

Algebraic Varieties and Automorphism Groups

July 7 – 11, 2014

Room 420, RIMS, Kyoto University

Program

July 7

- 10:00 – 11:00 Registration
- 11:00 – 12:00 **Masanori Ishida (Tohoku University)**
Cusp singularities and discrete groups generated by reflections
- 13:30 – 14:30 **Noboru Nakayama (Kyoto University)**
A variant of Shokurov's criterion of toric surface
- 14:50 – 15:50 **Osamu Fujino (Kyoto University)**
Subadditivity of the logarithmic Kodaira dimension for affine varieties
- 16:10 – 17:00 **De-Qi Zhang (National University of Singapore)**
Projective varieties of dimension n admitting the action of a free abelian group of rank $n - 1$

July 8

- 10:00 – 11:00 **Masayoshi Miyanishi (Kwansei Gakuin University)**
Unipotent group actions on projective varieties
- 11:20 – 12:20 **Hubert Flenner (Ruhr Universität Bochum)**
A Gromov Winkelman type theorem for flexible varieties
- 14:30 – 15:20 **Iwan Arzhantsev (Moscow State University)**
Cox rings, universal torsors, and infinite transitivity
- 15:40 – 16:40 **Alexsandre Perepechko (St. Petersburg State University)**
Flexibility of affine varieties covered by toric charts
- 18:00 – Welcome Party

July 9

- 10:00 – 11:00 **David Wright (Washington University)**
Amalgamations and automorphism groups
- 11:20 – 12:20 **Takashi Kishimoto (Saitama University)**
Birational tentative to produce a counter-example for Abhyankar-Sathaye embedding problem of dimension three

July 10

- 10:00 - 11:00 **Vladimir Popov (Steklov Institute)**
Finite group actions on algebraic varieties: a “social” approach
- 11:20 – 12:20 **Shigeru Kuroda (Tokyo Metropolitan University)**
Lnd-automorphisms and the Linearization Problem
- 13:30 – 14:20 **Lucy Moser-Jauslin (Université de Bourgogne)**
Extensions of automorphisms of rational hypersurfaces of complex affine space
- 14:40 – 15:40 **Jérémy Blanc (Universität Basel)**
Group of birational transformations generated by standard and linear transformations
- 16:00 – 17:00 **Masaki Hanamura (Tohoku University)**
Structure of birational automorphism groups of non uni-ruled varieties

July 11

- 10:00 – 11:00 **Rajendra V. Gurjar (Tata Institute of Fundamental Research)**
Pure extensions of commutative rings
- 11:20 – 12:10 **Daniel Daigle (University of Ottawa)**
Homogeneous locally nilpotent derivations of $k[x, y, z]$
- 13:40 – 14:30 **Pierrette Cassou-Noguès (Université Bordeaux I)**
Field generators
- 14:50 – 15:40 **Hideo Kojima (Niigata University)**
Open algebraic surfaces of non-negative logarithmic Kodaira dimension

Organizers:

Takashi Kishimoto (Saitama University), Hideo Kojima (Niigata University)
Kayo Masuda (Kwansei Gakuin University)*, Masayoshi Miyanishi (Kwansei Gakuin University)
Mikhail Zaidenberg (Université Grenoble I)
* chair

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Abstract

Iwan Arzhantsev (Moscow State University)

Cox rings, universal torsors, and infinite transitivity

Let X be an algebraic variety covered by open charts isomorphic to the affine space and let $q : X' \rightarrow X$ be the universal torsor over X . We prove that the automorphism group of the quasiaffine variety X' acts on X' infinitely transitively. Also we find wide classes of varieties X admitting such a covering. This is a joint work with Alexander Perepechko and Hendrik Suess.

Jérémy Blanc (Universität Basel)

Group of birational transformations generated by the standard and linear transformations

The famous Noether-Castelnuovo theorem says that the group of birational transformations of the plane is generated by linear automorphisms and the standard quadratic transformation, when the field is algebraically closed. In dimension higher or when working over non-closed field, the result fails, and the only reason for a map not being generated by these simple maps is that some non-rational hypersurface is contracted. I will describe other criteria, which are geometric but can also be viewed algebraically.

Pierrette Cassou-Noguès (Université Bordeaux I)

Field generators

In this talk we will discuss field generators and particularly bad field generators. We will explain what happened since Russell's paper: "Good and bad field generators" (J. Math. Kyoto Univ. (1977)) and give an account of our joint work in progress with Daniel Daigle.

Daniel Daigle (University of Ottawa)

Homogeneous locally nilpotent derivations of $\mathbf{k}[x, y, z]$

Let B be a polynomial ring in 3 variables over an algebraically closed field \mathbf{k} of characteristic zero. If \mathfrak{g} is any grading of B , define

$$\text{LND}(B, \mathfrak{g}) = \text{set of locally nilpotent derivations } D : B \rightarrow B \\ \text{that are homogeneous with respect to } \mathfrak{g}.$$

By a *positive grading* of B , we mean an \mathbb{N} -grading $B = \bigoplus_{n \in \mathbb{N}} B_n$ satisfying $B_0 = \mathbf{k}$. The talk will present recent progress in:

Problem. Describe $\text{LND}(B, \mathfrak{g})$ for each positive grading \mathfrak{g} of B .

Hubert Flenner (Ruhr Universität Bochum)

A Gromov Winkelmann type theorem for flexible varieties

An affine variety X over \mathbf{C} of dimension ≥ 2 is called *flexible* if its special automorphism group $\text{SAut}(X)$ acts transitively on the smooth locus X_{reg} . Here the special automorphism group $\text{SAut}(X)$ is the subgroup of the automorphism group $\text{Aut}(X)$ generated by all one-parameter unipotent subgroups. Given a normal, flexible, affine variety X and a closed subvariety Y in X of codimension at least 2, we show that the pointwise stabilizer subgroup of Y in the group $\text{SAut}(X)$ acts infinitely transitively on the complement $X \setminus Y$, that is, m -transitively for any $m \geq 1$. More generally we show such a result for any quasi-affine variety X and codimension ≥ 2 subset Y of X .

In the particular case of the affine space $X = \mathbf{A}^n$, $n \geq 2$, this yields a Theorem of Gromov and Winkelmann.

(Joint with Kaliman and Zaidenberg)

Osamu Fujino (Kyoto University)

Subadditivity of the logarithmic Kodaira dimension for affine varieties

We prove the subadditivity of the logarithmic Kodaira dimension for affine varieties. Our proof heavily depends on Nakayama's theory of ω -sheaves and $\widehat{\omega}$ -sheaves. We also use the minimal model program for projective klt pairs with big boundary divisor.

Rajendra V. Gurjar (Tata Institute of Fundamental Research)

Pure Extensions of Commutative Rings

We will discuss the notion of pure extensions of commutative rings. Several results proved from algebraic and geometric viewpoints will be discussed, as well as interesting examples of pure and non-pure extensions will be mentioned. We will also raise many open problems about pure extensions. This is a joint work with Sagnik Chakraborty and Masayoshi Miyanishi.

Masaki Hanamura (Tohoku University)

Structure of birational automorphism groups of non uni-ruled varieties

Masanori Ishida (Tohoku University)

Cusp singularities and discrete groups generated by reflections

Since $\text{GL}(r, \mathbf{Z})$ is the automorphism group of the algebraic torus $(\mathbf{C}^\times)^r$, any subgroup Γ of $\text{GL}(r, \mathbf{Z})$ acts on the algebraic torus, and the action may extend to some toric variety including it. For instance, if Γ is finite, then it acts on a projective toric variety.

We are interested in such a group Γ that the orbit of a point in the lattice \mathbf{Z}^r generates an open convex cone C in \mathbf{R}^r without lattice points on the boundary except at the origin. In such case, some subgroup $\Gamma' \subset \Gamma$ of finite index acts freely on the cone, and it defines a cusp singularity introduced by Tsuchihashi, using toric geometry.

However, not so many groups with this property are known. I will talk on examples for $r = 4$, and will comment on the relation with the works of Vinberg on infinite Coxeter groups.

Takashi Kishimoto (Saitama University)

Birational tentative to produce a counter-example for Abhyankar-Sathaye embedding problem of dimension three

The geometry on del Pezzo surfaces plays often important roles for not only birational geometry but also affine geometry. In this talk, we will talk of some birational tentative to produce a counter-example for Abhyankar-Sathaye embedding problem in dimension three by make use of smooth del Pezzo surfaces S of degree one with log-canonical threshold $\text{lct}(S) = 5/6$, in other words, S with an anti-canonical rational cuspidal curve, say $C \in |-K_S|$. More precisely, it is well known that a del Pezzo surface S of degree $(-K_S)^2 = 1$ is embedded in the weighted projective space $\mathbb{P}(1, 1, 2, 3)$ as a hypersurface of degree 6, so that we take a hyperplane $H \in |\mathcal{O}_{\mathbb{P}(1,1,2,3)}(1)|$ in such a way that $S \circ H = C$ scheme-theoretically. Then our main interest consists in a detailed observation of the pencil Λ spanned by S and $6H$, namely, we perform in an explicit manner the process to resolve $\text{Bs } \Lambda = C$ and then execute minimal model program (mmp) relative to the morphism defined by a resolved pencil. In fact, the output of a relative (mmp) is either a del Pezzo fibration of degree one $(\text{dP})_1$ or a Mori conic bundle (MCB). In consideration of the results due to Grinenko and Pukhlikov concerning birational rigidity on certain del Pezzo fibrations of low degree, it seems to be reasonable to expect, to some extent, that the output is actually an (MCB). Nevertheless, we give an example of S in which the corresponding output is $(\text{dP})_1$. Further, we see that provided we reach an (MCB) by choosing S suitably, we can construct a counter-example of Abhyankar-Sathaye embedding problem in dimension three. (However, we do not know so far any example of S such that the corresponding output of a relative (mmp) ends with an (MCB).)

Hideo Kojima (Niigata University)

Open algebraic surfaces of non-negative logarithmic Kodaira dimension

We discuss open algebraic surfaces of non-negative logarithmic Kodaira dimension over an algebraically closed field of arbitrary characteristic. After recalling structure theorems for open algebraic surfaces, we give some results on logarithmic plurigenera of open algebraic surfaces of non-negative logarithmic Kodaira dimension. In particular, we show that, for a smooth irrational open algebraic surface S , $\bar{\kappa}(S) \geq 0$ if and only if $\bar{P}_{12}(S) > 0$. We also give some results on open algebraic surfaces of $\bar{\kappa} = 0$.

Shigeru Kuroda (Tokyo Metropolitan University)

End-automorphisms and the Linearization Problem

Let R be a \mathbf{Q} -domain, and δ a nonzero locally nilpotent derivation of R . Then, an automorphism

$\exp \delta$ of the ring R is defined by

$$(\exp \delta)(a) = \sum_{l=0}^{\infty} \frac{\delta^l(a)}{l!} \quad \text{for each } a \in R.$$

Note that $R^\delta := \{a \in R \mid \delta(a) = 0\}$ is a subring of R , and $\exp \delta$ belongs to the automorphism group $\text{Aut}(R/R^\delta)$ of the R^δ -algebra R . Let $\text{LND}(R/R^\delta)$ be the set of locally nilpotent R^δ -derivations of R . Then, for each $D \in \text{LND}(R/R^\delta)$, we have $R^D \supset R^\delta$, and so $\exp D \in \text{Aut}(R/R^D) \subset \text{Aut}(R/R^\delta)$. Hence,

$$\mathcal{E}_\delta := \{\exp D \mid D \in \text{LND}(R/R^\delta)\}$$

is a subset of $\text{Aut}(R/R^\delta)$. In fact, \mathcal{E}_δ forms a normal subgroup of $\text{Aut}(R/R^\delta)$. In this talk, we study the quotient group

$$\text{Aut}(R/R^\delta)/\mathcal{E}_\delta.$$

For example, let k be a field of characteristic zero, and $R = k[x_1, x_2, x_3]$ the polynomial ring in three variables over k . Then,

$$\delta = x_2 \frac{\partial}{\partial x_1} - 2x_3 \frac{\partial}{\partial x_2}$$

is a locally nilpotent derivation of R with $R^\delta = k[x_1 x_3 + x_2^2, x_3]$. It is well known that the famous automorphism given by Nagata is an element of \mathcal{E}_δ for this δ . The k -automorphism ϕ of R defined by $\phi(x_2) = -x_2$ and $\phi(x_i) = x_i$ for $i = 1, 3$ belongs to $\text{Aut}(R/R^\delta) - \mathcal{E}_\delta$. In this case, $\text{Aut}(R/R^\delta)/\mathcal{E}_\delta$ is a cyclic group of order two generated the image of ϕ .

A special case of Kambayashi's Linearization Problem asks whether every finite-order k -automorphism of the polynomial ring $k[x_1, \dots, x_n]$ with $\bar{k} = k$ is linearizable. When $R = k[x_1, \dots, x_n]$, the study of $\text{Aut}(R/R^\delta)/\mathcal{E}_\delta$ is closely related to the study of the Linearization Problem. Actually, every element of $\text{Aut}(R/R^\delta) - \mathcal{E}_\delta$ has finite order unless R is a polynomial ring in one variable over R^δ . We discuss when \mathcal{E}_δ equals $\text{Aut}(R/R^\delta)$ from the view point of linearizability of elements of $\text{Aut}(R/R^\delta) - \mathcal{E}_\delta$ and triangularizability of δ . We also announce some recent results on the Linearization Problem.

Masayoshi Miyanishi (Kwansei Gakuin University)

Unipotent group actions on projective varieties

A theory of unipotent group actions on affine varieties, e.g., G_a -actions, can be carried over to projective varieties via the stratification by an effective ample divisor and its iterated intersections. We then have to consider a stratified vector field on a projective variety instead of a locally nilpotent derivation on the coordinate ring of an affine variety. We thereby exploit some new topological structure of Fano threefolds of rank one and index larger than one. This is a joint work with R.V. Gurjar and K. Masuda.

Lucy Moser-Jauslin (Université de Bourgogne)

Extensions of automorphisms of rational hypersurfaces of complex affine space

Given a hypersurface X of affine n -space, we consider the question of whether automorphisms of X extend to automorphisms of the ambient space. In general, many examples of non extension are known, but under certain specific circumstances, the existence of an extension can be guaranteed. We will be particularly interested in the case of certain rational threefolds in four-space, where it is known

that each automorphism does extend to the ambient space. We will show, however, that $\text{Aut}(X)$ cannot always be realised as a subgroup of the automorphism group of the ambient space in a natural way.

Noboru Nakayama (Kyoto University)

A variant of Shokurov's criterion of toric surface

For a log canonical pair (X, D) of a normal projective surface X and a reduced anti-canonical divisor D , Shokurov's criterion asserts that if the inequality $n(D) \geq \hat{\rho}(X) + 2$ holds for the number $n(D)$ of irreducible components of D and for the Weil-Picard number $\hat{\rho}(X)$, then X is a toric surface and D is a boundary divisor. In this talk, I will discuss the structure of (X, D) which satisfies the equality $n(D) = \hat{\rho}(X) + 1$.

Alexsandre Perepechko (St. Petersburg State University)

Flexibility of affine varieties covered by toric charts

The talk is based on the joint work with Hendrik Süß and Mateusz Michałek. We study two families of affine varieties covered by toric open charts and prove their flexibility.

Let X be an affine algebraic variety of dimension ≥ 2 defined over an algebraically closed field \mathbb{K} of characteristic zero, and let $\text{SAut } X$ be the subgroup of automorphism group $\text{Aut } X$ that is generated by the 1-parameter unipotent subgroups, i.e. actions of the additive group $\mathbb{G}_a = \mathbb{G}_a(\mathbb{K})$.

A variety X is called *flexible* if the tangent space to X at an arbitrary regular point $x \in X$ is generated by tangent vectors to orbits of \mathbb{G}_a -actions. This is equivalent to the infinite transitivity of the action of $\text{SAut } X$ on the regular locus $X_{\text{reg}} \subset X$, see [1].

Previously described classes of flexible varieties include:

affine cones over flag varieties, non-degenerate toric varieties of dimension ≥ 2 , and suspensions over flexible varieties [2]; affine cones over del Pezzo surfaces of degree ≥ 4 [5]; universal torsors over A -covered varieties [3].

We will prove flexibility of the following families of varieties:

1. affine cones over secant varieties of Veronese–Segre varieties;
2. total coordinate spaces of smooth projective T-varieties of complexity 1.

We use the construction from [4] that provides a certain correspondence between open cylindrical subsets on a projective variety Y and regular \mathbb{G}_a -actions on the affine cone over Y .

References

- [1] I.V. Arzhantsev, H. Flenner, S. Kaliman, F. Kutzschebauch, and M. Zaidenberg, *Flexible varieties and automorphism groups*, *Duke Math. J.* **162** (2013), no. 4, 767–823.
- [2] I.V. Arzhantsev, K. Kuyumzhiyan, M. Zaidenberg, *Flag varieties, toric varieties, and suspensions: three instances of infinite transitivity*, *Sbornik: Math.* **203** (2012), no. 7, 923–949.
- [3] I.V. Arzhantsev, A. Perepechko, and H. Süß, *Infinite transitivity on universal torsors*, arXiv:1302.2309, accepted to the *Journal of the LMS*, 17p.

- [4] T. Kishimoto, Yu. Prokhorov, and M. Zaidenberg, *Group actions on affine cones*, Montreal Centre de Recherches Mathématiques, CRM Proceedings and Lecture Notes **54** (2011), 123–163.
- [5] A.Yu. Perepechko. *Flexibility of affine cones over del Pezzo surfaces of degree 4 and 5*. *Func. Anal. Appl.* **47** (2013), no. 4, 45–52.

Vladimir Popov (Steklov Institute)

Finite group actions on algebraic varieties: a “social” approach

In the last two years there was an increasing activity in describing qualitative properties of finite automorphism groups of algebraic varieties. We shall discuss these developments.

David Wright (Washington University in St. Louis)

Amalgamations and Automorphism Groups

We review some facts about automorphism groups over polynomial rings. It is shown the the tame subgroup of the group of polynomials automorphisms of affine 3-space can be realized as the product of three subgroups, amalgamated along pairwise intersections, in a manner that generalizes the well-known amalgamated free product structure in dimension 2. The result follows from the defining relations for the tame subgroup given by U. U. Umirbaev.

De-Qi Zhang (National University of Singapore)

Projective varieties of dimension n admitting the action of a free abelian group of rank $n - 1$

Let X be a normal projective variety of dimension $n > 2$ admitting the action of the group $G := \mathbf{Z}^{n-1}$ (the free abelian group of rank $n - 1$) such that every non-trivial element of G is of positive entropy (i.e., its action on the Neron Severi group is not quasi-unipotent). Consider the statements:

- (1) X is not rationally connected.
- (2) X is G -equivariant birational to the quotient of a complex torus.
- (3) $K_X + D$ is pseudo-effective for some G -periodic effective fractional divisor D .
- (4) the Kodaira dimension $\kappa(X)$ is non-negative.
- (5) X is not uniruled.

We prove that (1) implies (2); (2) and (3) are equivalent. For applications, we use the known facts: (4) implies (5), and (5) implies (1).