d, "\psi_0"'] & H^2(G, \mathbb{Q}/\mathbb{Z}) \arrow[r, "\mathrm{res}"] & H^2(A, \mathbb{Q}/\mathbb{Z}) \\
H^2(AN/N, \mathbb{Q}/\mathbb{Z}) \arrow[rr, "\widetilde{\psi}", "\simeq"'] & & H^2(A/A\cap N, \mathbb{Q}/\mathbb{Z}) \arrow[u, "\psi_1"']
\end{tikzcd}$$

where ψ_0 is the restriction map, ψ_1 is the inflation map, $\widetilde{\psi}$ is the natural isomorphism.

(ii) $\Longrightarrow AN/N \simeq C_m \Longrightarrow H^2(C_m, \mathbb{Q}/\mathbb{Z}) = 0$.

$\Longrightarrow \psi_0$ is zero-map.

$\Longrightarrow$ res $\circ\, \psi\colon H^2(G/N, \mathbb{Q}/\mathbb{Z}) \to H^2(A, \mathbb{Q}/\mathbb{Z})$ is zero-map.

$\therefore \mathrm{Image}(\psi) \subset B_0(G)$

$\mathrm{Image}(\psi) \subset B_0(G)$ and $\mathrm{Image}(\psi) \neq 0$ (by (i)) $\Longrightarrow B_0(G) \neq 0$. $\square$

Proof (Φ_6): $B_0(G) = 0$

- $G = \Phi_6(211)a = \langle f_1, f_2, f_0, h_1, f_2\rangle, f_1^p = h_1, f_2^p = h_2,$
  $Z(G) = \langle h_1, h_2\rangle, f_0^p = h_1^p = h_2^p = 1$
  $[f_1, f_2] = f_0, [f_0, f_1] = h_1, [f_0, f_2] = h_2$

Hochschild-Serre 5-term exact sequence:

$$0 \to H^1(G/N, \mathbb{Q}/\mathbb{Z}) \to H^1(G, \mathbb{Q}/\mathbb{Z}) \to H^1(N, \mathbb{Q}/\mathbb{Z})^G \xrightarrow{\mathrm{tr}} H^2(G/N, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\psi} H^2(G, \mathbb{Q}/\mathbb{Z})$$

Proof (Φ_6): $B_0(G) = 0$

- $G = \Phi_6(211)a = \langle f_1, f_2, f_0, h_1, f_2\rangle, f_1^p = h_1, f_2^p = h_2,$
  $Z(G) = \langle h_1, h_2\rangle, f_0^p = h_1^p = h_2^p = 1$
  $[f_1, f_2] = f_0, [f_0, f_1] = h_1, [f_0, f_2] = h_2$

Hochschild-Serre 5-term exact sequence:

$$
\begin{array}{l}
0 \to H^1(G/N, \mathbb{Q}/\mathbb{Z}) \to H^1(G, \mathbb{Q}/\mathbb{Z}) \to H^1(N, \mathbb{Q}/\mathbb{Z})^G \xrightarrow{\mathrm{tr}} H^2(G/N, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\psi} H^2(G, \mathbb{Q}/\mathbb{Z}) \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow \\
\qquad\qquad\qquad \mathrm{Ker}\{H^2(G, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\mathrm{res}} H^2(N, \mathbb{Q}/\mathbb{Z})\} =: H^2(G, \mathbb{Q}/\mathbb{Z})_1 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad H^1(G/N, H^1(N, \mathbb{Q}/\mathbb{Z})) \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \lambda \downarrow \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad H^3(G/N, \mathbb{Q}/\mathbb{Z})
\end{array}
$$

- Explicit formula for λ is given
  by Dekimpe-Hartl-Wauters (2012)
- $N := \langle f_1, f_0, h_1, h_2\rangle \Longrightarrow G/N \simeq C_p \Longrightarrow H^2(G/N, \mathbb{Q}/\mathbb{Z}) = 0$
- $B_0(G) \subset H^2(G, \mathbb{Q}/\mathbb{Z})_1$
- We should show $H^2(G, \mathbb{Q}/\mathbb{Z})_1 = 0$ ($\iff$ λ: injective)

Noether's problem over $\mathbb{C}$ (3/3)

Theorem (Hoshi-Kang-Kunyavskii, 2013, Asian J. Math.)

Assume $|G| = p^5$ where p is odd prime.
$B_0(G) \neq 0 \iff G$ belongs to the isoclinism family Φ_{10}.

Theorem (Chu-Hoshi-Hu-Kang, 2015, J. Algebra) $|G| = 3^5 = 243$

If $B_0(G) = 0$, then $\mathbb{C}(G)$ is rational over $\mathbb{C}$ except for Φ_7.

- Non-rationality of Φ_7 is detected by $H^3_{\rm nr}(\mathbb{C}(G), \mathbb{Q}/\mathbb{Z})$ (later).
- Φ_5 and Φ_7 are very similar: $C = 1$ (Φ_5), $C = \omega$ (Φ_7).

$\mathbb{C}(G)$ is stably isomorphic to $\mathbb{C}(z_1, z_2, z_3, z_4, z_5, z_6, z_7, z_8, z_9)^{\langle f_1, f_2 \rangle}$

$$\begin{aligned}
f_1 : z_1 &\mapsto z_2, z_2 \mapsto \frac{1}{z_1 z_2}, z_3 \mapsto z_4, z_4 \mapsto \frac{1}{z_3 z_4}, \\
z_5 &\mapsto \frac{z_5}{z_1^2 z_3}, z_6 \mapsto \frac{z_1 z_6}{z_3}, z_7 \mapsto z_8, z_8 \mapsto \frac{1}{z_7 z_8}, z_9 \mapsto \frac{z_4 z_9}{z_1}, \\
f_2 : z_1 &\mapsto z_3, z_2 \mapsto z_4, z_3 \mapsto \frac{1}{z_1 z_3}, z_4 \mapsto \frac{1}{z_2 z_4}, \\
z_5 &\mapsto z_6, z_6 \mapsto \frac{1}{z_5 z_6}, z_7 \mapsto C\frac{z_4 z_7}{z_3}, z_8 \mapsto C\frac{z_8}{z_3 z_4^2}, z_9 \mapsto \frac{z_4 z_9}{z_1}.
\end{aligned}$$

Unramified Brauer group: purely monomial case (1/3)

Theorem (Hoshi-Kang-Yamasaki, 2023, Mem AMS) purely monomial

Let G be a finite group and M be a faithful G-lattice.
(1) If $\mathrm{rank}_{\mathbb{Z}} M \leq 3$, then $\mathrm{Br}_{\mathrm{nr}}(\mathbb{C}(M)^G) = 0$.
(2) When $\mathrm{rank}_{\mathbb{Z}} M = 4$, $\exists$ 5 M's with $\mathrm{Br}_{\mathrm{nr}}(\mathbb{C}(M)^G) \neq 0$.
(3) When $\mathrm{rank}_{\mathbb{Z}} M = 5$, $\exists$ 46 M's with $\mathrm{Br}_{\mathrm{nr}}(\mathbb{C}(M)^G) \neq 0$.
(4) When $\mathrm{rank}_{\mathbb{Z}} M = 6$, $\exists$ 1073 M's with $\mathrm{Br}_{\mathrm{nr}}(\mathbb{C}(M)^G) \neq 0$.

rank	# of G-lattices	# of unramified Brauer groups $\neq 0$
1	2	0
2	13	0
3	73	0
4	710	5
5	6079	46
6	85308	1073

- If M is of rank ≤ 6 and $\mathrm{Br}_{\mathrm{nr}}(\mathbb{C}(M^G)) \neq 0$, then G is solvable and non-abelian, and $\mathrm{Br}_{\mathrm{nr}}(\mathbb{C}(M)^G) \simeq \mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/3\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

Unramified Brauer group: purely monomial case (2/3)

Theorem (Hoshi-Kang-Yamasaki, 2023, Mem. AMS) $G = A_6$: simple

Embed $A_6 \simeq PSL_2(\mathbb{F}_9) \hookrightarrow S_{10}$. Let $N = \oplus_{1 \leq i \leq 10} \mathbb{Z} \cdot x_i$ be the S_{10}-lattice defined by $\sigma \cdot x_i = x_{\sigma(i)}$ for any $\sigma \in S_{10}$; it becomes an A_6-lattice by restricting the action of S_{10} to A_6. Define $M = N/(\mathbb{Z} \cdot \sum_{i=1}^{10} x_i)$ with $\mathrm{rank}_{\mathbb{Z}} M = 9$. $\exists A_6$-lattices $M = M_1, M_2, \ldots, M_6$ which are $\mathbb{Q}$-conjugate but not $\mathbb{Z}$-conjugate to each other; in fact, all these M_i form a single $\mathbb{Q}$-class, but this $\mathbb{Q}$-class consists of six $\mathbb{Z}$-classes. Then we have

$$H^2_{\mathrm{nr}}(A_6, M_1) \simeq H^2_{\mathrm{nr}}(A_6, M_3) \simeq \mathbb{Z}/2\mathbb{Z},\ H^2_{\mathrm{nr}}(A_6, M_i) = 0 \text{ for } i = 2, 4, 5, 6.$$

In particular, $\mathbb{C}(M_1)^{A_6}$ and $\mathbb{C}(M_3)^{A_6}$ are not retract rational over $\mathbb{C}$. Furthermore, M_1 and M_3 may be distinguished by Tate cohomologies:

$$H^1(A_6, M_1) = 0, \qquad \widehat{H}^{-1}(A_6, M_1) = \mathbb{Z}/10\mathbb{Z},$$
$$H^1(A_6, M_3) = \mathbb{Z}/5\mathbb{Z}, \qquad \widehat{H}^{-1}(A_6, M_3) = \mathbb{Z}/2\mathbb{Z}.$$

By using a result of Saltman (1987, J. Algebra, Corollary 3.3), as a corollary of Theorem above, we can get:

Theorem (Hoshi-Kang-Yamasaki, 2023, Mem. AMS) $G = A_6$: simple

Let $N_1 \simeq (C_{10})^9$ and $N_3 \simeq (C_2)^8 \times C_{10}$. Then, for $i = 1, 3$, $\mathrm{Br}_u(\mathbb{C}(N_i \rtimes A_6)) \simeq \mathbb{Z}/2\mathbb{Z}$ and Noether's problem for $N_i \rtimes A_6$ over $\mathbb{C}$ has a negative answer. Moreover, $\mathbb{C}(N_i \rtimes A_6)$ $(i = 1, 3)$ is not retract (stably) rational over $\mathbb{C}$.

- Noether's problem for A_6 over $\mathbb{Q}$ (resp. over $\mathbb{C}$) is still unsolved!

Unramified cohomology (1/4)

Colliot-Thélène and Ojanguren (1989) generalized the notion of the unramified Brauer group $\mathrm{Br}_{\mathrm{nr}}(K/\mathbb{C})$ to the unramified cohomology $H^i_{\mathrm{nr}}(K/\mathbb{C},\mu_n^{\otimes j})$ of degree $i\geq 1$:

Definition (Colliot-Thélène and Ojanguren, 1989, Invent. Math.)

Let $K/\mathbb{C}$ be a function field, that is finitely generated as a field over $\mathbb{C}$. The unramified cohomology group $H^i_{\mathrm{nr}}(K/\mathbb{C},\mu_n^{\otimes j})$ of K over $\mathbb{C}$ of degree $i\geq 1$ is defined to be

$$H^i_{\mathrm{nr}}(K/\mathbb{C},\mu_n^{\otimes j})=\bigcap_R \mathrm{Ker}\{r_R : H^i(K,\mu_n^{\otimes j})\to H^{i-1}(\Bbbk_R,\mu_n^{\otimes (j-1)})\}$$

where R runs over all the DVR of rank one such that $\mathbb{C}\subset R\subset K$ and $K=\mathrm{Quot}(R)$ and r_R is the residue map.

- Note that ${}_n\mathrm{Br}_{\mathrm{nr}}(K/\mathbb{C})\simeq H^2_{\mathrm{nr}}(K/\mathbb{C},\mu_n)$.

Proposition (Colliot-Thélène and Ojanguren, 1989)

If K and L are stably $\mathbb{C}$-isomorphic, then
$H^i_{\mathrm{nr}}(K/\mathbb{C}, \mu_n^{\otimes j}) \xrightarrow{\sim} H^i_{\mathrm{nr}}(L/\mathbb{C}, \mu_n^{\otimes j})$.
In particular, K is stably rational over $\mathbb{C}$, then $H^i_{\mathrm{nr}}(K/\mathbb{C}, \mu_n^{\otimes j}) = 0$.

- Moreover, if K is retract rational over $\mathbb{C}$, then $H^i_{\mathrm{nr}}(K/\mathbb{C}, \mu_n^{\otimes j}) = 0$.
- CTO (1989) $\exists$ $\mathbb{C}$-unirational field K with $\mathrm{trdeg}_{\mathbb{C}} K = 6$ s.t. $H^3_{\mathrm{nr}}(K/\mathbb{C}, \mu_2^{\otimes 3}) \neq 0$ and $\mathrm{Br}_{\mathrm{nr}}(K/\mathbb{C}) = 0$.
- Peyre (1993) gave a sufficient condition for $H^i_{\mathrm{nr}}(K/\mathbb{C}, \mu_p^{\otimes i}) \neq 0$:
- $\exists K$ s.t. $H^3_{\mathrm{nr}}(K/\mathbb{C}, \mu_p^{\otimes 3}) \neq 0$ and $\mathrm{Br}_{\mathrm{nr}}(K/\mathbb{C}) = 0$;
- $\exists K$ s.t. $H^4_{\mathrm{nr}}(K/\mathbb{C}, \mu_2^{\otimes 4}) \neq 0$ and $\mathrm{Br}_{\mathrm{nr}}(K/\mathbb{C}) = 0$.

Take the direct limit with respect to n:

$$H^i(K/\mathbb{C},\mathbb{Q}/\mathbb{Z}(j))=\varinjlim_{n} H^i(K/\mathbb{C},\mu_n^{\otimes j})$$

and we also define the unramified cohomology group

$$\begin{aligned}&H^i_{\mathrm{nr}}(K/\mathbb{C},\mathbb{Q}/\mathbb{Z}(j))\\&=\bigcap_R \mathrm{Ker}\{r_R : H^i(K/\mathbb{C},\mathbb{Q}/\mathbb{Z}(j))\to H^{i-1}(\Bbbk_R,\mathbb{Q}/\mathbb{Z}(j-1))\}.\end{aligned}$$

Then we have $\mathrm{Br}_{\mathrm{nr}}(K/\mathbb{C})\simeq H^2_{\mathrm{nr}}(K/\mathbb{C},\mathbb{Q}/\mathbb{Z}(1))$.

- The case $K=\mathbb{C}(G)$:

Theorem (Peyre, 2008, Invent. Math.) p: odd prime

$\exists$ p-group G of order p^{12} such that $B_0(G)=0$ and $H^3_{\mathrm{nr}}(\mathbb{C}(G),\mathbb{Q}/\mathbb{Z})\neq 0$.
In particular, $\mathbb{C}(G)$ is not (retract, stably) rational over $\mathbb{C}$.

- Asok (2013) generalized Peyre's argument (1993):

Theorem (Asok, 2013, Compos. Math.)

(1) For any $n > 0$, $\exists$ a smooth projective complex variety X that is $\mathbb{C}$-unirational, for which $H^i_{\mathrm{nr}}(\mathbb{C}(X), \mu_2^{\otimes i}) = 0$ for each $i < n$, yet $H^n_{\mathrm{nr}}(\mathbb{C}(X), \mu_2^{\otimes n}) \neq 0$, and so
X is not $\mathbb{A}^1$-connected, nor (retract, stably) rational over $\mathbb{C}$;
(2) For any prime l and any $n \geq 2$, $\exists$ a smooth projective rationally connected complex variety Y such that $H^n_{\mathrm{nr}}(\mathbb{C}(Y), \mu_l^{\otimes n}) \neq 0$.
In particular, Y is not $\mathbb{A}^1$-connected, nor (retract, stably) rational over $\mathbb{C}$.

- Namely, the triviality of the unramified Brauer group or the unramified cohomology of higher degree is just a necessary condition of $\mathbb{C}$-rationality of fields.
- It is interesting to consider an analog of above Theorem for quotient varieties V/G, e.g. $\mathbb{C}(V_{\mathrm{reg}}/G) = \mathbb{C}(G)$.

Unramified cohomology (3/4)

Theorem (Peyre, 2008, Invent. Math.) p: odd prime

$\exists$ p-group G of order p^{12} such that $B_0(G) = 0$ and $H^3_{\mathrm{nr}}(\mathbb{C}(G), \mathbb{Q}/\mathbb{Z}) \neq 0$.
In particular, $\mathbb{C}(G)$ is not (retract, stably) rational over $\mathbb{C}$.

Using Peyre's method, we improve this result:

Theorem (Hoshi-Kang-Yamasaki, 2016, J. Algebra) p: odd prime

$\exists$ p-group G of order p^{9} such that $B_0(G) = 0$ and $H^3_{\mathrm{nr}}(\mathbb{C}(G), \mathbb{Q}/\mathbb{Z}) \neq 0$.
In particular, $\mathbb{C}(G)$ is not (retract, stably) rational over $\mathbb{C}$.

On the other hand, CT and Voisin proved: ($\leftrightarrow$ integral Hodge conjecture)

Theorem (Colliot-Thélène and Voisin, 2012, Duke Math. J.)

Let X be a smooth projective rationally connected complex variety. Then
$H^3_{\mathrm{nr}}(X, \mathbb{Q}/\mathbb{Z}) \simeq \mathrm{Hdg}^4(X, \mathbb{Z})/\mathrm{Hdg}^4(X, \mathbb{Z})_{\mathrm{alg}}$.

Unramified cohomology (4/4)

- Using Peyre's formula [Peyre, 2008, Invent. Math.], we get:

Theorem (Hoshi-Kang-Yamasaki, 2020, J. Algebra) $|G|=3^5$

$H^3_{\rm nr}(\mathbb{C}(G),\mathbb{Q}/\mathbb{Z})\neq 0 \iff G$ belongs to the isoclinism family Φ_7.
In particular, $\mathbb{C}(G)$ is not rational over $\mathbb{C} \iff G$ belongs to Φ_7,Φ_{10}.

$\lvert G\rvert=3^5$	Φ_1	Φ_2	Φ_3	Φ_4	Φ_5	Φ_6	Φ_7	Φ_8	Φ_9	Φ_{10}
$H^2_{\rm nr}(\mathbb{C}(G),\mathbb{Q}/\mathbb{Z})$	0	0	0	0	0	0	0	0	0	$\mathbb{Z}/3\mathbb{Z}$
$H^3_{\rm nr}(\mathbb{C}(G),\mathbb{Q}/\mathbb{Z})$	0	0	0	0	0	0	$\mathbb{Z}/3\mathbb{Z}$	0	0	0

Theorem (Hoshi-Kang-Yamasaki, 2020, J. Algebra) $|G|=5^5$ or 7^5

$H^3_{\rm nr}(\mathbb{C}(G),\mathbb{Q}/\mathbb{Z})\neq 0 \iff G$ belongs to Φ_6, Φ_7 or Φ_{10}.

$\lvert G\rvert=p^5$ $(p=5,7)$	Φ_1	Φ_2	Φ_3	Φ_4	Φ_5	Φ_6	Φ_7	Φ_8	Φ_9	Φ_{10}
$H^2_{\rm nr}(\mathbb{C}(G),\mathbb{Q}/\mathbb{Z})$	0	0	0	0	0	0	0	0	0	$\mathbb{Z}/p\mathbb{Z}$
$H^3_{\rm nr}(\mathbb{C}(G),\mathbb{Q}/\mathbb{Z})$	0	0	0	0	0	$\mathbb{Z}/p\mathbb{Z}$	$\mathbb{Z}/p\mathbb{Z}$	0	0	$\mathbb{Z}/p\mathbb{Z}$

Noether's problem over $\mathbb{C}$ for 2-groups

- (Chu-Kang, 2001) G is p-group ($|G| \leq p^4$) $\Longrightarrow \mathbb{C}(G)$ is rational.
- (Chu-Hu-Kang-Prokhorov, 2008)
  $|G| = 32 = 2^5 \Longrightarrow \mathbb{C}(G)$ is rational.
- $\exists 267$ groups G of order $64 = 2^6$ which are classified into 27 isoclinism families $\Phi_1, \ldots, \Phi_{27}$.

Theorem (Chu-Hu-Kang-Kunyavskii, 2010) $|G| = 64 = 2^6$

(1) $B_0(G) \neq 0 \iff G$ belongs to Φ_{16}. ($\exists 9$ such G's)
Moreover, if $B_0(G) \neq 0$, then $B_0(G) \simeq C_2$.
(2) If $B_0(G) = 0$, then $\mathbb{C}(G)$ is rational except for Φ_{13}. ($\exists 5$ such G's)

- ([CHKK10], [HY14]) ($B_0(G) = 0$, but rationality unknown)
  If G belongs to Φ_{13}, then $\mathbb{C}(G)$ is stably $\mathbb{C}$-isomorphic to $L_{\mathbb{C}}^{(0)}$.
- ([CHKK10], [HKK14]) ($B_0(G) \simeq C_2$, not retract rational)
  If G belongs to Φ_{16}, then $\mathbb{C}(G)$ is stably $\mathbb{C}$-isomorphic to $L_{\mathbb{C}}^{(1)}$.

- ([CHKK10], [HY14]) ($B_0(G)=0$, but rationality unknown)
  If G belongs to Φ_{13}, then $\mathbb{C}(G)$ is stably $\mathbb{C}$-isomorphic to $L_{\mathbb{C}}^{(0)}$.
- ([CHKK10], [HKK14]) ($B_0(G)\simeq C_2$, not retract rational)
  If G belongs to Φ_{16}, then $\mathbb{C}(G)$ is stably $\mathbb{C}$-isomorphic to $L_{\mathbb{C}}^{(1)}$.

Definition (The fields $L_{\mathbb{C}}^{(0)}$ and $L_{\mathbb{C}}^{(1)}$)

(i) The field $L_{\mathbb{C}}^{(0)}$ is defined to be $\mathbb{C}(X_1,X_2,X_3,X_4,X_5,X_6)^H$ where $H=\langle\sigma_1,\sigma_2\rangle\simeq C_2\times C_2$ act on $\mathbb{C}(X_1,X_2,X_3,X_4,X_5,X_6)$ by

$$\sigma_1: X_1\mapsto X_3,\ X_2\mapsto \frac{1}{X_1X_2X_3},\ X_3\mapsto X_1,\ X_4\mapsto X_6,\ X_5\mapsto \frac{1}{X_4X_5X_6},\ X_6\mapsto X_4,$$
$$\sigma_2: X_1\mapsto X_2,\ X_2\mapsto X_1,\ X_3\mapsto \frac{1}{X_1X_2X_3},\ X_4\mapsto X_5,\ X_5\mapsto X_4,\ X_6\mapsto \frac{1}{X_4X_5X_6}.$$

(ii) The field $L_{\mathbb{C}}^{(1)}$ is defined to be $\mathbb{C}(X_1,X_2,X_3,X_4)^{\langle\tau\rangle}$ where $\langle\tau\rangle\simeq C_2$ acts on $\mathbb{C}(X_1,X_2,X_3,X_4)$ by

$$\tau: X_1\mapsto -X_1,\ X_2\mapsto \frac{X_4}{X_2},\ X_3\mapsto \frac{(X_4-1)(X_4-X_1^2)}{X_3},\ X_4\mapsto X_4.$$

- ([CHKK10], [HY14]) ($B_0(G)=0$, but rationality unknown)
  If G belongs to Φ_{13}, then $\mathbb{C}(G)$ is stably $\mathbb{C}$-isomorphic to $L_{\mathbb{C}}^{(0)}$.
- ([CHKK10], [HKK14]) ($B_0(G)\simeq C_2$, not retract rational)
  If G belongs to Φ_{16}, then $\mathbb{C}(G)$ is stably $\mathbb{C}$-isomorphic to $L_{\mathbb{C}}^{(1)}$.

- $L_{\mathbb{C}}^{(0)}=\mathbb{C}(z_1,z_2,z_3,z_4,u_4,u_5,u_6)$ where
$$(z_1^2-a)(z_4^2-d)=(z_2^2-b)(z_3^2-c),$$
$$a=u_4(u_4-1), b=u_4-1, c=u_4(u_4-u_6^2), d=u_5^2(u_4-u_6^2).$$
- $L_{\mathbb{C}}^{(0)}=\mathbb{C}(u,v,t,w_3,w_4,w_5,w_6)$ where
$$u^2-tv^2=-\left(w_4^2(w_5^2-1)t^2+(w_3^2-w_3^2w_5^2+1)t-w_5^2\right)$$
$$\cdot\left(w_4^2w_6^2t^2-(w_4^2+w_3^2w_6^2)t+w_3^2-w_6^2+1\right).$$
- $L_{\mathbb{C}}^{(0)}=\mathbb{C}(m_0,\ldots,m_6)$ where
$$m_0^2=(4m_3+m_3m_4^2+m_4^2)(m_3-m_5^2+1)$$
$$\cdot(m_1^2m_3+m_6^2-1)(4m_3+m_1^2m_2^2m_3+m_2^2m_6^2).$$
- $L_{\mathbb{C}}^{(1)}=\mathbb{C}(u,v,t,w_3,w_4)$ where
$$u^2-tv^2=(tw_4^2-w_3^2+1)(t+tw_4^2-w_3^2).$$

- $\exists 2328$ groups G of order $128 = 2^7$ which are classified into 115 isoclinism families $\Phi_1, \ldots, \Phi_{115}$.

Theorem (Moravec, 2012, Amer. J. Math.) $|G| = 128 = 2^7$

$B_0(G) \neq 0$ if and only if G belongs to the isoclinism family Φ_{16}, Φ_{30}, Φ_{31}, Φ_{37}, Φ_{39}, Φ_{43}, Φ_{58}, Φ_{60}, Φ_{80}, Φ_{106} or Φ_{114}. If $B_0(G) \neq 0$, then

$$B_0(G) \simeq \begin{cases} C_2 & (\Phi_{16}, \Phi_{31}, \Phi_{37}, \Phi_{39}, \Phi_{43}, \Phi_{58}, \Phi_{60}, \Phi_{80}, \Phi_{106}, \Phi_{114}) \\ C_2 \times C_2 & (\Phi_{30}). \end{cases}$$

In particular, $\mathbb{C}(G)$ is not (retract, stably) rational over $\mathbb{C}$.

	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	Total
Family	Φ_{16}	Φ_{31}	Φ_{37}	Φ_{39}	Φ_{43}	Φ_{58}	Φ_{60}	Φ_{80}	Φ_{106}	Φ_{114}	Φ_{30}	
$B_0(G)$					C_2						$C_2 \times C_2$	
$\#$ G's	48	55	18	6	26	20	10	9	2	2	34	220

- $\boxed{\text{Q.}}$ Birational classification of $\mathbb{C}(G)$?
  In particular, what happens when $B_0(G) \neq 0$?
  How many $\mathbb{C}(G)$'s exist up to stably $\mathbb{C}$-isomorphism?

Theorem (Hoshi, 2016, J. Algebra) $|G| = 128 = 2^7$

Assume that $B_0(G) \neq 0$.
Then $\mathbb{C}(G)$ and $L_{\mathbb{C}}^{(m)}$ are stably $\mathbb{C}$-isomorphic where

$$m = \begin{cases} 1 & \text{if } G \text{ belongs to } \Phi_{16}, \Phi_{31}, \Phi_{37}, \Phi_{39}, \Phi_{43}, \Phi_{58}, \Phi_{60} \text{ or } \Phi_{80}, \\ 2 & \text{if } G \text{ belongs to } \Phi_{106} \text{ or } \Phi_{114}, \\ 3 & \text{if } G \text{ belongs to } \Phi_{30}. \end{cases}$$

In particular, $\mathrm{Br}_{\mathrm{nr}}(L_{\mathbb{C}}^{(1)}) \simeq \mathrm{Br}_{\mathrm{nr}}(L_{\mathbb{C}}^{(2)}) \simeq C_2$ and $\mathrm{Br}_{\mathrm{nr}}(L_{\mathbb{C}}^{(3)}) \simeq C_2 \times C_2$ and hence $L_{\mathbb{C}}^{(1)}$, $L_{\mathbb{C}}^{(2)}$ and $L_{\mathbb{C}}^{(3)}$ are not (retract, stably) rational over $\mathbb{C}$.

- $L_{\mathbb{C}}^{(1)} \not\approx L_{\mathbb{C}}^{(3)}$, $L_{\mathbb{C}}^{(2)} \not\approx L_{\mathbb{C}}^{(3)}$ (not stably $\mathbb{C}$-isomorphic) because their unramified Brauer groups are not isomorphic.
- However, we do not know whether $L_{\mathbb{C}}^{(1)} \sim L_{\mathbb{C}}^{(2)}$.
- If not, evaluate the higher unramified cohomologies $H_{\mathrm{nr}}^i (i \geq 3)$? (Peyre's formula can not work for $|G| = 2^m$)

Definition (The fields $L_{\mathbb{C}}^{(2)}$ and $L_{\mathbb{C}}^{(3)}$)

(i) The field $L_{\mathbb{C}}^{(2)}$ is defined to be $\mathbb{C}(X_1,X_2,X_3,X_4,X_5,X_6)^{\langle\rho\rangle}$ where $\langle\rho\rangle\simeq C_4$ acts on $\mathbb{C}(X_1,X_2,X_3,X_4,X_5,X_6)$ by

$$\rho: X_1\mapsto X_2, X_2\mapsto -X_1, X_3\mapsto X_4, X_4\mapsto X_3,$$
$$X_5\mapsto X_6, X_6\mapsto \frac{(X_1^2X_2^2-1)(X_1^2X_3^2+X_2^2-X_3^2-1)}{X_5}.$$

(ii) The field $L_{\mathbb{C}}^{(3)}$ is defined to be $\mathbb{C}(X_1,X_2,X_3,X_4,X_5,X_6,X_7)^{\langle\lambda_1,\lambda_2\rangle}$ where $\langle\lambda_1,\lambda_2\rangle\simeq C_2\times C_2$ acts on $\mathbb{C}(X_1,X_2,X_3,X_4,X_5,X_6,X_7)$ by

$$\lambda_1: X_1\mapsto X_1, X_2\mapsto \frac{X_1}{X_2}, X_3\mapsto\frac{1}{X_1X_3}, X_4\mapsto\frac{X_2X_4}{X_1X_3},$$
$$X_5\mapsto -\frac{X_1X_6^2-1}{X_5}, X_6\mapsto -X_6, X_7\mapsto X_7,$$
$$\lambda_2: X_1\mapsto\frac{1}{X_1}, X_2\mapsto X_3, X_3\mapsto X_2, X_4\mapsto\frac{(X_1X_6^2-1)(X_1X_7^2-1)}{X_4},$$
$$X_5\mapsto -X_5, X_6\mapsto -X_1X_6, X_7\mapsto -X_1X_7.$$

§3. (general) quasi-monomial actions

Notion of "quasi-monomial" actions
is defined in Hoshi-Kang-Kitayama [HKK14], J. Algebra (2014).

Theorem ([HKK14]) 1-dim. quasi-monomial actions

(1) purely quasi-monomial $\Longrightarrow K(x)^G$ is rational over k.
(2) $K(x)^G$ is rational over k excpet for the case: $\exists N \leq G$ such that
(i) $G/N = \langle\sigma\rangle \simeq C_2$;
(ii) $K(x)^N = k(\alpha)(y)$, $\alpha^2 = a \in K^\times$, $\sigma(\alpha) = -\alpha$ (if char k $\neq 2$),
$\alpha^2 + \alpha = a \in K$, $\sigma(\alpha) = \alpha + 1$ (if char $k = 2$);
(iii) $\sigma \cdot y = b/y$ for some $b \in k^\times$.
For the exceptional case, $K(x)^G = k(\alpha)(y)^{G/N}$ is rational over $k \iff$
Hilbert symbol $(a,b)_k = 0$ (if char $k \neq 2$), $[a,b)_k = 0$ (if char $k = 2$).
Moreover, $K(x)^G$ is not rational over $k \Longrightarrow$ not unirational over k.

Theorem ([HKK14]) 2-dim. purely quasi-monomial actions

$N=\{\sigma\in G\mid \sigma(x)=x,\ \sigma(y)=y\}$, $H=\{\sigma\in G\mid \sigma(\alpha)=\alpha(\forall\alpha\in K)\}$.
$K(x,y)^G$ is rational over k except for:
(1) char $k\neq 2$ and (2) (i) $(G/N,HN/N)\simeq(C_4,C_2)$ or (ii) (D_4,C_2).
For the exceptional case, we have $k(x,y)=k(u,v)$:
(i) $(G/N,HN/N)\simeq(C_4,C_2)$,
$K^N=k(\sqrt{a})$, $G/N=\langle\sigma\rangle\simeq C_4$, $\sigma:\sqrt{a}\mapsto-\sqrt{a}$, $u\mapsto\frac{1}{u}$, $v\mapsto-\frac{1}{v}$;
(ii) $(G/N,HN/N)\simeq(D_4,C_2)$;
$K^N=k(\sqrt{a},\sqrt{b})$, $G/N=\langle\sigma,\tau\rangle\simeq D_4$, $\sigma:\sqrt{a}\mapsto-\sqrt{a}$, $\sqrt{b}\mapsto\sqrt{b}$,
$u\mapsto\frac{1}{u}$, $v\mapsto-\frac{1}{v}$, $\tau:\sqrt{a}\mapsto\sqrt{a}$, $\sqrt{b}\mapsto-\sqrt{b}$, $u\mapsto u$, $v\mapsto -v$.
Case (i), $K(x,y)^G$ is rational over k $\iff$ Hilbert symbol $(a,-1)_k=0$.
Case (ii), $K(x,y)^G$ is rational over k $\iff$ Hilbert symbol $(a,-b)_k=0$.
Moreover, $K(x,y)^G$ is not rational over $k\Longrightarrow$
$\mathrm{Br}(k)\neq 0$ and $K(x,y)^G$ is not unirational over k.

Galois-theoretic interpretation:
(i) rational over k $\iff$ $k(\sqrt{a})$ may be embedded into C_4-ext. of k.
(ii) rational over k $\iff$ $k(\sqrt{a},\sqrt{b})$ may be embedded into D_4-ext. of k.

Application to purely monomial actions (1/2)

Theorem ([HKK14]), 4-dim. purely monomial

Let M be a G-lattice with $\mathrm{rank}_{\mathbb{Z}} M = 4$ and G act on $k(M)$ by purely monomial k-automorphisms. If M is decomposable,
i.e. $M = M_1 \oplus M_2$ as $\mathbb{Z}[G]$-modules where $1 \leq \mathrm{rank}_{\mathbb{Z}} M_1 \leq 3$,
then $k(M)^G$ is rational over k.

- When $\mathrm{rank}_{\mathbb{Z}} M_1 = 1, \mathrm{rank}_{\mathbb{Z}} M_2 = 3$,
  it is easy to see $k(M)^G$ is rational.
- When $\mathrm{rank}_{\mathbb{Z}} M_1 = \mathrm{rank}_{\mathbb{Z}} M_2 = 2$, we may apply Theorem of 2-dim.
  to $k(M) = k(x_1, x_2, y_1, y_2) = k(x_1, x_2)(y_1, y_2) = K(y_1, y_2)$.

Theorem ([HKK14]) char $k \neq 2$

Let $C_2 = \langle \tau \rangle$ act on the rational function field $k(x_1, x_2, x_3, x_4)$ by k-automorphisms defined as

$$\tau : x_1 \mapsto -x_1,\ x_2 \mapsto \tfrac{x_4}{x_2},\ x_3 \mapsto \tfrac{(x_4-1)(x_4-x_1^2)}{x_3},\ x_4 \mapsto x_4.$$

Then $k(x_1, x_2, x_3, x_4)^{C_2}$ is not retract rational over k.
In particular, it is not rational over k.

Theorem A ([HKK14]) char $k \neq 2$, 5-dim. purely monomial

Let $D_4 = \langle \rho, \tau \rangle$ act on the rational function field $k(x_1, x_2, x_3, x_4, x_5)$ by k-automorphisms defined as

$$\rho : x_1 \mapsto x_2,\ x_2 \mapsto x_1,\ x_3 \mapsto \tfrac{1}{x_1 x_2 x_3},\ x_4 \mapsto x_5,\ x_5 \mapsto \tfrac{1}{x_4},$$
$$\tau : x_1 \mapsto x_3,\ x_2 \mapsto \tfrac{1}{x_1 x_2 x_3},\ x_3 \mapsto x_1,\ x_4 \mapsto x_5,\ x_5 \mapsto x_4.$$

Then $k(x_1, x_2, x_3, x_4, x_5)^{D_4}$ is not retract rational over k.
In particular, it is not rational over k.

Application to purely monomial actions (2/2)

Theorem ([HKK14]), 5-dim. purely monomial

Let M be a G-lattice and G act on $k(M)$ by purely monomial k-automorphisms. Assume that
(i) $M = M_1 \oplus M_2$ as $\mathbb{Z}[G]$-modules where $\mathrm{rank}_{\mathbb{Z}} M_1 = 3$ and $\mathrm{rank}_{\mathbb{Z}} M_2 = 2$,
(ii) either M_1 or M_2 is a faithful G-lattice.
Then $k(M)^G$ is rational over k except for the case as in Theorem A.

- we may apply Theorem of 2-dim. to $k(M) = k(x_1, x_2, x_3, y_1, y_2) = k(x_1, x_2, x_3)(y_1, y_2) = K(y_1, y_2)$.

More recent results

- 3-dim. purely quasi-monomial actions (Hoshi-Kitayama, 2020, Kyoto J. Math.)

§4. Rationality problem for algebraic tori (2-dim., 3-dim.)

$G \simeq \mathrm{Gal}(K/k) \curvearrowright K(x_1, \ldots, x_n)$: purely quasi-monomial, $K(x_1, \ldots, x_n)^G$ may be regarded as the function field of algebraic torus T over k which splits over K ($T \otimes_k K \simeq \mathbb{G}_m^n$).

- T is unirational over k, i.e. $K(x_1, \ldots, x_n)^G \subset k(t_1, \ldots, t_n)$.
- $\exists 13$ $\mathbb{Z}$-coujugacy subgroups $G \leq \mathrm{GL}_2(\mathbb{Z})$.

Theorem (Voskresenskii, 1967) 2-dim. algebraic tori T

T is rational over k.

- $\exists 73$ $\mathbb{Z}$-coujugacy subgroups $G \leq \mathrm{GL}_3(\mathbb{Z})$.

Theorem (Kunyavskii, 1990) 3-dim. algebraic tori T

(i) T is rational over k $\iff$ T is stably rational over k $\iff$ T is retract rational over k $\iff$ $\exists G$: 58 groups;
(ii) T is not rational over k $\iff$ T is not stably rational over k $\iff$ T is not retract rational over k $\iff$ $\exists G$: 15 groups.

Rationality of algebraic tori (4-dim., 5-dim.)

- $\exists 710$ $\mathbb{Z}$-coujugacy subgroups $G \leq \mathrm{GL}_4(\mathbb{Z})$.

Theorem (Hoshi-Yamasaki, 2017, Mem. AMS) 4-dim. alg. tori T

(i) T is stably rational over k $\iff$ $\exists G$: 487 groups;
(ii) T is not stably but retract rational over k $\iff$ $\exists G$: 7 groups;
(iii) T is not retract rational over k $\iff$ $\exists G$: 216 groups.

- $\exists 6079$ $\mathbb{Z}$-coujugacy subgroups $G \leq \mathrm{GL}_5(\mathbb{Z})$.

Theorem (Hoshi-Yamasaki, 2017, Mem. AMS) 5-dim. alg. tori T

(i) T is stably rational over k $\iff$ $\exists G$: 3051 groups;
(ii) T is not stably but retract rational over k $\iff$ $\exists G$: 25 groups;
(iii) T is not retract rational over k $\iff$ $\exists G$: 3003 groups.

- (Voskresenskii's conjecture) any stably rational torus is rational.
- $\exists 85308$ $\mathbb{Z}$-coujugacy subgroups $G \leq \mathrm{GL}_6(\mathbb{Z})$!

Proof: Flabby (Flasque) resolution (1/2)

- $\boxed{\text{The function field of } n\text{-dim. } T \overset{\text{identified}}{\longleftrightarrow} L(M)^G,\ G \leq \mathrm{GL}(n,\mathbb{Z})}$
- M: G-lattice, i.e. f.g. $\mathbb{Z}$-free $\mathbb{Z}[G]$-module.

Definition

(i) M is permutation $\overset{\text{def}}{\iff} M \simeq \oplus_{1\leq i\leq m}\mathbb{Z}[G/H_i]$.

(ii) M is stably permutation $\overset{\text{def}}{\iff} M \oplus \exists P \simeq P'$, P, P': permutation.

(iii) M is invertible $\overset{\text{def}}{\iff} M \oplus \exists M' \simeq P$: permutation.

(iv) M is coflabby $\overset{\text{def}}{\iff} H^1(H, M) = 0$ $(\forall H \leq G)$.

(v) M is flabby $\overset{\text{def}}{\iff} \widehat{H}^{-1}(H, M) = 0$ $(\forall H \leq G)$. ($\widehat{H}$: Tate cohomology)

- "permutation"
  $\Longrightarrow$ "stably permutation"
  $\Longrightarrow$ "invertible"
  $\Longrightarrow$ "flabby and coflabby".

Proof: Flabby (Flasque) resolution (2/2)

Commutative monoid $\mathcal{M}$

$M_1 \sim M_2 \overset{\text{def}}{\iff} M_1 \oplus P_1 \simeq M_2 \oplus P_2$ ($\exists P_1, \exists P_2$: permutation).
$\Longrightarrow$ commutative monoid $\mathcal{M}$: $[M_1] + [M_2] := [M_1 \oplus M_2]$, $0 = [P]$.

Theorem (Endo-Miyata, 1974, Colliot-Thélène-Sansuc, 1977)

$\exists P$: permutation, $\exists F$: flabby such that

$$0 \to M \to P \to F \to 0\text{: flabby resolution of } M.$$

$[M]^{fl} := [F]$, $[M]^{fl}$ is invertible $\overset{\text{def}}{\iff}$ $[M]^{fl} = [E]$ ($\exists E$: invertible).

Theorem (Endo-Miyata, 1973, Voskresenskii, 1974, Saltman, 1984)

(EM73) $[M]^{fl} = 0 \iff L(M)^G$ is stably rational over k.
(Vos74) $[M]^{fl} = [M']^{fl} \iff L(M)^G(x_1, \ldots, x_m) \simeq L(M')^G(y_1, \ldots, y_n)$.
(Sal84) $[M]^{fl}$ is invertible $\iff L(M)^G$ is retract rational over k.

Our contribution

- We give a procedure to compute a flabby resolution of M, in particular $[M]^{fl} = [F]$, effectively (with smaller rank after base change) by computer software GAP.
- The function `IsFlabby` (resp. `IsCoflabby`) may determine whether M is flabby (resp. coflabby).
- The function `IsInvertibleF` may determine whether $[M]^{fl} = [F]$ is invertible ($\leftrightarrow$ whether $L(M)^G$ (resp. T) is retract rational).
- We provide some functions for checking a possibility of isomorphism

$$\left(\bigoplus_{i=1}^{r} a_i \, \mathbb{Z}[G/H_i]\right) \oplus a_{r+1} F \simeq \bigoplus_{i=1}^{r} b_i' \, \mathbb{Z}[G/H_i] \qquad (*)$$

  by computing some invariants (e.g. trace, $\widehat{Z}^0$, $\widehat{H}^0$) of both sides.
- [HY17, Example 10.7]. $G \simeq S_5 \leq \mathrm{GL}(5,\mathbb{Z})$ with number $(5,946,4)$ $\Longrightarrow \mathrm{rank}(F) = 17$ and $\mathrm{rank}(*) = 88$ holds $\Longrightarrow [F] = 0 \Longrightarrow L(M)^G$ (resp. T) is stably rational over k.

Application

Corollary ($[F]=[M]^{fl}$: invertible case, $G\simeq S_5, F_{20}$)

$\exists T$, T'; 4-dim. not stably rational algebraic tori over k such that $T\not\approx T'$ (birational) and $T\times T'$: 8-dim. stably rational over k.
$\because -[M]^{fl}=[M']^{fl}\neq 0$.

Prop. ([HY17], Krull-Schmidt fails for permutation D_6-lattices)

$\{1\}$, $C_2^{(1)}$, $C_2^{(2)}$, $C_2^{(3)}$, C_3, C_2^2, C_6, $S_3^{(1)}$, $S_3^{(2)}$, D_6: conj. subgroups of D_6.

$$\mathbb{Z}[D_6]\oplus\mathbb{Z}[D_6/C_2^2]^{\oplus 2}\oplus\mathbb{Z}[D_6/C_6]\oplus\mathbb{Z}[D_6/S_3^{(1)}]\oplus\mathbb{Z}[D_6/S_3^{(2)}]$$
$$\simeq\mathbb{Z}[D_6/C_2^{(1)}]\oplus\mathbb{Z}[D_6/C_2^{(2)}]\oplus\mathbb{Z}[D_6/C_2^{(3)}]\oplus\mathbb{Z}[D_6/C_3]\oplus\mathbb{Z}^{\oplus 2}.$$

- D_6 is the smallest example exhibiting the failure of K-S:

Theorem (Dress, 1973)

Krull-Schmidt holds for permutation G-lattices $\iff$ $G/O_p(G)$ is cyclic where $O_p(G)$ is the maximal normal p-subgroup of G.

Krull-Schmidt and Direct sum cancelation

Theorem (Hindman-Klingler-Odenthal, 1998) Assume $G \neq D_8$

Krull-Schmidt holds for G-lattices $\iff$ (i) $G = C_p$ ($p \leq 19$; prime), (ii) $G = C_n$ ($n = 1, 4, 8, 9$), (iii) $G = V_4$ or (iv) $G = D_4$.

Theorem (Endo-Hironaka, 1979)

Direct sum cancellation holds, i.e. $M_1 \oplus N \simeq M_2 \oplus N \Longrightarrow M_1 \simeq M_2$, $\Longrightarrow G$ is abelian, dihedral, A_4, S_4 or A_5 (*).

- via projective class group (see Swan (1988) Corollary 1.3, Section 7).
- Except for (*) $\Longrightarrow$ Direct sum cancelation fails $\Longrightarrow$ K-S fails

Theorem ([HY17]) $G \leq \mathrm{GL}(n, \mathbb{Z})$ (up to conjugacy)

(i) $n \leq 4 \Longrightarrow$ K-S holds.
(ii) $n = 5$. K-S fails $\iff$ 11 groups G (among 6079 groups).
(iii) $n = 6$. K-S fails $\iff$ 131 groups G (among 85308 groups).

Special case: $T = R^{(1)}_{K/k}(\mathbb{G}_m)$; norm one tori (1/5)

- Rationality problem for $T = R^{(1)}_{K/k}(\mathbb{G}_m)$ is investigated by S. Endo, Colliot-Thélène and Sansuc, W. Hürlimann, L. Le Bruyn, A. Cortella and B. Kunyavskii, N. Lemire and M. Lorenz, M. Florence, etc.

Theorem (Endo-Miyata, 1974), (Saltman, 1984)

Let K/k be a finite Galois field extension and $G = \mathrm{Gal}(K/k)$.
(i) T is retract k-rational $\iff$ all the Sylow subgroups of G are cyclic;
(ii) T is stably k-rational $\iff$ G is a cyclic group, or a direct product of a cyclic group of order m and a group $\langle \sigma, \tau \,|\, \sigma^n = \tau^{2^d} = 1, \tau\sigma\tau^{-1} = \sigma^{-1} \rangle$, where $d, m \geq 1, n \geq 3, m, n$: odd, and $(m, n) = 1$.

Theorem (Endo, 2011)

Let K/k be a finite non-Galois, separable field extension and L/k be the Galois closure of K/k. Assume that the Galois group of L/k is nilpotent. Then the norm one torus $T = R^{(1)}_{K/k}(\mathbb{G}_m)$ is not retract k-rational.

- Let K/k be a finite non-Galois, separable field extension
- Let L/k be the Galois closure of K/k.
- Let $G = \mathrm{Gal}(L/k)$ and $H = \mathrm{Gal}(L/K) \leq G$.

Theorem (Endo, 2011)

Assume that all the Sylow subgroups of G are cyclic.
Then T is retract k-rational.
$T = R^{(1)}_{K/k}(\mathbb{G}_m)$ is stably k-rational $\iff G = D_n$, n odd $(n \geq 3)$ or $C_m \times D_n$, m, n odd $(m, n \geq 3)$, $(m, n) = 1$, $H \leq D_n$ with $|H| = 2$.

Special case: $T = R^{(1)}_{K/k}(\mathbb{G}_m)$; norm one tori (3/5)

Theorem (Endo, 2011) $\dim T = n-1$

Assume that $\mathrm{Gal}(L/k) = S_n$, $n \geq 3$, and $\mathrm{Gal}(L/K) = S_{n-1}$ is the stabilizer of one of the letters in S_n.
(i) $R^{(1)}_{K/k}(\mathbb{G}_m)$ is retract k-rational $\iff$ n is a prime;
(ii) $R^{(1)}_{K/k}(\mathbb{G}_m)$ is (stably) k-rational $\iff$ $n = 3$.

Theorem (Endo, 2011) $\dim T = n-1$

Assume that $\mathrm{Gal}(L/k) = A_n$, $n \geq 4$, and $\mathrm{Gal}(L/K) = A_{n-1}$ is the stabilizer of one of the letters in A_n.
(i) $R^{(1)}_{K/k}(\mathbb{G}_m)$ is retract k-rational $\iff$ n is a prime;
(ii) $\exists t \in \mathbb{N}$ s.t. $[R^{(1)}_{K/k}(\mathbb{G}_m)]^{(t)}$ is stably k-rational $\iff$ $n = 5$.

- $[R^{(1)}_{K/k}(\mathbb{G}_m)]^{(t)}$: the product of t copies of $R^{(1)}_{K/k}(\mathbb{G}_m)$.

Special case: $T = R^{(1)}_{K/k}(\mathbb{G}_m)$; norm one tori (4/5)

Theorem ([HY17], Rationality for $R^{(1)}_{K/k}(\mathbb{G}_m)$ (dim. 4, $[K:k]=5$))

Let K/k be a separable field extension of degree 5 and L/k be the Galois closure of K/k. Assume that $G = \mathrm{Gal}(L/k)$ is a transitive subgroup of S_5 and $H = \mathrm{Gal}(L/K)$ is the stabilizer of one of the letters in G. Then the rationality of $R^{(1)}_{K/k}(\mathbb{G}_m)$ is given by

G		$L(M) = L(x_1, x_2, x_3, x_4)^G$
$5T1$	C_5	stably k-rational
$5T2$	D_5	stably k-rational
$5T3$	F_{20}	not stably but retract k-rational
$5T4$	A_5	stably k-rational
$5T5$	S_5	not stably but retract k-rational

- This theorem is already known except for the case of A_5 (Endo).
- Stably k-rationality for the case A_5 is asked by S. Endo (2011).

Special case: $T = R^{(1)}_{K/k}(\mathbb{G}_m)$; norm one tori (5/5)

Corollary of (Endo, 2011) and [HY17]

Assume that $\mathrm{Gal}(L/k) = A_n$, $n \geq 4$, and $\mathrm{Gal}(L/K) = A_{n-1}$ is the stabilizer of one of the letters in A_n. Then
$R^{(1)}_{K/k}(\mathbb{G}_m)$ is stably k-rational $\iff$ $n = 5$.

More recent results on stably/retract k-rational classification for T

- $G \leq S_n$ $(n \leq 10)$ and $G \neq 9T27 \simeq PSL_2(\mathbb{F}_8)$,
  $G \leq S_p$ and $G \neq PSL_2(\mathbb{F}_{2^e})$ $(p = 2^e + 1 \geq 17$; Fermat prime$)$
  (Hoshi-Yamasaki, 2021, Israel J. Math.)
- $G \leq S_n$ $(n = 12, 14, 15)$ $(n = 2^e)$
  (Hoshi-Hasegawa-Yamasaki, 2020, Math. Comp.)

$Ш(T)$ and Hasse norm principle over number fields k

- (Hoshi-Kanai-Yamasaki, 2022, Math. Comp.,
  2023, J. Number Theory, 2024, J. Algebra, and arXiv:2210.09119)

§5. Norm one tori and Hasse norm principle (HNP)

- k : a global field, i.e. a number field or a finite extension of $\mathbb{F}_q(t)$.

Definition (Hasse norm principle)

Let k be a global field. K/k be a finite extension and $\mathbb{A}_K^\times$ be the idele group of K. We say that the Hasse norm principle holds for K/k if

$$\mathrm{Obs}(K/k) := (N_{K/k}(\mathbb{A}_K^\times) \cap k^\times)/N_{K/k}(K^\times) = 1$$

where $N_{K/k}$ is the norm map.

Theorem (Hasse's norm theorem 1931)

If K/k is a cyclic extension of a number field, then

$$\mathrm{Obs}(K/k) = 1.$$

Example (Hasse [Has31]): $\mathrm{Obs}(\mathbb{Q}(\sqrt{-39}, \sqrt{-3})/\mathbb{Q}) \simeq \mathbb{Z}/2\mathbb{Z}$.
$\mathrm{Obs}(\mathbb{Q}(\sqrt{2}, \sqrt{-1})/\mathbb{Q}) = 1$.

In both cases, Galois group $G \simeq V_4$ (Klein four-group).

Tate's theorem (1967)

For any Galois extension K/k, Tate gave:

Theorem (Tate 1967, *in Alg. Num. Th. ed. by Cassels and Fröhlich*)

Let K/k be a finite Galois extension with Galois group $\mathrm{Gal}(K/k) \simeq G$. Let V_k be the set of all places of k and G_v be the decomposition group of G at $v \in V_k$. Then

$$\mathrm{Obs}(K/k) \simeq \mathrm{Coker}\{\bigoplus_{v\in V_k} \widehat{H}^{-3}(G_v,\mathbb{Z}) \xrightarrow{\mathrm{cores}} \widehat{H}^{-3}(G,\mathbb{Z})\}$$

where $\widehat{H}$ is the Tate cohomology. In particular, In particular, the Hasse norm principle holds for K/k if and only if the restriction map $H^3(G,\mathbb{Z}) \xrightarrow{\mathrm{res}} \bigoplus_{v\in V_k} H^3(G_v,\mathbb{Z})$ is injective.

- If $G \simeq C_n$ is cyclic, then $H^3(C_n,\mathbb{Z}) \simeq H^1(C_n,\mathbb{Z}) = 0$ and hence the Hasse's original theorem follows.
- If $G \simeq V_4$, then $\mathrm{Obs}(K/k) = 0 \iff {}^{\exists}v \in V_k$ such that $G_v = V_4$ $(H^3(V_4,\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z})$ (v: should be ramified).

Known results for HNP (1/2)

The HNP for Galois extensions K/k was investigated by
Gerth [Ger77], [Ger78], Gurak [Gur78a], [Gur78b], [Gur80],
Morishita [Mor90], Horie [Hor93], Takeuchi [Tak94], Kagawa [Kag95], etc.

- (Gurak 1978; Endo-Miyata 1975 + Ono 1963)
  If all the Sylow subgroups of $\mathrm{Gal}(K/k)$ is cyclic, then $\mathrm{Obs}(K/k) = 0$.

However, for non-Galois extensions K/k, very little is known whether the Hasse norm principle holds:

- (Bartels 1981) $[K : k] = p$; prime $\Rightarrow$ HNP for K/k holds.
- (Bartels 1981) $[K : k] = n$ and Galois closure $\mathrm{Gal}(L/k) \simeq D_n$
  $\Rightarrow$ HNP for K/k holds.
- (Voskresenskii-Kunyavskii 1984) $[K : k] = n$ and $\mathrm{Gal}(L/k) \simeq S_n$
  $\Rightarrow$ HNP for K/k holds.
- (Macedo 2020) $[K : k] = n$ and $\mathrm{Gal}(L/k) \simeq A_n$
  $\Rightarrow$ HNP for K/k holds if $n \geq 5$; $n = 6$ using Hoshi-Yamasaki [HY17].

Ono's theorem (1963)

- T : algebraic k-torus, i.e. $T \times_k \overline{k} \simeq (\mathbb{G}_{m,\overline{k}})^n$.
- $\text{Ш}(T) := \mathrm{Ker}\{H^1(k,T) \xrightarrow{\mathrm{res}} \bigoplus_{v \in V_k} H^1(k_v,T)\}$: Shafarevich-Tate gp.
- The norm one torus $R^{(1)}_{K/k}(\mathbb{G}_m)$ of K/k:

$$1 \longrightarrow R^{(1)}_{K/k}(\mathbb{G}_m) \longrightarrow R_{K/k}(\mathbb{G}_{m,K}) \xrightarrow{\mathrm{N}_{K/k}} \mathbb{G}_{m,k} \longrightarrow 1$$

  where $R_{K/k}$ is the Weil restriction.
- $R^{(1)}_{K/k}(\mathbb{G}_m)$ is biregularly isomorphic to the norm hyper surface $f(x_1,\ldots,x_n)=1$ where $f \in k[x_1,\ldots,x_n]$ is the norm form of K/k.

Theorem (Ono 1963, Ann. of Math.)

Let K/k be a finite extension and $T = R^{(1)}_{K/k}(\mathbb{G}_m)$. Then

$$\text{Ш}(T) \simeq \mathrm{Obs}(K/k).$$

Known results for HNP (2/2)

- $T = R^{(1)}_{K/k}(\mathbb{G}_m)$.
- $Ш(T) \simeq \mathrm{Obs}(K/k)$.

Theorem (Kunyavskii 1984)

Let $[K:k]=4$, $G=\mathrm{Gal}(L/k)\simeq 4Tm$ $(1\le m\le 5)$.
Then $Ш(T)=0$ except for $4T2$ and $4T4$. For $4T2\simeq V_4$, $4T4\simeq A_4$,
(i) $Ш(T)\le \mathbb{Z}/2\mathbb{Z}$;
(ii) $Ш(T)=0 \Leftrightarrow {}^{\exists}v\in V_k$ *such that* $V_4\le G_v$.

Theorem (Drakokhrust-Platonov 1987)

Let $[K:k]=6$, $G=\mathrm{Gal}(L/k)\simeq 6Tm$ $(1\le m\le 16)$.
Then $Ш(T)=0$ except for $6T4$ and $6T12$. For $6T4\simeq A_4$, $6T12\simeq A_5$,
(i) $Ш(T)\le \mathbb{Z}/2\mathbb{Z}$;
(ii) $Ш(T)=0 \Leftrightarrow {}^{\exists}v\in V_k$ such that $V_4\le G_v$.

Voskresenskii's theorem (1969) (1/2)

Theorem (Voskresenskii 1969)

Let k be a global field, T be an algebraic k-torus and X be a smooth k-compactification of T. Then there exists an exact sequence

$$0 \to A(T) \to H^1(k, \operatorname{Pic} \overline{X})^\vee \to Ш(T) \to 0$$

where $M^\vee = \operatorname{Hom}(M, \mathbb{Q}/\mathbb{Z})$ is the Pontryagin dual of M.

- The group $A(T) := \left(\prod_{v \in V_k} T(k_v)\right) \Big/ \overline{T(k)}$ is called the kernel of the weak approximation of T.
- T : retract rational $\Longleftrightarrow$ $[\widehat{T}]^{fl} = [\operatorname{Pic} \overline{X}]$ is invertible
  $\Longrightarrow$ $\operatorname{Pic} \overline{X}$ is flabby and coflabby
  $\Longrightarrow$ $H^1(k, \operatorname{Pic} \overline{X})^\vee = 0 \quad \Longrightarrow$ $A(T) = Ш(T) = 0$.
- when $T = R^{(1)}_{K/k}(\mathbb{G}_m)$, by Ono's theorem $Ш(T) \simeq \operatorname{Obs}(K/k)$,
  T : retract k-rational $\Longrightarrow$ $\operatorname{Obs}(K/k) = 0$ (HNP for K/k holds).

Voskresenskii's theorem (1969) (2/2)

- when $T = R^{(1)}_{K/k}(\mathbb{G}_m)$, by Ono's theorem $Ш(T) \simeq \mathrm{Obs}(K/k)$, T : retract k-rational $\Longrightarrow$ $\mathrm{Obs}(K/k) = 0$ (HNP for K/k holds).
- when $T = R^{(1)}_{K/k}(\mathbb{G}_m)$, $\widehat{T} = J_{G/H}$ where $J_{G/H} = (I_{G/H})^\circ = \mathrm{Hom}(I_{G/H}, \mathbb{Z})$ is the dual lattice of $I_{G/H} = \mathrm{Ker}(\varepsilon)$ and $\varepsilon : \mathbb{Z}[G/H] \to \mathbb{Z}$ is the augmentation map.
- (Hoshi-Yamasaki, 2018, Hasegawa-Hoshi-Yamasaki, 2020) For $[K : k] = n \leq 17$ except $9T27 \simeq \mathrm{PSL}_2(\mathbb{F}_8)$, the classificasion of stably/retract rational $R^{(1)}_{K/k}(\mathbb{G}_m)$ was given.
- $H^1(k, \mathrm{Pic}\,\overline{X}) \simeq \mathrm{Br}(X)/\mathrm{Br}(k) \simeq \mathrm{Br}_{\mathrm{nr}}(k(X)/k)/\mathrm{Br}(k)$ where $\mathrm{Br}(X)$ is the étale cohomological/Azumaya Brauer group of X by Colliot-Thélène-Sansuc 1987.

- $\exists\ 2, 13, 73, 710, 6079$ cases of alg. k-tori T of $\dim(T) = 1, 2, 3, 4, 5$.
- X: a smooth k-compactification of T, $\overline{X} = X \times_k \overline{k}$.

Theorem 1 ([HKY22, Theorem 1.5 and Theorem 1.6])

(i) $\dim(T) = 4$. Among the 216 cases (of 710) of not retract k-rational T,

$$H^1(k, \mathrm{Pic}\,\overline{X}) \simeq \begin{cases} 0 & (194 \text{ of } 216), \\ \mathbb{Z}/2\mathbb{Z} & (20 \text{ of } 216), \\ (\mathbb{Z}/2\mathbb{Z})^{\oplus 2} & (2 \text{ of } 216). \end{cases}$$

(ii) $\dim(T) = 5$. Among 3003 cases (of 6079) of not retract k-rational T,

$$H^1(k, \mathrm{Pic}\,\overline{X}) \simeq \begin{cases} 0 & (2729 \text{ of } 3003), \\ \mathbb{Z}/2\mathbb{Z} & (263 \text{ of } 3003), \\ (\mathbb{Z}/2\mathbb{Z})^{\oplus 2} & (11 \text{ of } 3003). \end{cases}$$

- Kunyavskii (1984) showed that among the 15 cases (of 73) of not retract k-rational T of $\dim(T) = 3$, $H^1(k, \mathrm{Pic}\,\overline{X}) = 0$ $(13 \text{ of } 15)$, $H^1(k, \mathrm{Pic}\,\overline{X}) \simeq \mathbb{Z}/2\mathbb{Z}$ $(2 \text{ of } 15)$.

Main theorems 1,2,3,4,5 (2/3)

- k : a field, K/k : a separable field extension of $[K:k]=n$.
- $T=R^{(1)}_{K/k}(\mathbb{G}_m)$ with $\dim(T)=n-1$.
- X : a smooth k-compactification of T.
- L/k : Galois closure of K/k, $G:=\mathrm{Gal}(L/k)$ and $H=\mathrm{Gal}(L/K)$ with $[G:H]=n \Longrightarrow G=nTm\leq S_n$: transitive.

Theorem 2 ([HKY22, Theorem 1.5], [HKY23, Theorem 1.1])

Let $2\leq n\leq 15$ be an integer. Then $H^1(k,\mathrm{Pic}\,\overline{X})\neq 0 \Longleftrightarrow G=nTm$ is given as in [HKY22, Table 1] $(n\neq 12)$ or [HKY23,Table 1] $(n=12)$.

- The number of transitive subgroups nTm of S_n $(2\leq n\leq 16)$ up to conjugacy (with $H^1(k,\mathrm{Pic}\,\overline{X})\neq 0$) is given as follows:

n	2	3	4	5	6	7	8	9	10
# of nTm	1	2	5	5	16	7	50	34	45
(with $H^1(k,\mathrm{Pic}\,\overline{X})\neq 0$)	0	0	2	0	2	0	15	7	3

n	11	12	13	14	15	16
# of nTm	8	301	9	63	104	1954
(with $H^1(k,\mathrm{Pic}\,\overline{X})\neq 0$)	0	64	0	1	2	853

[HKY22, Table 1]: $H^1(k, \mathrm{Pic}\,\overline{X}) \simeq H^1(G, [J_{G/H}]^{fl}) \neq 0$ where $G = nTm$ with $2 \leq n \leq 15$ and $n \neq 12$

G	$H^1(k, \mathrm{Pic}\,\overline{X}) \simeq H^1(G, [J_{G/H}]^{fl})$
$4T2 \simeq V_4$	$\mathbb{Z}/2\mathbb{Z}$
$4T4 \simeq A_4$	$\mathbb{Z}/2\mathbb{Z}$
$6T4 \simeq A_4$	$\mathbb{Z}/2\mathbb{Z}$
$6T12 \simeq A_5$	$\mathbb{Z}/2\mathbb{Z}$
$8T2 \simeq C_4 \times C_2$	$\mathbb{Z}/2\mathbb{Z}$
$8T3 \simeq (C_2)^3$	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 3}$
$8T4 \simeq D_4$	$\mathbb{Z}/2\mathbb{Z}$
$8T9 \simeq D_4 \times C_2$	$\mathbb{Z}/2\mathbb{Z}$
$8T11 \simeq (C_4 \times C_2) \rtimes C_2$	$\mathbb{Z}/2\mathbb{Z}$
$8T13 \simeq A_4 \times C_2$	$\mathbb{Z}/2\mathbb{Z}$
$8T14 \simeq S_4$	$\mathbb{Z}/2\mathbb{Z}$
$8T15 \simeq C_8 \rtimes V_4$	$\mathbb{Z}/2\mathbb{Z}$
$8T19 \simeq (C_2)^3 \rtimes C_4$	$\mathbb{Z}/2\mathbb{Z}$
$8T21 \simeq (C_2)^3 \rtimes C_4$	$\mathbb{Z}/2\mathbb{Z}$
$8T22 \simeq (C_2)^3 \rtimes V_4$	$\mathbb{Z}/2\mathbb{Z}$
$8T31 \simeq ((C_2)^4 \rtimes C_2) \rtimes C_2$	$\mathbb{Z}/2\mathbb{Z}$
$8T32 \simeq ((C_2)^3 \rtimes V_4) \rtimes C_3$	$\mathbb{Z}/2\mathbb{Z}$
$8T37 \simeq \mathrm{PSL}_3(\mathbb{F}_2) \simeq \mathrm{PSL}_2(\mathbb{F}_7)$	$\mathbb{Z}/2\mathbb{Z}$
$8T38 \simeq (((C_2)^4 \rtimes C_2) \rtimes C_2) \rtimes C_3$	$\mathbb{Z}/2\mathbb{Z}$

[HKY22, Table 1]: $H^1(k, \mathrm{Pic}\,\overline{X}) \simeq H^1(G, [J_{G/H}]^{fl}) \neq 0$ where $G = nTm$ with $2 \leq n \leq 15$ and $n \neq 12$

G	$H^1(k, \mathrm{Pic}\,\overline{X}) \simeq H^1(G, [J_{G/H}]^{fl})$
$9T2 \simeq (C_3)^2$	$\mathbb{Z}/3\mathbb{Z}$
$9T5 \simeq (C_3)^2 \rtimes C_2$	$\mathbb{Z}/3\mathbb{Z}$
$9T7 \simeq (C_3)^2 \rtimes C_3$	$\mathbb{Z}/3\mathbb{Z}$
$9T9 \simeq (C_3)^2 \rtimes C_4$	$\mathbb{Z}/3\mathbb{Z}$
$9T11 \simeq (C_3)^2 \rtimes C_6$	$\mathbb{Z}/3\mathbb{Z}$
$9T14 \simeq (C_3)^2 \rtimes Q_8$	$\mathbb{Z}/3\mathbb{Z}$
$9T23 \simeq ((C_3)^2 \rtimes Q_8) \rtimes C_3$	$\mathbb{Z}/3\mathbb{Z}$
$10T7 \simeq A_5$	$\mathbb{Z}/2\mathbb{Z}$
$10T26 \simeq \mathrm{PSL}_2(\mathbb{F}_9) \simeq A_6$	$\mathbb{Z}/2\mathbb{Z}$
$10T32 \simeq S_6$	$\mathbb{Z}/2\mathbb{Z}$
$14T30 \simeq \mathrm{PSL}_2(\mathbb{F}_{13})$	$\mathbb{Z}/2\mathbb{Z}$
$15T9 \simeq (C_5)^2 \rtimes C_3$	$\mathbb{Z}/5\mathbb{Z}$
$15T14 \simeq (C_5)^2 \rtimes S_3$	$\mathbb{Z}/5\mathbb{Z}$

Theorem 3 ([HKY25, Theorem 1.1]) $[K:k]=16$

Assume that $G=\mathrm{Gal}(L/k)=16Tm$ $(1\leq m\leq 1954)$ is a transitive subgroup of S_{16} and $H=\mathrm{Gal}(L/K)$ with $[G:H]=16$. Then

$$H^1(k,\mathrm{Pic}\,\overline{X})=\begin{cases}\mathbb{Z}/2\mathbb{Z} & \text{if } m \text{ is given as in [HKY25, Table 1-1] (774 cases)},\\ (\mathbb{Z}/2\mathbb{Z})^{\oplus 2} & \text{if } m=7,10,11,46,58,61,73,76,82,87,89,107,113,118,\\ & 120,128,129,138,142,162,164,165,178,183,206,297,308,\\ & 319,414,731,1080 \text{ (31 cases)},\\ (\mathbb{Z}/2\mathbb{Z})^{\oplus 3} & \text{if } m=2,9,18,20,23,25,67,69,83,92,98,101,127,173,\\ & 197,202,212,241,246,270,295,301,313,358,372,440,463,\\ & 466,604,632,649,656,794,801,1082,1187,1378 \text{ (37 cases)},\\ (\mathbb{Z}/2\mathbb{Z})^{\oplus 4} & \text{if } m=64 \text{ (1 case)},\\ (\mathbb{Z}/2\mathbb{Z})^{\oplus 6} & \text{if } m=3 \text{ (1 case)},\\ \mathbb{Z}/4\mathbb{Z} & \text{if } m=4,51,63,143,185,323,375,430,769 \text{ (9 cases)},\\ 0 & \text{otherwise (1101 cases)}.\end{cases}$$

- $16T64\simeq (C_2)^4\rtimes C_3$ with $H^1(k,\mathrm{Pic}\,\overline{X})\simeq(\mathbb{Z}/2\mathbb{Z})^4$.
- $16T3\simeq (C_2)^4$ with $H^1(k,\mathrm{Pic}\,\overline{X})\simeq(\mathbb{Z}/2\mathbb{Z})^6$.
- $16T4\simeq (C_4)^2$, $16T51\simeq (C_4)^2\rtimes C_2$, $16T63\simeq (C_4)^2\rtimes C_3$, $16T143\simeq (C_4)^2\rtimes C_4$, $16T185\simeq (C_4)^2\rtimes C_6$, $16T430\simeq (C_4)^2\rtimes Q_{12}$ with $H^1(k,\mathrm{Pic}\,\overline{X})\simeq\mathbb{Z}/4\mathbb{Z}$.

Main theorems 1,2,3,4,5 (3/3)

- k : a number field, K/k : a separable field extension of $[K:k]=n$.
- $T = R^{(1)}_{K/k}(\mathbb{G}_m)$, X : a smooth k-compactification of T.

Theorem 4 ([HKY22, Th 1.18], [HKY23, Th 1.3], [HKY25, Th 1.4])

Let $2 \leq n \leq 16$ be an integer. For $G = nTm$ with $H^1(k, \mathrm{Pic}\,\overline{X}) \neq 0$, assume G is primitive ($\exists$ 22 cases), i.e. $H \leq G$: maximal, when $n = 16$,

$$Ш(T) = 0 \iff G = nTm \text{ satisfies } \boxed{\text{some conditions}} \text{ of } G_v$$

where G_v is the decomposition group of G at v.

- By Ono's theorem $Ш(T) \simeq \mathrm{Obs}(K/k)$, Theorem 4 gives a necessary and sufficient condition for HNP holds for K/k.

Theorem 5 ([HKY22, Theorem 1.17])

Assume that $G = M_n \leq S_n$ $(n = 11, 12, 22, 23, 24)$ is the Mathieu group of degree n. Then $H^1(k, \mathrm{Pic}\,\overline{X}) = 0$. In particular, $Ш(T) = 0$.

Examples of Theorem 4

Example ($G = 8T4 \simeq D_4$, $8T13 \simeq A_4 \times C_2$, $8T14 \simeq S_4$, $8T37 \simeq \mathrm{PSL}_2(\mathbb{F}_7)$, $10T7 \simeq A_5$, $14T30 \simeq \mathrm{PSL}_2(\mathbb{F}_{13})$)

$Ш(T) = 0 \iff {}^{\exists}v \in V_k$ such that $V_4 \leq G_v$.

Example ($G = 10T26 \simeq \mathrm{PSL}_2(\mathbb{F}_9)$)

$Ш(T) = 0 \iff {}^{\exists}v \in V_k$ such that $D_4 \leq G_v$.

Example ($G = 10T32 \simeq S_6 \leq S_{10}$)

$Ш(T) = 0 \iff {}^{\exists}v \in V_k$ such that
(i) $V_4 \leq G_v$ where $N_{\widetilde{G}}(V_4) \simeq C_8 \rtimes (C_2 \times C_2)$ for the normalizer $N_{\widetilde{G}}(V_4)$ of V_4 in $\widetilde{G}$ with the normalizer $\widetilde{G} = N_{S_{10}}(G) \simeq \mathrm{Aut}(G)$ of G in S_{10} or
(ii) $D_4 \leq G_v$ where $D_4 \leq [G, G] \simeq A_6$.

- $45/165$ subgroups $V_4 \leq G$ satisfy (i).
- $45/180$ subgroups $D_4 \leq G$ satisfy (ii).