Degree three unramifed cohomology groups and Noether's problem for groups of order 243

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Hoshi-Kang-Yamasaki, Degree three unramified cohomology groups and Noether's problem for groups of order 243, arXiv:1710.01958, 61 pages.

$\mathrm{Br}_{\mathrm{nr}}(X/\mathbb{C}) \simeq H^3(X,\mathbb{Z})_{\mathrm{tors}}$; Artin-Mumford invariant

$H^3_{\mathrm{nr}}(X,\mathbb{Q}/\mathbb{Z}) \simeq \mathrm{Hdg}^4(X,\mathbb{Z})/\mathrm{Hdg}^4(X,\mathbb{Z})_{\mathrm{alg}} \leftrightarrow$ integral Hodge conjecture

cf. Colliot-Thélène and Voisin, Duke Math. J. **161** (2012) 735–801.

Theorem (Colliot-Thélène and Voisin, 2012, Duke Math. J.)

For any smooth projective complex variety X, there is an exact sequence

$$0 \to H^3_{\mathrm{nr}}(X, \mathbb{Z}) \otimes \mathbb{Q}/\mathbb{Z} \to H^3_{\mathrm{nr}}(X, \mathbb{Q}/\mathbb{Z}) \to \mathrm{Tors}(Z^4(X)) \to 0$$

where

$$Z^4(X) = \mathrm{Hdg}^4(X, \mathbb{Z})/\mathrm{Hdg}^4(X, \mathbb{Z})_{\mathrm{alg}}$$

and the lower index "alg" means that we consider the group of integral Hodge classes which are algebraic. In particular, if X is rationally connected, then we have

$$H^3_{\mathrm{nr}}(X, \mathbb{Q}/\mathbb{Z}) \simeq \mathrm{Hdg}^4(X, \mathbb{Z})/\mathrm{Hdg}^4(X, \mathbb{Z})_{\mathrm{alg}}.$$

- We show $H^3_{\mathrm{nr}}(X, \mathbb{Q}/\mathbb{Z}) \simeq \mathbb{Z}/3\mathbb{Z}$

  for some $X = \mathbb{P}^n/G$ with $|G| = 3^5 = 243$.

§1. Noether's problem/$\mathbb{C}$ and unram. Brauer group (1/4)

- k; field, G; finite group
- $G \curvearrowright k$; trivial, $G \curvearrowright k(x_g \mid g \in G)$; permutation.
- $k(G) := k(x_g \mid g \in G)^G$; invariant field

Noether's problem (Emmy Noether, 1913)

Is $k(G)$ rational over k?, i.e. $k(G) \simeq k(t_1, \ldots, t_n)$?

- Is the quotient variety $\mathbb{P}^n/G$ rational over k?

- Assume $G = A$; abelian group.
- (Fisher, 1915) $\mathbb{C}(A)$ is rational over $\mathbb{C}$.

Noether's problem/$\mathbb{C}$ and unram. Brauer group (2/4)

Let G be a p-group. $\mathbb{C}(G) := \mathbb{C}(x_g \mid g \in G)^G$.

- (Saltman, 1984, Invent. Math.)
  For $\forall p$; prime, $\exists$ meta-abelian p-group G of order p^9
  such that $\mathbb{C}(G)$ is not retract rational over $\mathbb{C}$.
- (Bogomolov, 1988)
  For $\forall p$; prime, $\exists$ p-group G of order p^6
  such that $\mathbb{C}(G)$ is not retract rational over $\mathbb{C}$.

Indeed they showed $\mathrm{Br}_{\mathrm{nr}}(\mathbb{C}(G)/\mathbb{C}) \neq 0$; unramified Brauer group

- rational $\Longrightarrow$ stably rational $\Longrightarrow$ retract rational $\Longrightarrow$ $\mathrm{Br}_{\mathrm{nr}}(\mathbb{C}(G)) = 0$.

not rational $\Leftarrow$ not stably rational $\Leftarrow$ not retract rational $\Leftarrow$ $\mathrm{Br}_{\mathrm{nr}}(\mathbb{C}(G)) \neq 0$.

Unramified Brauer group (1/2)

Definition (Unramified Brauer group) Saltman (1984)

Let $k \subset K$ be an extension of fields.

$$\mathrm{Br}_{\mathrm{nr}}(K/k) := \bigcap_{\substack{k \subset R \subset K:\mathrm{DVR} \\ \mathrm{and}\ Q(R)=K}} \mathrm{Image}\{\mathrm{Br}(R) \to \mathrm{Br}(K)\}.$$

- If K is retract rational over k, then $\mathrm{Br}(k) \xrightarrow{\sim} \mathrm{Br}_{\mathrm{nr}}(K/k)$. In particular, if K is retract rational over $\mathbb{C}$, then $\mathrm{Br}_{\mathrm{nr}}(K/\mathbb{C}) = 0$.
- For a smooth projective variety X over $\mathbb{C}$ with function field K, $\mathrm{Br}_{\mathrm{nr}}(K/\mathbb{C}) \simeq H^3(X, \mathbb{Z})_{\mathrm{tors}}$ which is given by Artin-Mumford (1972).

Unramified Brauer group (2/2)

- $K = \mathbb{C}(G)$.

Theorem (Bogomolov 1988, Saltman 1990) $\mathrm{Br}_{\mathrm{nr}}(\mathbb{C}(G)/\mathbb{C}) \simeq B_0(G)$

Let G be a finite group. Then $\mathrm{Br}_{\mathrm{nr}}(\mathbb{C}(G)/\mathbb{C})$ is isomorphic to

$$B_0(G) = \bigcap_{A \leq G:\text{bicyclic}} \mathrm{Ker}\{\mathrm{res} : H^2(G, \mathbb{Q}/\mathbb{Z}) \to H^2(A, \mathbb{Q}/\mathbb{Z})\}.$$

- $\mathbb{C}(G)$: "retract rational" $\Longrightarrow B_0(G) = 0$.
  $B_0(G) \neq 0 \Longrightarrow \mathbb{C}(G)$: not (retract) rational over k.
- $B_0(G) \leq H^2(G, \mathbb{Q}/\mathbb{Z}) \simeq H_2(G, \mathbb{Z})$; Schur multiplier.
- $B_0(G)$ is called Bogomolov multiplier.

Noether's problem/$\mathbb{C}$ and unram. Brauer group (3/4)

- (Chu-Kang, 2001) G is p-group ($|G| \leq p^4$) $\Longrightarrow \mathbb{C}(G)$ is rational.

Theorem (Moravec, 2012, Amer. J. Math.)

Assume $|G| = 3^5 = 243$. $B_0(G) \neq 0 \iff G = G(243, i)$, $28 \leq i \leq 30$.
In particular, $\exists 3$ groups G such that $\mathbb{C}(G)$ is not retract rational over $\mathbb{C}$.

- $\exists G$: 67 groups such that $|G| = 243$.

Theorem (H-Kang-Kunyavskii, 2013, Asian J. Math.)

Assume $|G| = p^5$ where p is odd prime.
$B_0(G) \neq 0 \iff G$ belongs to the isoclinism family Φ_{10}.
In particular, $\exists$ gcd$(4, p-1)+$ gcd$(3, p-1) + 1$ (resp. $\exists 3$) groups G of order p^5 $(p \geq 5)$ (resp. $p = 3$) s.t. $\mathbb{C}(G)$ is not retract rational over $\mathbb{C}$.

- $\exists 2p + 61+$ gcd$(4, p-1) + 2$ gcd$(3, p-1)$ groups such that $|G| = p^5 (p \geq 5)$. $(\exists \Phi_1, \ldots, \Phi_{10})$

From the proof (1/3)

Definition (isoclinic)

p-groups G_1 and G_2 are isoclinic $\overset{\text{def}}{\iff}$
isom. $\theta : G_1/Z(G_1) \overset{\sim}{\to} G_2/Z(G_2)$, $\phi : [G_1, G_1] \overset{\sim}{\to} [G_2, G_2]$ such that

$$\begin{tikzcd} G_1/Z(G_1) \times G_1/Z(G_1) \arrow[r, "(\theta,\theta)", "\simeq"'] \arrow[d, "{[\ ,\ ]}"'] \arrow[dr, phantom, "\circlearrowleft"] & G_2/Z(G_2) \times G_2/Z(G_2) \arrow[d, "{[\ ,\ ]}"] \\ {[G_1, G_1]} \arrow[r, "\phi", "\simeq"'] & {[G_2, G_2]} \end{tikzcd}$$

Invariants

- lower central series
- $\#$ of conj. classes with precisely p^i members
- $\#$ of irr. complex rep. of G of degree p^i

From the proof (2/3)

- $|G| = p^4 (p > 2)$. $\exists 15$ groups (Φ_1, Φ_2, Φ_3)
- $|G| = 2^4 = 16$. $\exists 14$ groups (Φ_1, Φ_2, Φ_3)
- $|G| = p^5 (p > 3)$. $\exists 2p + 61 + (4, p-1) + 2 \times (3, p-1)$ groups $(\Phi_1, \ldots, \Phi_{10})$

	Φ_1	Φ_2	Φ_3	Φ_4	Φ_5	Φ_6	Φ_7	Φ_8
$\#$	7	15	13	$p+8$	2	$p+7$	5	1
$(p=3)$						7		

	Φ_9	Φ_{10}
$\#$	$2 + (3, p-1)$	$1 + (4, p-1) + (3, p-1)$
$(p=3)$		3

From the proof (3/3)

[H-Kang-Kunyavskii, Question 1.11] (2013)

Let G_1 and G_2 be isoclinic p-groups.
Is it true that the fields $k(G_1)$ and $k(G_2)$ are stably isomorphic,
i.e. $k(G_1)(\exists s_1, \ldots, \exists s_m) \simeq k(G_2)(\exists t_1, \ldots, \exists t_n)$,
or, at least, that $B_0(G_1) \simeq B_0(G_2)$?

Theorem (Moravec, 2013)

G_1 and G_2 are isoclinic $\Longrightarrow B_0(G_1) \simeq B_0(G_2)$.

Theorem (Bogomolov-Böhning, 2013)

G_1 and G_2 are isoclinic $\Longrightarrow \mathbb{C}(G_1)$ and $\mathbb{C}(G_2)$ are stably isomorphic.

Noether's problem/$\mathbb{C}$ and unram. Brauer group (4/4)

Theorem (H-Kang-Kunyavskii, 2013, Asian J. Math.)

Assume $|G| = p^5$ where p is odd prime.
$B_0(G) \neq 0 \iff G$ belongs to the isoclinism family Φ_{10}.

Theorem (Chu-H-Hu-Kang, 2015, J. Algebra) $|G| = 3^5 = 243$

If $B_0(G) = 0$, then $\mathbb{C}(G)$ is rational over $\mathbb{C}$ except for Φ_7.

- Rationality of Φ_7 was unknown.
- Φ_5 and Φ_7 are very similar: $C = 1$ (Φ_5), $C = \omega$ (Φ_7).

$\mathbb{C}(G)$ is stably isomorphic to $\mathbb{C}(z_1, z_2, z_3, z_4, z_5, z_6, z_7, z_8, z_9)^{\langle f_1, f_2 \rangle}$

$$\begin{aligned}
f_1 : z_1 &\mapsto z_2, z_2 \mapsto \frac{1}{z_1 z_2}, z_3 \mapsto z_4, z_4 \mapsto \frac{1}{z_3 z_4},\\
z_5 &\mapsto \frac{z_5}{z_1^2 z_3}, z_6 \mapsto \frac{z_1 z_6}{z_3}, z_7 \mapsto z_8, z_8 \mapsto \frac{1}{z_7 z_8}, z_9 \mapsto \frac{z_4 z_9}{z_1},\\
f_2 : z_1 &\mapsto z_3, z_2 \mapsto z_4, z_3 \mapsto \frac{1}{z_1 z_3}, z_4 \mapsto \frac{1}{z_2 z_4},\\
z_5 &\mapsto z_6, z_6 \mapsto \frac{1}{z_5 z_6}, z_7 \mapsto C \frac{z_4 z_7}{z_3}, z_8 \mapsto C \frac{z_8}{z_3 z_4^2}, z_9 \mapsto \frac{z_4 z_9}{z_1}.
\end{aligned}$$

- From $\mathrm{Br}_{\mathrm{nr}}(K/\mathbb{C})$ to $H^i_{\mathrm{nr}}(K/\mathbb{C}, \mathbb{Q}/\mathbb{Z})$.

Definition (Colliot-Thélène and Ojanguren, 1989, Invent. Math.)

Let $K/\mathbb{C}$ be a function field, that is finitely generated as a field over $\mathbb{C}$. The unramified cohomology group $H^i_{\mathrm{nr}}(K/\mathbb{C}, \mu_n^{\otimes j})$ of K over $\mathbb{C}$ of degree $i \geq 1$ is defined to be

$$H^i_{\mathrm{nr}}(K/\mathbb{C}, \mu_n^{\otimes j}) = \bigcap_{\substack{\mathbb{C}\subset R\subset K:\,\mathrm{DVR\ of\ rank\ one}\\ \mathrm{and}\ \ Q(R)=K}} \mathrm{Ker}\{H^i(K, \mu_n^{\otimes j}) \xrightarrow{r} H^{i-1}(\Bbbk_R, \mu_n^{\otimes (j-1)})\}.$$

- If K is the function field of a complete smooth variety over k, then

$$H^i_{\mathrm{nr}}(K/\mathbb{C}, \mu_n^{\otimes j}) = \bigcap_{\substack{\mathbb{C}\subset R\subset K:\,\mathrm{DVR\ of\ rank\ one}\\ \mathrm{and}\ \ Q(R)=K}} \mathrm{Image}\{H^i_{\text{ét}}(R, \mu_n^{\otimes j}) \to H^i_{\text{ét}}(K, \mu_n^{\otimes j})\}.$$

- Note that ${}_n\mathrm{Br}_{\mathrm{nr}}(K/\mathbb{C}) \simeq H^2_{\mathrm{nr}}(K/\mathbb{C}, \mu_n)$.

Noether's problem/$\mathbb{C}$ and unram. cohomology (2/7)

Theorem (Colliot-Thélène and Ojanguren, 1989)

If K and L are stably $\mathbb{C}$-isomorphic, then
$H^i_{\rm nr}(K/\mathbb{C}, \mu_n^{\otimes j}) \xrightarrow{\sim} H^i_{\rm nr}(L/\mathbb{C}, \mu_n^{\otimes j})$.
In particular, K is stably $\mathbb{C}$-rational, then $H^i_{\rm nr}(K/\mathbb{C}, \mu_n^{\otimes j}) = 0$.

- Moreover, if K is retract $\mathbb{C}$-rational, then $H^i_{\rm nr}(K/\mathbb{C}, \mu_n^{\otimes j}) = 0$.
- CTO (1989) $\exists K$ $({\rm trdeg}_{\mathbb{C}} K = 6)$ s.t. $H^3_{\rm nr}(K/\mathbb{C}, \mu_2^{\otimes 3}) \neq 0$.
- Peyre (1993) gave a sufficient condition for $H^i_{\rm nr}(K/\mathbb{C}, \mu_p^{\otimes i}) \neq 0$:
- $\exists K$ s.t. $H^3_{\rm nr}(K/\mathbb{C}, \mu_p^{\otimes 3}) \neq 0$ and ${\rm Br}_{\rm nr}(K/\mathbb{C}) = 0$;
- $\exists K$ s.t. $H^4_{\rm nr}(K/\mathbb{C}, \mu_2^{\otimes 4}) \neq 0$ and ${\rm Br}_{\rm nr}(K/\mathbb{C}) = 0$.

Noether's problem/$\mathbb{C}$ and unram. cohomology (3/7)

- Asok (2013) generalized Peyre's argument (1993):

Theorem (Asok, 2013, Compos. Math.)

(1) For any $n > 0$, $\exists$ a smooth projective complex variety X that is $\mathbb{C}$-unirational, for which $H^i_{\rm nr}(\mathbb{C}(X), \mu_2^{\otimes i}) = 0$ for each $i < n$, yet $H^n_{\rm nr}(\mathbb{C}(X), \mu_2^{\otimes n}) \neq 0$, and so
X is not $\mathbb{A}^1$-connected, nor (retract, stably) $\mathbb{C}$-rational;
(2) For any prime l and any $n \geq 2$, $\exists$ a smooth projective rationally connected complex variety Y such that $H^n_{\rm nr}(\mathbb{C}(Y), \mu_l^{\otimes n}) \neq 0$.
In particular, Y is not $\mathbb{A}^1$-connected, nor (retract, stably) $\mathbb{C}$-rational.

- Namely, $H^i_{\rm nr}(\mathbb{C}(X), \mu_n^{\otimes j}) = 0$ is just a necessary condition for $\mathbb{C}$-rationality.
- It is interesting to consider an analog of above for quotient varieties V/G, e.g. $\mathbb{C}(\mathbb{P}^n/G) = \mathbb{C}(G)$ (Noether's problem).

Noether's problem/$\mathbb{C}$ and unram. cohomology (4/7)

Take the direct limit with respect to n:

$$H^i(K/\mathbb{C}, \mathbb{Q}/\mathbb{Z}(j)) = \varinjlim_n H^i(K/\mathbb{C}, \mu_n^{\otimes j})$$

and we also define the unramified cohomology group

$$H^i_{\mathrm{nr}}(K/\mathbb{C}, \mathbb{Q}/\mathbb{Z}(j)) = \bigcap_R \mathrm{Ker}\{H^i(K, \mathbb{Q}/\mathbb{Z}(j)) \xrightarrow{r} H^{i-1}(\Bbbk_R, \mathbb{Q}/\mathbb{Z}(j-1))\}.$$

Then we have $\mathrm{Br}_{\mathrm{nr}}(K/\mathbb{C}) \simeq H^2_{\mathrm{nr}}(K/\mathbb{C}, \mathbb{Q}/\mathbb{Z}(1))$.

- The case $K = \mathbb{C}(G)$:

Theorem (Peyre, 2008, Invent. Math.) p : any odd prime

$\exists$ p-group G of order p^{12} such that $B_0(G) = 0$ and $H^3_{\mathrm{nr}}(\mathbb{C}(G), \mathbb{Q}/\mathbb{Z}) \neq 0$.
In particular, $\mathbb{C}(G)$ is not (retract, stably) $\mathbb{C}$-rational.

Noether's problem/$\mathbb{C}$ and unram. cohomology (5/7)

Theorem (Peyre, 2008, Invent. Math.) p : any odd prime

$\exists$ p-group G of order p^{12} such that $B_0(G) = 0$ and $H^3_{\mathrm{nr}}(\mathbb{C}(G), \mathbb{Q}/\mathbb{Z}) \neq 0$.
In particular, $\mathbb{C}(G)$ is not (retract, stably) $\mathbb{C}$-rational.

Using Peyre's method, we improved this result:

Theorem (H-Kang-Yamasaki, 2016, J. Algebra) p : any odd prime

$\exists$ p-group G of order p^9 such that $B_0(G) = 0$ and $H^3_{\mathrm{nr}}(\mathbb{C}(G), \mathbb{Q}/\mathbb{Z}) \neq 0$.
In particular, $\mathbb{C}(G)$ is not (retract, stably) $\mathbb{C}$-rational.

Noether's problem/$\mathbb{C}$ and unram. cohomology (6/7)

- Using Saltman (1995) and Peyre's method (2008, Invent. Math.), we get our main results:

Theorem (H-Kang-Yamasaki, arXiv:1710.01958) $|G| = 3^5$

$H^3_{\rm nr}(\mathbb{C}(G), \mathbb{Q}/\mathbb{Z}) \neq 0 \iff G$ belongs to the isoclinism family Φ_7.
Moreover, if $H^3_{\rm nr}(\mathbb{C}(G)/\mathbb{C}, \mathbb{Q}/\mathbb{Z}) \neq 0$, then $H^3_{\rm nr}(\mathbb{C}(G)/\mathbb{C}, \mathbb{Q}/\mathbb{Z}) \simeq \mathbb{Z}/3\mathbb{Z}$.

$\lvert G\rvert = 3^5$	Φ_1	Φ_2	Φ_3	Φ_4	Φ_5	Φ_6	Φ_7	Φ_8	Φ_9	Φ_{10}
$H^2_{\rm nr}(\mathbb{C}(G), \mathbb{Q}/\mathbb{Z})$	0	0	0	0	0	0	0	0	0	$\mathbb{Z}/3\mathbb{Z}$
$H^3_{\rm nr}(\mathbb{C}(G), \mathbb{Q}/\mathbb{Z})$	0	0	0	0	0	0	$\mathbb{Z}/3\mathbb{Z}$	0	0	0

Corollary (H-Kang-Yamasaki, arXiv:1710.01958) $|G| = 3^5$

$\mathbb{C}(G)$ is not rational over $\mathbb{C} \iff G$ belongs to Φ_7, Φ_{10}.

- $H^3_{\rm nr}(X, \mathbb{Q}/\mathbb{Z}) \simeq {\rm Hdg}^4(X, \mathbb{Z})/{\rm Hdg}^4(X, \mathbb{Z})_{\rm alg}$ (CT-Voisin, 2012).

- For $p \geq 5$?

Theorem (H-Kang-Yamasaki, arXiv:1710.01958) $|G| = 5^5$ or 7^5

$H^3_{\rm nr}(\mathbb{C}(G), \mathbb{Q}/\mathbb{Z}) \neq 0 \iff G$ belongs to Φ_6, Φ_7 or Φ_{10}.
Moreover, if $H^3_{\rm nr}(\mathbb{C}(G)/\mathbb{C}, \mathbb{Q}/\mathbb{Z}) \neq 0$, then $H^3_{\rm nr}(\mathbb{C}(G)/\mathbb{C}, \mathbb{Q}/\mathbb{Z}) \simeq \mathbb{Z}/p\mathbb{Z}$.

$\lvert G\rvert = p^5$ $(p = 5, 7)$	Φ_1	Φ_2	Φ_3	Φ_4	Φ_5	Φ_6	Φ_7	Φ_8	Φ_9	Φ_{10}
$H^2_{\rm nr}(\mathbb{C}(G), \mathbb{Q}/\mathbb{Z})$	0	0	0	0	0	0	0	0	0	$\mathbb{Z}/p\mathbb{Z}$
$H^3_{\rm nr}(\mathbb{C}(G), \mathbb{Q}/\mathbb{Z})$	0	0	0	0	0	$\mathbb{Z}/p\mathbb{Z}$	$\mathbb{Z}/p\mathbb{Z}$	0	0	$\mathbb{Z}/p\mathbb{Z}$

$\lvert G\rvert = 3^5$	Φ_1	Φ_2	Φ_3	Φ_4	Φ_5	Φ_6	Φ_7	Φ_8	Φ_9	Φ_{10}
$H^2_{\rm nr}(\mathbb{C}(G), \mathbb{Q}/\mathbb{Z})$	0	0	0	0	0	0	0	0	0	$\mathbb{Z}/3\mathbb{Z}$
$H^3_{\rm nr}(\mathbb{C}(G), \mathbb{Q}/\mathbb{Z})$	0	0	0	0	0	0	$\mathbb{Z}/3\mathbb{Z}$	0	0	0