Degree three unramified cohomology groups and Noether’s problem for groups of order 243

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Table of contents

1 Noether’s problem over $\mathbb{C}$ and unramified Brauer groups

2 Noether’s problem over $\mathbb{C}$ and unramified cohomology groups


$$\text{Br}_{nr}(X/\mathbb{C}) \cong H^3(X, \mathbb{Z})_{\text{tors}}; \text{ Artin-Mumford invariant}$$

$$H^3_{nr}(X, \mathbb{Q}/\mathbb{Z}) \cong \text{Hdg}^4(X, \mathbb{Z})/\text{Hdg}^4(X, \mathbb{Z})_{\text{alg}} \leftrightarrow \text{integral Hodge conjecture}$$


For any smooth projective complex variety $X$, there is an exact sequence

$$0 \rightarrow H^3_{nr}(X, \mathbb{Z}) \otimes \mathbb{Q}/\mathbb{Z} \rightarrow H^3_{nr}(X, \mathbb{Q}/\mathbb{Z}) \rightarrow \text{Tors}(Z^4(X)) \rightarrow 0$$

where

$$Z^4(X) = \text{Hdg}^4(X, \mathbb{Z})/\text{Hdg}^4(X, \mathbb{Z})_{\text{alg}}$$

and the lower index "alg" means that we consider the group of integral Hodge classes which are algebraic. In particular, if $X$ is rationally connected, then we have

$$H^3_{nr}(X, \mathbb{Q}/\mathbb{Z}) \cong \text{Hdg}^4(X, \mathbb{Z})/\text{Hdg}^4(X, \mathbb{Z})_{\text{alg}}.$$ 

We show $H^3_{nr}(X, \mathbb{Q}/\mathbb{Z}) \cong \mathbb{Z}/3\mathbb{Z}$ for some $X = \mathbb{P}^n/G$ with $|G| = 3^5 = 243$. 
§1. Noether’s problem/\(\mathbb{C}\) and unram. Brauer group (1/4)

- \(k\); field, \(G\); finite group
- \(G \curvearrowright k\); trivial, \(G \curvearrowright k(x_g \mid g \in G)\); permutation.
- \(k(G) := k(x_g \mid g \in G)^G\); invariant field

**Noether’s problem (Emmy Noether, 1913)**

Is \(k(G)\) rational over \(k\)?, i.e. \(k(G) \simeq k(t_1, \ldots, t_n)\)?

- Is the quotient variety \(\mathbb{P}^n/G\) rational over \(k\)?
- Assume \(G = A\); abelian group.
- (Fisher, 1915) \(\mathbb{C}(A)\) is rational over \(\mathbb{C}\).
Noether’s problem/$\mathbb{C}$ and unram. Brauer group (2/4)

Let $G$ be a $p$-group. $\mathbb{C}(G) := \mathbb{C}(x_g \mid g \in G)^G$.

- (Saltman, 1984, Invent. Math.)
  For $\forall p$; prime, $\exists$ meta-abelian $p$-group $G$ of order $p^9$
  such that $\mathbb{C}(G)$ is not retract rational over $\mathbb{C}$.

- (Bogomolov, 1988)
  For $\forall p$; prime, $\exists$ $p$-group $G$ of order $p^6$
  such that $\mathbb{C}(G)$ is not retract rational over $\mathbb{C}$.

Indeed they showed $\text{Br}_{\text{nr}}(\mathbb{C}(G)/\mathbb{C}) \neq 0$; unramified Brauer group

- rational $\Rightarrow$ stably rational $\Rightarrow$ retract rational $\Rightarrow$ $\text{Br}_{\text{nr}}(\mathbb{C}(G)) = 0$.
- not rational $\iff$ not stably rational $\iff$ not retract rational $\iff$ $\text{Br}_{\text{nr}}(\mathbb{C}(G)) \neq 0$. 
Unramified Brauer group (1/2)

**Definition (Unramified Brauer group) Saltman (1984)**

Let \( k \subset K \) be an extension of fields.

\[
\text{Br}_{nr}(K/k) := \bigcap_{k \subset R \subset K: \text{DVR} \quad \text{and} \quad Q(R) = K} \text{Image}\{\text{Br}(R) \rightarrow \text{Br}(K)\}.
\]

- If \( K \) is retract rational over \( k \), then \( \text{Br}(k) \xrightarrow{\sim} \text{Br}_{nr}(K/k) \).
  In particular, if \( K \) is retract rational over \( \mathbb{C} \), then \( \text{Br}_{nr}(K/\mathbb{C}) = 0 \).

- For a smooth projective variety \( X \) over \( \mathbb{C} \) with function field \( K \),
  \( \text{Br}_{nr}(K/\mathbb{C}) \cong H^3(X, \mathbb{Z})_{\text{tors}} \) which is given by Artin-Mumford (1972).
Unramified Brauer group (2/2)

- \( K = C(G) \).

**Theorem (Bogomolov 1988, Saltman 1990)** \( \text{Br}_{\text{nr}}(\mathbb{C}(G)/\mathbb{C}) \cong B_0(G) \)

Let \( G \) be a finite group. Then \( \text{Br}_{\text{nr}}(\mathbb{C}(G)/\mathbb{C}) \) is isomorphic to

\[
B_0(G) = \bigcap_{A \leq G: \text{bicyclic}} \ker \{ \text{res} : H^2(G, \mathbb{Q}/\mathbb{Z}) \to H^2(A, \mathbb{Q}/\mathbb{Z}) \}.
\]

- \( C(G) : \) “retract rational” \( \implies B_0(G) = 0 \).
- \( B_0(G) \neq 0 \implies C(G) : \) not (retract) rational over \( k \).
- \( B_0(G) \leq H^2(G, \mathbb{Q}/\mathbb{Z}) \cong H_2(G, \mathbb{Z}) ; \) Schur multiplier.
- \( B_0(G) \) is called Bogomolov multiplier.
Noether’s problem/$\mathbb{C}$ and unram. Brauer group ($3/4$)

- (Chu-Kang, 2001) $G$ is $p$-group ($|G| \leq p^4 \implies \mathbb{C}(G)$ is rational.

**Theorem (Moravec, 2012, Amer. J. Math.)**

Assume $|G| = 3^5 = 243$. $B_0(G) \neq 0 \iff G = G(243, i)$, $28 \leq i \leq 30$.

In particular, $\exists 3$ groups $G$ such that $C(G)$ is not retract rational over $\mathbb{C}$.

- $\exists G$: 67 groups such that $|G| = 243$.

**Theorem (H-Kang-Kunyavskii, 2013, Asian J. Math.)**

Assume $|G| = p^5$ where $p$ is odd prime.

$B_0(G) \neq 0 \iff G$ belongs to the isoclinism family $\Phi_{10}$.

In particular, $\exists \gcd(4, p - 1) + \gcd(3, p - 1) + 1$ (resp. $\exists 3$) groups $G$ of order $p^5$ ($p \geq 5$) (resp. $p = 3$) s.t. $C(G)$ is not retract rational over $\mathbb{C}$.

- $\exists 2p + 61 + \gcd(4, p - 1) + 2 \gcd(3, p - 1)$ groups such that $|G| = p^5 (p \geq 5)$. ($\exists \Phi_1, \ldots, \Phi_{10}$)
From the proof (1/3)

**Definition (isoclinic)**

$p$-groups $G_1$ and $G_2$ are **isoclinic** $\iff$ isom. $\theta : G_1/Z(G_1) \sim G_2/Z(G_2)$, $\phi : [G_1, G_1] \sim [G_2, G_2]$ such that

$$
\begin{array}{c}
G_1/Z(G_1) \times G_1/Z(G_1) \\
\downarrow \left[ [ , ] \right]
\end{array}
\xrightarrow{\sim}
\begin{array}{c}
G_2/Z(G_2) \times G_2/Z(G_2) \\
\downarrow \left[ [ , ] \right]
\end{array}
\xrightarrow{\phi \left[ [ , ] \right]} [G_2, G_2]
$$

**Invariants**

- lower central series
- $\#$ of conj. classes with precisely $p^i$ members
- $\#$ of irr. complex rep. of $G$ of degree $p^i$
From the proof (2/3)

1. $|G| = p^4 (p > 2)$. There exist 15 groups $(\Phi_1, \Phi_2, \Phi_3)$

2. $|G| = 2^4 = 16$. There exist 14 groups $(\Phi_1, \Phi_2, \Phi_3)$

3. $|G| = p^5 (p > 3)$. There exist $2p + 61 + (4, p - 1) + 2 \times (3, p - 1)$ groups $(\Phi_1, \ldots, \Phi_{10})$

|    | $\Phi_1$ | $\Phi_2$ | $\Phi_3$ | $\Phi_4$ | $\Phi_5$ | $\Phi_6$ | $\Phi_7$ | $\Phi_8$
<table>
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<tr>
<td>#</td>
<td>7</td>
<td>15</td>
<td>13</td>
<td>$p + 8$</td>
<td>2</td>
<td>$p + 7$</td>
<td>5</td>
<td>1</td>
</tr>
<tr>
<td>$(p = 3)$</td>
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|    | $\Phi_9$ | $\Phi_{10}$
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<tbody>
<tr>
<td>#</td>
<td>$2 + (3, p - 1)$</td>
<td>$1 + (4, p - 1) + (3, p - 1)$</td>
</tr>
<tr>
<td>$(p = 3)$</td>
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<td>$3$</td>
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Let $G_1$ and $G_2$ be isoclinic $p$-groups. Is it true that the fields $k(G_1)$ and $k(G_2)$ are stably isomorphic, i.e. $k(G_1)(\exists s_1, \ldots, \exists s_m) \simeq k(G_2)(\exists t_1, \ldots, \exists t_n)$, or, at least, that $B_0(G_1) \simeq B_0(G_2)$?

**Theorem (Moravec, 2013)**

$G_1$ and $G_2$ are isoclinic $\implies B_0(G_1) \simeq B_0(G_2)$.

**Theorem (Bogomolov-Böhning, 2013)**

$G_1$ and $G_2$ are isoclinic $\implies \mathbb{C}(G_1)$ and $\mathbb{C}(G_2)$ are stably isomorphic.
Noether’s problem/$\mathbb{C}$ and unram. Brauer group (4/4)

Theorem (H-Kang-Kunyavskii, 2013, Asian J. Math.)

Assume $|G| = p^5$ where $p$ is odd prime.

$B_0(G) \neq 0 \iff G$ belongs to the isoclinism family $\Phi_{10}$.

Theorem (Chu-H-Hu-Kang, 2015, J. Algebra) $|G| = 3^5 = 243$

If $B_0(G) = 0$, then $\mathbb{C}(G)$ is rational over $\mathbb{C}$ except for $\Phi_7$.

- Rationality of $\Phi_7$ was unknown.
- $\Phi_5$ and $\Phi_7$ are very similar: $C = 1$ ($\Phi_5$), $C = \omega$ ($\Phi_7$).

$\mathbb{C}(G)$ is stably isomorphic to $\mathbb{C}(z_1, z_2, z_3, z_4, z_5, z_6, z_7, z_8, z_9)\langle f_1, f_2 \rangle$

$$f_1 : z_1 \mapsto z_2, z_2 \mapsto \frac{1}{z_1 z_2}, z_3 \mapsto z_4, z_4 \mapsto \frac{1}{z_3 z_4},$$
$$z_5 \mapsto \frac{z_5}{z_2 z_3}, z_6 \mapsto \frac{z_1 z_6}{z_3}, z_7 \mapsto z_8, z_8 \mapsto \frac{1}{z_7 z_8}, z_9 \mapsto \frac{z_4 z_9}{z_1},$$

$$f_2 : z_1 \mapsto z_3, z_2 \mapsto z_4, z_3 \mapsto \frac{1}{z_1 z_3}, z_4 \mapsto \frac{1}{z_2 z_4},$$
$$z_5 \mapsto z_6, z_6 \mapsto \frac{1}{z_5 z_6}, z_7 \mapsto C \frac{z_4 z_7}{z_3}, z_8 \mapsto C \frac{z_8}{z_3 z_2^4}, z_9 \mapsto \frac{z_4 z_9}{z_1}.$$
From $\text{Br}_{nr}(K/\mathbb{C})$ to $H^i_{nr}(K/\mathbb{C}, \mathbb{Q}/\mathbb{Z})$.

**Definition (Colliot-Thélène and Ojanguren, 1989, Invent. Math.)**

Let $K/\mathbb{C}$ be a function field, that is finitely generated as a field over $\mathbb{C}$. The **unramified cohomology group** $H^i_{nr}(K/\mathbb{C}, \mu \otimes j)$ of $K$ over $\mathbb{C}$ of degree $i \geq 1$ is defined to be

$$H^i_{nr}(K/\mathbb{C}, \mu \otimes j) = \bigcap_{\mathbb{C} \subset R \subset K: \text{DVR of rank one and } Q(R) = K} \text{Ker}\{H^i(K, \mu \otimes j) \to H^{i-1}(\mathbb{A}_R, \mu \otimes (j-1))\}.$$

If $K$ is the function field of a complete smooth variety over $k$, then

$$H^i_{nr}(K/\mathbb{C}, \mu \otimes j) = \bigcap_{\mathbb{C} \subset R \subset K: \text{DVR of rank one and } Q(R) = K} \text{Image}\{H^i_{\text{ét}}(R, \mu \otimes j) \to H^i_{\text{ét}}(K, \mu \otimes j)\}.$$

Note that $n\text{Br}_{nr}(K/\mathbb{C}) \simeq H^2_{nr}(K/\mathbb{C}, \mu_n)$. 

§2. Noether’s problem/$\mathbb{C}$ and unram. cohomology (1/7)
Theorem (Colliot-Thélène and Ojanguren, 1989)

If $K$ and $L$ are stably $\mathbb{C}$-isomorphic, then

\[
H^i_{\text{nr}}(K/\mathbb{C}, \mu_n^\otimes j) \sim H^i_{\text{nr}}(L/\mathbb{C}, \mu_n^\otimes j).
\]

In particular, $K$ is stably $\mathbb{C}$-rational, then

\[
H^i_{\text{nr}}(K/\mathbb{C}, \mu_n^\otimes j) = 0.
\]

- Moreover, if $K$ is retract $\mathbb{C}$-rational, then $H^i_{\text{nr}}(K/\mathbb{C}, \mu_n^\otimes j) = 0$.
- CTO (1989) $\exists K$ (trdeg$_{\mathbb{C}}K = 6$) s.t. $H^3_{\text{nr}}(K/\mathbb{C}, \mu_2^\otimes 3) \neq 0$.
- Peyre (1993) gave a sufficient condition for $H^i_{\text{nr}}(K/\mathbb{C}, \mu_p^\otimes i) \neq 0$:
  - $\exists K$ s.t. $H^3_{\text{nr}}(K/\mathbb{C}, \mu_p^\otimes 3) \neq 0$ and $\text{Br}_{\text{nr}}(K/\mathbb{C}) = 0$;
  - $\exists K$ s.t. $H^4_{\text{nr}}(K/\mathbb{C}, \mu_2^\otimes 4) \neq 0$ and $\text{Br}_{\text{nr}}(K/\mathbb{C}) = 0$. 

**Theorem (Asok, 2013, Compos. Math.)**

(1) For any $n > 0$, $\exists$ a smooth projective complex variety $X$ that is $\mathbb{C}$-unirational, for which $H^i_{nr}(\mathbb{C}(X), \mu_{2^i}) = 0$ for each $i < n$, yet $H^n_{nr}(\mathbb{C}(X), \mu_{2^n}) \neq 0$, and so $X$ is not $\mathbb{A}^1$-connected, nor (retract, stably) $\mathbb{C}$-rational;

(2) For any prime $l$ and any $n \geq 2$, $\exists$ a smooth projective rationally connected complex variety $Y$ such that $H^n_{nr}(\mathbb{C}(Y), \mu_{l^n}) \neq 0$. In particular, $Y$ is not $\mathbb{A}^1$-connected, nor (retract, stably) $\mathbb{C}$-rational.

- Namely, $H^i_{nr}(\mathbb{C}(X), \mu_{n^j}) = 0$ is just a necessary condition for $\mathbb{C}$-rationality.

- It is interesting to consider an analog of above for quotient varieties $V/G$, e.g. $\mathbb{C}(\mathbb{P}^n/G) = \mathbb{C}(G)$ (Noether’s problem).
Take the direct limit with respect to $n$:

$$H^i(K/\mathbb{C}, \mathbb{Q}/\mathbb{Z}(j)) = \lim_{\rightarrow n} H^i(K/\mathbb{C}, \mu_n \otimes j)$$

and we also define the unramified cohomology group

$$H^i_{nr}(K/\mathbb{C}, \mathbb{Q}/\mathbb{Z}(j)) = \bigcap_{R} \ker\{ H^i(K, \mathbb{Q}/\mathbb{Z}(j)) \rightarrow H^{i-1}(\mathbb{A}_R, \mathbb{Q}/\mathbb{Z}(j-1)) \}.$$

Then we have $\text{Br}_{nr}(K/\mathbb{C}) \simeq H^2_{nr}(K/\mathbb{C}, \mathbb{Q}/\mathbb{Z}(1))$.

- The case $K = \mathbb{C}(G)$:


$\exists$ $p$-group $G$ of order $p^{12}$ such that $B_0(G) = 0$ and $H^3_{nr}(\mathbb{C}(G), \mathbb{Q}/\mathbb{Z}) \neq 0$. In particular, $\mathbb{C}(G)$ is not (retract, stably) $\mathbb{C}$-rational.
Noether’s problem over $\mathbb{C}$ and unram. cohomology (5/7)


$\exists$ $p$-group $G$ of order $p^{12}$ such that $B_0(G) = 0$ and $H^3_{nr}(\mathbb{C}(G), \mathbb{Q}/\mathbb{Z}) \neq 0$. In particular, $\mathbb{C}(G)$ is not (retract, stably) $\mathbb{C}$-rational.

Using Peyre’s method, we improved this result:

\textbf{Theorem (H-Kang-Yamasaki, 2016, J. Algebra)} $p$: any odd prime

$\exists$ $p$-group $G$ of order $p^9$ such that $B_0(G) = 0$ and $H^3_{nr}(\mathbb{C}(G), \mathbb{Q}/\mathbb{Z}) \neq 0$. In particular, $\mathbb{C}(G)$ is not (retract, stably) $\mathbb{C}$-rational.

**Theorem (H-Kang-Yamasaki, arXiv:1710.01958)** \(|G| = 3^5\)

\[ H^3_{nr}(\mathbb{C}(G), \mathbb{Q}/\mathbb{Z}) \neq 0 \iff G \text{ belongs to the isoclinism family } \Phi_7. \]

Moreover, if \( H^3_{nr}(\mathbb{C}(G)/\mathbb{C}, \mathbb{Q}/\mathbb{Z}) \neq 0 \), then \( H^3_{nr}(\mathbb{C}(G)/\mathbb{C}, \mathbb{Q}/\mathbb{Z}) \cong \mathbb{Z}/3\mathbb{Z}. \)

\[
\begin{array}{cccccccccc}
|G| = 3^5 & \Phi_1 & \Phi_2 & \Phi_3 & \Phi_4 & \Phi_5 & \Phi_6 & \Phi_7 & \Phi_8 & \Phi_9 & \Phi_{10} \\
H^2_{nr}(\mathbb{C}(G), \mathbb{Q}/\mathbb{Z}) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbb{Z}/3\mathbb{Z} \\
H^3_{nr}(\mathbb{C}(G), \mathbb{Q}/\mathbb{Z}) & 0 & 0 & 0 & 0 & 0 & 0 & \mathbb{Z}/3\mathbb{Z} & 0 & 0 & 0
\end{array}
\]

**Corollary (H-Kang-Yamasaki, arXiv:1710.01958)** \(|G| = 3^5\)

\( \mathbb{C}(G) \text{ is not rational over } \mathbb{C} \iff G \text{ belongs to } \Phi_7, \Phi_{10}. \)

\[ H^3_{nr}(X, \mathbb{Q}/\mathbb{Z}) \cong \text{Hdg}^4(X, \mathbb{Z})/\text{Hdg}^4(X, \mathbb{Z})_{\text{alg}} \text{ (CT-Voisin, 2012).} \]
For $p \geq 5$?

**Theorem (H-Kang-Yamasaki, arXiv:1710.01958)** $|G| = 5^5$ or $7^5$

$H^3_{nr}(\mathbb{C}(G), \mathbb{Q}/\mathbb{Z}) \neq 0 \iff G \text{ belongs to } \Phi_6, \Phi_7 \text{ or } \Phi_{10}$.

Moreover, if $H^3_{nr}(\mathbb{C}(G)/\mathbb{C}, \mathbb{Q}/\mathbb{Z}) \neq 0$, then $H^3_{nr}(\mathbb{C}(G)/\mathbb{C}, \mathbb{Q}/\mathbb{Z}) \simeq \mathbb{Z}/p\mathbb{Z}$.

| $|G| = p^5$ ($p = 5, 7$) | $\Phi_1$ | $\Phi_2$ | $\Phi_3$ | $\Phi_4$ | $\Phi_5$ | $\Phi_6$ | $\Phi_7$ | $\Phi_8$ | $\Phi_9$ | $\Phi_{10}$ |
|-----------------------------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| $H^2_{nr}(\mathbb{C}(G), \mathbb{Q}/\mathbb{Z})$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $\mathbb{Z}/p\mathbb{Z}$ |
| $H^3_{nr}(\mathbb{C}(G), \mathbb{Q}/\mathbb{Z})$ | 0 | 0 | 0 | 0 | 0 | $\mathbb{Z}/p\mathbb{Z}$ | $\mathbb{Z}/p\mathbb{Z}$ | 0 | 0 | $\mathbb{Z}/p\mathbb{Z}$ |

| $|G| = 3^5$ | $\Phi_1$ | $\Phi_2$ | $\Phi_3$ | $\Phi_4$ | $\Phi_5$ | $\Phi_6$ | $\Phi_7$ | $\Phi_8$ | $\Phi_9$ | $\Phi_{10}$ |
|-----------------------------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| $H^2_{nr}(\mathbb{C}(G), \mathbb{Q}/\mathbb{Z})$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $\mathbb{Z}/3\mathbb{Z}$ |
| $H^3_{nr}(\mathbb{C}(G), \mathbb{Q}/\mathbb{Z})$ | 0 | 0 | 0 | 0 | 0 | $\mathbb{Z}/3\mathbb{Z}$ | 0 | 0 | 0 | 0 |