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A framework of verified eigenvalue bounds for self-adjoint differential operators

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ABSTRACT

For eigenvalue problems of self-adjoint differential operators, a universal framework is proposed to give explicit lower and upper bounds for the eigenvalues. In the case of the Laplacian operator, by applying Crouzeix–Raviart finite elements, an efficient algorithm is developed to bound the eigenvalues for the Laplacian defined in 1D, 2D and 3D spaces. Moreover, for nonconvex domains, for which case there may exist singularities of eigenfunctions around re-entrant corners, the proposed algorithm can easily provide eigenvalue bounds. By further adopting the interval arithmetic, the explicit eigenvalue bounds from numerical computations can be mathematically correct.

1. Introduction

The eigenvalue problem plays an important role in both natural and engineering sciences. In this paper, we consider the class of self-adjoint eigenvalue problems, the eigenvalues of which are real numbers, and propose a universal framework to give lower and upper bounds for eigenvalues.

For a long time, the numerical analysis for eigenvalue problems, for example, of Laplacian eigenvalues, has been well documented in literature. Most classical research focuses on the qualitative analysis of numerical schemes, such as convergence order. But, quantitative analysis, for example, explicit eigenvalue bounds, has not drawn much interest from researchers.

Recently, explicit eigenvalue bounds have become more indispensable, especially in adaptive computing of the finite element method (FEM) and in the computer-assisted proof for nonlinear differential equations. For example, a good indicator for the error of approximate solutions requires the explicit error estimation for various interpolation operators. The estimation of error constants is reduced to eigenvalue problems of Laplace and biharmonic operators; see, [10,13]. In addition, verifying the solution for nonlinear differential equations requires eigenvalue bounds of the controlling differential operators; see, e.g., [17,19,22].

Generally, we can easily obtain upper bounds for eigenvalues by using Rayleigh–Ritz's method, but lower eigenvalue bounds remain difficult to find. Theoretical analysis of eigenvalue bounds, which is independent of numerical scheme selection, includes the early work of Kato, Weinstein and Stenger, Lehmann, Beattie and Goerisch, Behnke and Goerisch, Goerisch [9,23,11,1,4,7]. These theories provide nice eigenvalue bounds, assuming there are rough a priori bounds for the eigenvalues. A good choice to provide the necessary a priori eigenvalue bounds is the homotopy method proposed by Plum [18], which considers the connection between the base problem—with a known spectrum—to the objective problem. With a domain transformation, this method can even deal with the domain of general shapes. However, to apply the homotopy method in solving practical problems, we need case-by-case efforts in setting up the homotopy process.

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In practical computation, such methods as the FEM, finite difference method, and intermediate method, can result in good eigenvalue approximation. However, most of these methods have difficulties in dealing with domains of general shapes if rigorous eigenvalue bounds are wanted, and the indices of eigenvalues are not easy to verify; see [15] for a review of such methods and the reference therein. In Liu and Oishi [15], by applying the hypercircle equation technique, the authors developed an algorithm that can provide guaranteed lower and upper eigenvalue bounds for the Laplacian. By inheriting the advantage of FEMs, such an algorithm can naturally deal with eigenvalue problem over domains of arbitrary shape.

In this paper, we extend the algorithm of Liu and Oishi [15] to more general cases defined by abstract bilinear forms. Moreover, the proposed method enables the utility of non-conforming FEMs. For non-conforming finite elements, the numerical results themselves can be lower bounds for eigenvalues if the mesh size is small enough from the asymptotic analysis (see, e.g., [24,16]). But the necessary small enough condition usually cannot be verified explicitly. In our proposed algorithm, based on the computation results of non-conforming FEMs, guaranteed lower eigenvalue bounds are possible, even for a very raw mesh. The proposed algorithm can deal with the Laplace and Biharmonic eigenvalue problems. In this paper, we focus on the Laplacian eigenvalue problems.

For the eigenvalue problem of Laplacian, the Crouzeix–Raviart finite element is adopted to give lower eigenvalue bounds (see details in Section 3): Let λ_k be the kth eigenvalue and $\lambda_{h,k}$ the kth approximate eigenvalue. A lower bound of λ_k is given as

$$\frac{\lambda_{h,k}}{1 + C_h^2 \lambda_{h,k}} \leqslant \lambda_k,$$

where C_h is a constant related to error estimation for the Crouzeix–Raviart interpolation Π_h; see definition in Section 3.1. Let the diameter of an element K be h. The constant C_h is the one to make the following estimation hold:

$$\|u - \Pi_h u\|_{0,K} \leqslant C_h |u - \Pi_h u|_{1,K}.$$

Here,

- $C_h = h/\pi$ when K is an interval in $\mathbf{R}^1$, which is an already known result;
- $C_h = 0.1893h$ for a triangle element K in $\mathbf{R}^2$;
- $C_h = 0.3804h$ for a tetrahedron element K in $\mathbf{R}^3$.

Moreover, the selection of C_h for $\mathbf{R}^1$ is optimal and the value $C_h = 0.1893h$ for $\mathbf{R}^2$ is very near to optimal.

When this research was almost finished, we found independent results of Carstensen and Gallistl [5,6], which also use non-conforming FEMs to give lower eigenvalue bounds, but a separation condition is needed. As explained in Remark 3.1, the separation condition is in fact not needed. Also, our results give better estimation of the constant C_h for eigenvalue problems of the Laplacian in the 2D case.

The remainder of this paper is organized as follows: In Section 2, we introduce the eigenvalue problem defined in an abstract form along with the main theorem that provides lower eigenvalue bounds. In Section 3, the eigenvalue problem of the Laplacian is considered in $\mathbf{R}^m (m = 1, 2, 3)$. In Section 4, an optimal estimation of the error constant C_h for 2D case is given. In Section 5, the computation results are presented. Finally, in Section 6, we state our conclusions and discuss the scope for future work.

2. Abstractly defined eigenvalue problems and lower eigenvalue bounds

Let Ω be a domain of $\mathbf{R}^m$ ($m = 1, 2, 3$). We show the assumptions for function spaces to be used in the main theorem on eigenvalue bounds.

A1 V is a Hilbert space of real function on Ω with the inner product $M(\cdot,\cdot)$ and the corresponding norm $\|\cdot\|_M := \sqrt{M(\cdot,\cdot)}$.
A2 $N(\cdot,\cdot)$ is another inner product of V. The corresponding norm $\|\cdot\|_N := \sqrt{N(\cdot,\cdot)}$ is compact for V with respect to $\|\cdot\|_M$, i.e., every sequence in V which is bounded in $\|\cdot\|_M$ has a subsequence which is Cauchy in $\|\cdot\|_N$.

To deal with conforming or non-conforming finite element spaces in eigenvalue evaluations, we further take the following assumptions.

A3 V^h is a finite dimensional space of real function over Ω, $\mathrm{Dim}(V^h) = n$ (the value of n is fixed). Notice that V^h may not be a subspace of V. Define $V(h) := V + V^h = \{v + v_h | v \in V, v_h \in V^h\}$.
A4 Bilinear forms $M_h(\cdot,\cdot)$ and $N_h(\cdot,\cdot)$ on $V(h)$ are extension of $M(\cdot,\cdot)$ and $N(\cdot,\cdot)$ to $V(h)$ such that
– $M_h(u,v) = M(u,v),\ N_h(u,v) = N(u,v)$ for all $u,\ v \in V$.
– $M_h(\cdot,\cdot)$ and $N_h(\cdot,\cdot)$ are symmetric and positive definite on $V(h)$.

The assumption A4 implies that $M_h(\cdot,\cdot)$ and $N_h(\cdot,\cdot)$ are also inner products of $V(h)$. For purpose of simplicity, the extended bilinear forms $M_h(\cdot,\cdot)$ and $N_h(\cdot,\cdot)$ are still denoted by $M(\cdot,\cdot)$ and $N(\cdot,\cdot)$ and the corresponding norms are denoted by $\|\cdot\|_M$ and

$\|\cdot\|_N$, respectively. Since $\mathrm{Dim}(V^h) < \infty$, it is easy to see $V(h)$ is also a Hilbert space along with the inner product $M(\cdot,\cdot)$ and the norm $\|\cdot\|_M$. Moreover, $\|\cdot\|_N$ is compact in $V(h)$ with respect to $\|\cdot\|_M$.

Consider the objective eigenvalue problem defined by $M(\cdot,\cdot)$ and $N(\cdot,\cdot)$ over V: Find $u \in V$ and $\lambda \in R$ such that,

$$M(u,v) = \lambda N(u,v) \quad \forall v \in V. \tag{1}$$

From arguments of compactness (see, e.g., Section 8 of Babuska [8]), the eigenpair of (1) can be denoted by $\{\lambda_k, u_k\}$ $(k = 1,2,\ldots,\infty)$ with $0 < \lambda_1 \leqslant \lambda_2 \ldots$ and $N(u_i,u_j) = \delta_{ij}$ (δ_{ij}: Kronecker's delta).

Let us define the eigenvalue problem over finite dimensional space V^h: Find $u_h \in V^h$ and $\lambda_h \in R$ such that,

$$M(u_h,v_h) = \lambda_h N(u_h,v_h) \quad \forall v_h \in V^h. \tag{2}$$

Let $\{(\lambda_{h,k},u_{h,k})\}$ $(k = 1,2,\ldots,n)$ be the eigenpair of (2) with $0 < \lambda_{h,1} \leqslant \lambda_{h,2} \ldots \leqslant \lambda_{h,n}$. The eigenvalues $\lambda_{h,k}$ can be calculated rigorously by solving a matrix eigenvalue problem.

Let $R(\cdot)$ be the Rayleigh quotient defined over $V(h)$: for $v \in V(h)$

$$R(v) := \frac{M(v,v)}{N(v,v)}.$$

Therefore, the stationary values and stationary points of R over V and V^h correspond to the eigenpairs of eigenvalue problems (1) and (2), respectively. Also, the min–max principle holds for both λ_k and $\lambda_{h,k}$:

$$\lambda_k = \min_{S^k} \max_{v\in S^k} R(v), \quad \lambda_{h,k} = \min_{S^{h,k}} \max_{v_h\in S^{h,k}} R(v_h), \tag{3}$$

where S^k and $S^{h,k}$ are k-dimensional subspaces of V and V^h, respectively.

We show the main theorem that provides lower eigenvalue bounds.

Theorem 2.1. *Let $P_h : V(h) \to V^h$ be the projection with respect to inner product $M(\cdot,\cdot)$, i.e., for any $u \in V(h)$*

$$M(u - P_h u, v_h) = 0 \quad \forall v_h \in V^h. \tag{4}$$

Suppose the following error estimation holds for P_h: for any $u \in V$,

$$\|u - P_h u\|_N \leqslant C_h \|u - P_h u\|_M. \tag{5}$$

Let λ_k and $\lambda_{h,k}$ be the ones defined in (1) and (2). Then, we have

$$\frac{\lambda_{h,k}}{1+\lambda_{h,k}C_h^2} \leqslant \lambda_k \quad (k = 1,2,\ldots,n). \tag{6}$$

Proof. Since $\|\cdot\|_N$ is compact in $V(h)$ with respect to $\|\cdot\|_M$, resulting from the argument of compactness (see Section 8 of Babuska [8]), there exists $(0 <)\bar{\lambda}_1 \leqslant \bar{\lambda}_2 \leqslant \ldots$ such that

$$\bar{\lambda}_k = \min_{S^k\subset V(h)} \max_{v\in S^k} R(v) = \max_{W\subset V(h),\dim(W)\leqslant k-1} \min_{v\in W^\perp} R(v), \tag{7}$$

where S^k denotes any k-dimensional subspace of $V(h)$; $W^\perp$ denotes the orthogonal complement of W in $V(h)$ respect to $M(\cdot,\cdot)$. Since $V \subset V(h)$, we have $\lambda_k \geqslant \bar{\lambda}_k$ due to the min–max principle. Further, by choosing W in (7) as $E_{h,k-1} := \mathrm{span}\{u_{1,h},\ldots,u_{h,k-1}\}$, a lower bound for λ_k is obtained:

$$\lambda_k \geqslant \bar{\lambda}_k \geqslant \min_{v\in E_{h,k-1}^\perp} R(v). \tag{8}$$

Let $E_{h,k-1}^{\perp,h}$ denote the orthogonal complement of $E_{h,k}$ in V^h with respect to $M(\cdot,\cdot)$, i.e., $V^h = E_{h,k-1} \oplus E_{h,k-1}^{\perp,h}$. Then $V(h)$ can be decomposed by:

$$V(h) = V^h \oplus V^{h^\perp} = E_{h,k-1} \oplus E_{h,k-1}^{\perp,h} \oplus V^{h^\perp}.$$

Moreover, we have $E_{h,k-1}^\perp = E_{h,k-1}^{\perp,h} \oplus V^{h^\perp}$. For any $v \in E_{h,k-1}^\perp$, we have

$$v = P_h v + (I - P_h)v, \quad P_h v \in E_{h,k-1}^{\perp,h}, \quad (I-P_h)v \in V^{h^\perp}.$$

Therefore, we have $\|P_h v\|_N \leqslant \lambda_{h,k}^{-1/2} \|P_h u\|_M$ by noticing that

$$\lambda_{h,k} = \min_{v\in E_{h,k-1}^{\perp,h}} R(v).$$

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Further, from the condition (5) of Theorem 2.1, we have

$$\|v\|_N \leqslant \|P_h v\|_N + \|v - P_h v\|_N \leqslant \lambda_{h,k}^{-1/2}\|P_h v\|_M + C_h\|v - P_h v\|_M,$$

which leads to

$$\|v\|_N^2 \leqslant \left(\lambda_{h,k}^{-1} + C_h^2\right)\left(\|P_h v\|_M^2 + \|v - P_h v\|_M^2\right) = \left(\lambda_{h,k}^{-1} + C_h^2\right)\|v\|_M^2.$$

Hence, we obtain

$$R(v) \geqslant \lambda_{h,k}/\left(1 + C_h^2\lambda_{h,k}\right) \quad \text{for any } v \in E_{h,k-1}^{\perp}.$$

Using (8), we can draw the conclusion in (6). □

Remark 2.1. The technique in the above proof is an extension of the one given by Liu and Oishi [14], which applies the max–min principle to develop explicit lower eigenvalue bounds along with conforming FEM spaces. In Liu and Oishi [15], another proof based on the min–max principle is given for conforming FEM spaces; the technique therein can also be used to give the same result as Theorem 2.1.

Remark 2.2. The results of Carstensen and his coauthors give same lower bounds like (6), but a so-called "separation condition" is required:

$$C_h \leqslant (\sqrt{1 + 1/k} - 1)/\sqrt{\lambda_k}.$$

As shown in the Theorem 2.1, such a condition is not needed. Also, in solving the eigenvalue problem of the Laplacian in 2D case, the estimation of C_h given by Carstensen and Gallistl [5] is $C_h \leqslant 0.43955h$, which is refined to be $C_h \leqslant 0.2983h$ in Carstensen and Gallistl [5]. Both results are rough than our estimation $C_h \leqslant 0.1893h$; see Section 5.

To obtain eigenvalue bounds by using Theorem 2.1, there are two tasks.

(a) Selection of V^h. The finite element space will be adopted to construct V^h. Moreover, we prefer to a special kind of V^h: for each K, the restriction of $P_h u$ on K, i.e., $(P_h u)|_K$, is an interpolation of u on K.
(b) The concrete value of C_h in the error estimation (5) for P_h. With special selection of P_h, the projection error estimation can be reduced to the interpolation error estimation on each element, which can be easily solved.

Upper eigenvalue bounds. The upper bounds for eigenvalues can be easily obtained if the finite dimensional space V^h is a conforming one, i.e., $V^h \subset V$. In this case, the approximate eigenvalue $\lambda_{h,k}$ of (2) gives upper bound for λ_k in (1) from the min–max principle.

3. Eigenvalue problems for Laplace operators

3.1. Preliminary

Let $\Omega \subset \mathbf{R}^m$ $(m = 1, 2, 3)$ be a bounded polyhedron domain. In the rest of this paper, we apply Theorem 2.1 to give lower bounds for the concrete eigenvalue problem of the Laplacian over Ω:

$$-\Delta u = \lambda u \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega. \tag{9}$$

Notation for function spaces. Let $L_2(\Omega)$ be the standard Lebesgue function space and $H^k(\Omega)$ $(k = 0, 1, 2, \ldots)$ the kth order Sobolev function space, which contains all functions that have up to kth order derivatives in $L_2(\Omega)$. The semi-norm and norm for function in $H^k(\Omega)$ are denoted by $|\cdot|_{k,\Omega}$ and $\|\cdot\|_{k,\Omega}$, respectively.

We take the following settings to apply Theorem 2.1:

$$V = \{v \in H^1(\Omega) | v = 0 \text{ on } \partial\Omega\}, \quad M(u, v) = \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x, \quad N(u, v) = \int_\Omega uv \,\mathrm{d}x.$$

The variational form for eigenvalue problem (9) is defined as follows: Find $u \in V$ and $\lambda \in \mathbf{R}$ such that

$$M(u, v) = \lambda N(u, v), \quad \forall v \in V. \tag{10}$$

Finite element space V^h. For an interval $I \subset \mathbf{R}^1$, $\mathbf{K}^h$ denotes the subdivision of I with subintervals. In $\mathbf{R}^m$ $(m \geqslant 2)$, let $\mathbf{K}^h$ be a regular subdivision of Ω with m-simplexes; that is, any two surfaces S_i and S_j of elements of $\mathbf{K}^h$ satisfies $S_i \cap S_j = S_i = S_j$ or $\mu(m-1, S_i \cap S_j) = 0$, where $\mu(k, \cdot)$ is the measure in R^k $(k \geqslant 1)$.

Define the finite element space V^h by

$$\begin{aligned} V^h := \{ v_h | v_h &\text{ is piecewise linear on each element } K \text{ of } \mathbf{K}^h; \\ v_h &\text{ is continuous on the centroid of each inter-element surface } S; \\ v_h &\text{ vanishes on the centroid of boundary surfaces.} \} \end{aligned} \tag{11}$$

In the 2D case, such a FEM space is just the Crouzeix–Raviart FEM space. In case of 1D space, the simplex reduces to an interval and the surfaces are just the ends of intervals. The extension of M and N to $V(h)$ is defined by

$$M_h(u,v) := \sum_{K\in\mathbf{K}^h}\int_K \nabla u\cdot\nabla v \,\mathrm{d}x, \quad N_h(u,v) := \int_\Omega uv\,\mathrm{d}x.$$

It is easy to see that the above setting of $V,\ V^h,\ M(\cdot,\cdot),\ N(\cdot,\cdot),\ M_h(\cdot,\cdot),\ N_h(\cdot,\cdot)$ satisfy the assumption A1–A4.

Interpolation Π_h. The interpolation function $\Pi_h : H^1(\Omega)\to V^h$ is defined element-wisely. For each $K\in\mathbf{K}^h$, the $m+1$ surfaces of which are denoted by S_i $(i=1,2,\ldots,m+1)$, $\Pi_h u|_K$ is a linear function such that:

$$\int_{S_i} \Pi_h u - u \,\mathrm{d}s = 0, \quad i=1,2,\ldots,m+1. \tag{12}$$

For any $v_h\in V^h$, noticing that $\partial v_h/\partial n$ is a constant function on surfaces S_i and $\Delta v_h = 0$ inside K, we have

$$\int_K \nabla(\Pi_h u - u)\cdot\nabla v_h \,\mathrm{d}x = \sum_{i=1}^{m+1}\int_{S_i}(\Pi_h u - u)\frac{\partial v_h}{\partial n}\,\mathrm{d}s - \int_K(\Pi_h u - u)\Delta v_h\,\mathrm{d}x = 0.$$

Therefore, we have the following orthogonality

$$\sum_{K\in\mathbf{K}^h}\int_K \nabla(\Pi_h u - u)\cdot\nabla v_h\,\mathrm{d}x = M_h(\Pi_h u - u, v_h) = 0 \quad \forall v_h\in V^h. \tag{13}$$

Thus the projection $P_h : V(h)\to V^h$ is nothing else but the interpolation Π_h.

In the following subsection, we will consider several model eigenvalue problems and show how to estimate C_h in (5) for 1D, 2D and 3D cases. Because the projection is also an interpolation operator, we only need to evaluate the interpolation error constant $C(K;\mathbf{R}^m)$ defined on element K:

$$\|u-\Pi_h u\|_{0,K} \leqslant C(K;\mathbf{R}^m)|u-\Pi_h u|_{1,K}. \tag{14}$$

Thus C_h can be taken as $C_h := \max_{K\in\mathbf{K}^h} C(K;\mathbf{R}^m)$.

3.2. Laplace operator in 1D

Although the eigenvalue problem in 1D space is trivial, we consider it here to show the efficiency of Theorem 2.1. Let $\Omega := I = (0,1)$. Let

$$V := \{v\in H^1(I) | v(0)=v(1)=0\}, \quad M(u,v) = \int_I u^{(1)}v^{(1)}\mathrm{d}x, \quad N(u,v) = \int_I uv\mathrm{d}x.$$

Denote by I_h the subdivision of I with the nodes $\{x_i\}_{i=0}^n : x_0(=0) < x_1 < \ldots < x_n(=1)$. V^h is the space of all continuous piece-wise linear functions that vanish at $x=0$ and $x=1$. Since $V^h\subset V$, we take $M_h := M,\ N_h := N$. The interpolation defined by (12) is merely the Lagrange interpolation $\Pi_h : V\to V^h$,

$$(\Pi_h u)(x_i) = u(x_i), \quad i=0,1,\ldots,n.$$

Error estimation for projection operator. On each sub-interval $I_i = (x_i, x_{i+1})$ $(i=0,1,\ldots,n-1)$, let $h_i := x_{i+1}-x_i$. Define Rayleigh quotient $R(v) := \int_{I_i} {v^{(1)}}^2\mathrm{d}s \Big/ \int_{I_i} v^2\mathrm{d}s$. Notice that,

$$\min_{v\in H^1(I_i)} R(\Pi_h v - v) = \min_{w\in H^1(I_i), w(0)=w(1)=0} R(w) = R\left(\sin\frac{\pi(x-x_i)}{h}\right) = \pi^2/h_i^2.$$

Thus, we have

$$\|\Pi_h u - u\|_{0,I_i} \leqslant h_i/\pi |\Pi_h u - u|_{1,I_i}. \tag{15}$$

Therefore, the optimal constant C_h for (5) is $C_h := h/\pi$ with $h = \max_i h_i$.

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3.3. Laplace operator in 2D

Let us consider the Laplace operator over a 2D polygonal domain Ω. The subdivision $\mathbf{K}^h$ for Ω is now a regular triangulation of Ω. The FEM space V^h here is the Crouzeix–Raviart FEM space with the boundary condition:

$$V^h := \{v_h | v_h \text{ is piecewise linear and continuous on mid-points of interior edges; } v_h \text{ vanishes on the mid-point of boundary edges of } \mathbf{K}^h.\}$$

Notice that V^h is a non-conforming FEM space because $V^h \neg\subset H^1(\Omega)$. Except for the above V^h, we also introduce the results based on the piecewise linear conforming FEM space $V^h_{conf} (\subset V)$. The projections that project $V(h)$ to V^h and V^h_{conf} are denoted by P_h and $P_{h,conf}$, respectively.

Conforming FEM space V^h_{conf}. The projection operator $P_{h,conf} : V \to V^h_{conf}$ cannot be an interpolation operator anymore. Also, if the domain has a re-entrant corner, there may exist singularities of the eigenfunction u_i, which makes the a priori estimation not easy to obtain. Liu and Oishi [15] considers the following boundary value problem over a polygonal domain Ω of arbitrary shape

$$-\Delta u = f \text{ in } \Omega; \quad u = 0 \text{ on } \Gamma_1; \quad \frac{\partial u}{\partial n} = 0 \text{ on } \Gamma_2. \quad (\Gamma_1 \cup \Gamma_2 = \partial\Omega, \Gamma_1 \cap \Gamma_2 = \emptyset)$$

By adopting the hyper-circle equation method, [15] gives the explicit value of C_h for the a priori error estimation:

$$\|u - P_{h,conf} u\|_{0,\Omega} \leqslant C_h |u - P_{h,conf} u|_{1,\Omega} \leqslant C_h^2 \|f\|_{0,\Omega}.$$

The calculation of C_h is reduced to solving a matrix eigenvalue problem. An example based on V^h_{conf} along with explicit values of C_h is displayed in Section 5.

Non-conforming FEM space V^h. The projection operator $P_h : V(h) \to V^h$ is just the Crouzeix–Raviart interpolation $\Pi_h : V \to V^h$: On each element K, the edges of which are denoted by e_1, e_2, e_3, for $u \in V$

$$(\Pi_h u)|_K \in P_1(K), \quad \int_{e_i} \Pi_h u - u ds = 0 \quad (i = 1, 2, 3).$$

The constant $C(K; \mathbf{R}^2)$ in (14) is given by

$$C(K; \mathbf{R}^2) = \sup_{v \in H^1(K)} \frac{\|v - \Pi_h v\|_{0,K}}{|v - \Pi_h v|_{1,K}} = \sup_{v \in V_e(K)} \frac{\|v\|_{0,K}}{|v|_{1,K}},$$

where

$$V_e(K) := \left\{ v \in H^1(K) \Big| \int_{e_i} v ds = 0, i = 1, 2, 3 \right\}. \tag{16}$$

To give an estimation of $C(K; \mathbf{R}^2)$, u is decomposed as follows.

$$u = (u - \mathrm{Avg(u)}) + \mathrm{Avg}(u) \quad (\mathrm{Avg}(u) : \text{average of } u \text{ over } K).$$

From the result of Payne and Weinberger [12], it is known that over triangle K, the longest edge length denoted by h, for $u \in H^1(K)$,

$$\|u - \mathrm{Avg(u)}\|_{0,K} \leqslant \frac{h}{3.8317} |u|_{1,K}. \tag{17}$$

The following Lemma 3.1 and Theorem 3.2 provide an estimation for $C(K; \mathbf{R}^2)$.

Lemma 3.1. *Let the three vertices of triangle K be ABC. Given $u \in H^1(K)$, $\int_{AB} u ds = 0$, we have*

$$\int_K u dx dy \leqslant 0.443 \sqrt{|K|} \max(|AC|, |BC|) \|\nabla u\|_{0,K}.$$

Proof. Suppose that AB is on the x-axis and $A = (0, 0)$. The coordinates of B and C are denoted by $(x_B, 0)$, (x_C, y_C). Particularly, let $H := y_C$.

For a point $\mathbf{x} = (x, y)$ in K, suppose the line passing through C and $\mathbf{x}$ intersects AB at $\hat{\mathbf{x}} = (\hat{x}, 0)$; see Fig. 1. Then, the point $\mathbf{x} = (x, y)$ can be represented by new parameters $(\hat{x}, s)$, that is:

$$\mathbf{x} = (x, y) = \hat{\mathbf{x}} + s(C - \hat{\mathbf{x}}) = (\hat{x} + (x_C - \hat{x})s, sH) \quad (\hat{x} \in [0, x_B], s \in [0, 1]).$$

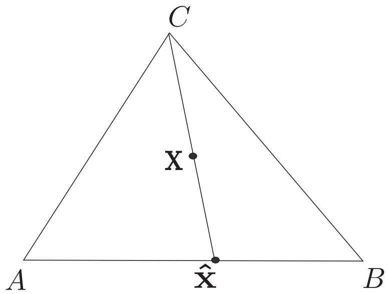

Fig. 1. New parameterization of a point in K.

Therefore, the integration of $u(\mathbf{x})$ can be represented by

$$\int_K u(\mathbf{x})\mathrm{d}x\mathrm{d}y = \int_0^{x_B}\int_0^1 u(\mathbf{x})\cdot(1-s)H\ \mathrm{d}s\ \mathrm{d}\hat{x}. \tag{18}$$

Let $\vec{\alpha}$ be the unit vector from $\hat{\mathbf{x}}$ to $\mathbf{x}$. Consider the line integral on $\hat{\mathbf{x}}\mathbf{x}$.

$$u(\mathbf{x}) = u(\hat{\mathbf{x}}) + \int_0^1 \frac{\partial u(\hat{\mathbf{x}}+t(\mathbf{x}-\hat{\mathbf{x}}))}{\partial\vec{\alpha}}\ \mathrm{d}t\cdot|\hat{\mathbf{x}}-\mathbf{x}|$$

Notice that $\int_0^{x_B} u(\hat{\mathbf{x}})\mathrm{d}\hat{x} = 0$. Substitute (3.3) into (18); we have:

$$\int_K u\ \mathrm{d}x\ \mathrm{d}y = \int_0^{x_B}\int_0^1\int_0^1 \frac{\partial u(\hat{\mathbf{x}}+t(\mathbf{x}-\hat{\mathbf{x}}))}{\partial\vec{\alpha}}\ \mathrm{d}t\cdot|\hat{\mathbf{x}}-\mathbf{x}|\cdot(1-s)H\ \mathrm{d}s\ \mathrm{d}\hat{x}.$$

Notice that $|\mathbf{x}-\hat{\mathbf{x}}| = s|C-\hat{\mathbf{x}}| \leqslant s\cdot\max(|AC|,|BC|)$. Let $\tilde{t} = s\cdot t$, then

$$\int_K u\ \mathrm{d}x\ \mathrm{d}y \leqslant \max(|AC|,|BC|)\int_0^{x_B}\int_0^1\int_0^s \frac{\partial u(\hat{\mathbf{x}}+\tilde{t}(C-\hat{\mathbf{x}}))}{\partial\vec{\alpha}}\ \mathrm{d}\tilde{t}\cdot(1-s)H\ \mathrm{d}s\ \mathrm{d}\hat{x}. \tag{19}$$

Denote by $\tilde{\mathbf{x}} := \hat{\mathbf{x}}+\tilde{t}(C-\hat{\mathbf{x}})$. Apply the Cauchy–Schwarz inequality,

$$\int_0^s \frac{\partial u(\tilde{\mathbf{x}})}{\partial\vec{\alpha}}\ \mathrm{d}\tilde{t} \leqslant \left\{\int_0^s\left(\frac{\partial u(\tilde{\mathbf{x}})}{\partial\vec{\alpha}}\right)^2 H\cdot(1-\tilde{t})\ \mathrm{d}\tilde{t}\right\}^{1/2}\left\{\int_0^s\frac{\mathrm{d}\tilde{t}}{H\cdot(1-\tilde{t})}\right\}^{1/2} \leqslant \left\{\int_0^1\left(\frac{\partial u(\tilde{\mathbf{x}})}{\partial\vec{\alpha}}\right)^2 H\cdot(1-\tilde{t})\ \mathrm{d}\tilde{t}\right\}^{1/2}\sqrt{\frac{-\log(1-s)}{H}}.$$

Thus,

$$\begin{aligned}\int_0^1\int_0^s \frac{\partial u(\tilde{\mathbf{x}})}{\partial\vec{\alpha}}\ \mathrm{d}\tilde{t}(1-s)H\ \mathrm{d}s &\leqslant \int_0^1\left\{\int_0^1\left(\frac{\partial u(\tilde{\mathbf{x}})}{\partial\vec{\alpha}}\right)^2 H\cdot(1-\tilde{t})\ \mathrm{d}\tilde{t}\right\}^{1/2}\sqrt{\frac{-\log(1-s)}{H}}(1-s)H\ \mathrm{d}s\\ &= \left\{\int_0^1\left(\frac{\partial u(\tilde{\mathbf{x}})}{\partial\vec{\alpha}}\right)^2 H\cdot(1-\tilde{t})\ \mathrm{d}\tilde{t}\right\}^{1/2}\cdot G_0\cdot\sqrt{H},\end{aligned} \tag{20}$$

where $G_0 = \int_0^1\sqrt{\log(1-s)}(1-s)\ \mathrm{d}s \approx 0.3133$.

Substitute (20) into (19),

$$\begin{aligned}\int_K u\ \mathrm{d}x\ \mathrm{d}y &\leqslant \max(|AC|,|BC|)\int_0^{x_B}\left\{\int_0^1\left(\frac{\partial u(\tilde{\mathbf{x}})}{\partial\vec{\alpha}}\right)^2 H\cdot(1-\tilde{t})\ \mathrm{d}\tilde{t}\right\}^{1/2} G_0\sqrt{H}\ \mathrm{d}\hat{x}\\ &\leqslant \max(|AC|,|BC|)\left\{\int_0^{x_B}\int_0^1\left(\frac{\partial u(\tilde{\mathbf{x}})}{\partial\vec{\alpha}}\right)^2 H\cdot(1-\tilde{t})\ \mathrm{d}\tilde{t}\ \mathrm{d}\hat{x}\right\}^{1/2} G_0\sqrt{H}\sqrt{|AB|} = \sqrt{2}G_0\sqrt{|K|}\max(|AC|,|BC|)\cdot|u|_{1,K}.\end{aligned}$$

Therefore,

$$\int_K u\ \mathrm{d}x\ \mathrm{d}y \leqslant 0.443\sqrt{|K|}\max(|AC|,|BC|)|u|_{1,K}. \qquad \square$$

Theorem 3.2. *Given* $u\in V_e(K)$, *the following error estimation holds*

$$\|u\|_{0,K} \leqslant 0.346h|u|_{1,K}.$$

Thus, we have $C(K;\mathbf{R}^2) \leqslant 0.346h$.

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Proof. From Lemma 6.1 in the appendix, we can select a $P \in K$ such that

$$\max(|PA|, |PB|, |PC|) \leqslant \frac{\sqrt{3}}{3} h.$$

Denote by K_1, K_2, and K_3 the three sub-triangles of K. Apply Lemma 3.1 to the integration of u on each sub-triangle K_i. Then,

$$\int_K u \, dx \, dy \leqslant 0.4431 \frac{\sqrt{3}}{3} h \sum_{i=1}^{3} \sqrt{|K_i|} \cdot \|\nabla u\|_{0,K_i} \leqslant 0.25582 h \sqrt{|K|} \cdot |u|_{1,K}.$$

Therefore, by further using the estimation in (17), we have

$$\|u\|_{0,K}^2 = \|u - \mathrm{Avg}(u)\|_0^2 + \|\mathrm{Avg}(u)\|_0^2 \leqslant \left(\frac{1}{3.8317^2} + 0.22582^2 \right) h^2 |u|_{1,K}^2 < (0.346 h |u|_{1,K})^2. \qquad \square$$

Remark 3.1. The estimation for $C(K; \mathbf{R}^2)$ is also found in Carstensen and Gedicke [6], where the bound is $C(K; \mathbf{R}^2) \leqslant \sqrt{0.1932} h \approx 0.4396 h$ and [5], where the bound is $C(K; \mathbf{R}^2) \leqslant 0.2983 h$.

Remark 3.2. Both the projection P_h and $P_{h,conf}$ introduced here have explicit projection error estimation, which does not involve the second derivatives of the eigenfunction. Therefore, even for the eigenvalue problem with singularities, explicit lower bounds and upper bounds for the eigenvalues are possible.

3.4. Laplace operator in 3D

Given a domain Ω in $\mathbf{R}^3$, we consider the tetrahedra mesh of Ω. The FEM space V^h is here the 3D version of Crouzeix–Raviart FEM space. To give a rough bound for $C(K; \mathbf{R}^3)$, we first provide a similar result to Lemma 3.1.

Lemma 3.3. *Given a tetrahedron K with vertices as ABCD. Suppose on the boundary triangle ABC, $u \in H^1(K)$ satisfies $\int_{ABC} u ds = 0$. Then,*

$$\int_K u dx \leqslant \frac{\sqrt{3}\pi}{16} \max(|DA|, |DB|, |DC|) \cdot \sqrt{|K|} \cdot |u|_{1,K}.$$

Proof. The technique we use is almost the same as the one in Lemma 3.1. We only show the sketch. Let consider one surface $S = ABC$ and its opposite vertex D. Suppose S is on the xy-plane. Let H be the height of D. Given a point $\mathbf{x} = (x, y, z) \in K$, define $\hat{\mathbf{x}} = (\hat{x}, \hat{y})$ by the intersection of S and the line passing $\mathbf{x}$ and D. Then $\mathbf{x}$ can be reparameterized by $\mathbf{x} = \hat{\mathbf{x}} + s(D - \hat{\mathbf{x}})$. We have

$$\int_K u \, dx \, dy \, dz = \int_S \int_0^1 u(1-s)^2 H ds dS,$$

where dS is an infinitesimal integration on surface S.

Then, by analogous arguments as Lemma 3.1, we can show that

$$\mathrm{Avg}(u) := \int_K u dx dy dz \leqslant \sqrt{3}\sqrt{K} G_1 \max(|DA|, |DB|, |DC|) \|\nabla u\|_{0,K},$$

where

$$G_1 = \int_0^1 \left\{ \int_0^s \frac{1}{(1-t)^2} dt \right\}^{1/2} (1-s)^2 ds = \int_0^1 s^{1/2} (1-s)^{3/2} ds = \frac{\pi}{16}. \qquad \square$$

Theorem 3.4. *Let the maximum edge length of tetrahedron K be h, then $C(K; \mathbf{R}^3) \leqslant 0.3804 h$, that is,*

$$\|\Pi_h u - u\|_{0,K} \leqslant 0.3804 h |\Pi_h u - u|_{1,K}.$$

Proof. Let $v = \Pi_h u - u$, then on each surface of K, we have $\int_S v dS = 0$. From Lemma 6.2 in Appendix, we can choose $P \in K$ such that

$$\max(|PA|, |PB|, |PC|, |PD|) \leqslant \frac{\sqrt{6}}{4} h.$$

For the sub-tetrahedrons $K_1 := PABC,\ K_2 := PABD,\ K_3 := PACD,\ K_4 := PBCD$, apply Lemma 3.3 to each K_i then we have:

$$\int_K u dx = \sum_{i=1}^{4}\int_{K_i} u dx \leqslant \frac{\sqrt{3}\pi}{16}\frac{\sqrt{6}}{4}h\sum_{i=1}^{4}\sqrt{|K_i|}|u|_{1,K_i} \leqslant \frac{3\sqrt{2}\pi}{64}h\sqrt{|K|}|u|_{1,K}.$$

The result of Payne and Weinberger, Bebendorf [12,2] shows that $\|u - \mathrm{Avg}(u)\|_{0,K} \leqslant 1/\pi h|u|_{1,K}$. Thus,

$$\|u\|_{0,K}^2 = \|u - \mathrm{Avg}(u)\|_{0,K}^2 + \|\mathrm{Avg}(u)\|_{0,K}^2 \leqslant \left(\frac{1}{\pi^2} + \frac{9\pi^2}{2048}\right)h^2|u|_{1,K}^2 \leqslant (0.3804h|u|_{1,K})^2. \qquad \square$$

4. Optimal estimation of $C(K)$ for 2D case

The constant $C(K)$ is determined by solving eigenvalue problems. In this section, we apply the lower bound provided by Theorem 2.1 to give a sharp bound for constant $C(K)$. For the 2D case, an optimal bound for $C(K)$ is obtained as $C(K) \leqslant 0.1893h$.

The optimal estimation is done in two steps. First, we evaluate $C(K)$ for several reference elements by verified computation. Second, the variation of constant $C(K)$ corresponding to the perturbation of reference elements is estimated by theoretical analysis.

4.1. Evaluation of $C(K)$ for reference elements

Given an element K, the edges of which are denoted by $e_1,\ e_2$ and e_3, recall the definition of $V_e(K)$ in (16). Define the eigenvalue problem for $C(K)$: Find $u \in V_e(K)$ and $\lambda > 0$ such that

$$\int_K \nabla u \cdot \nabla v \,\mathrm{d}x = \lambda \int_K uv \,\mathrm{d}x \quad \forall v \in V_e(K). \tag{21}$$

Let $\lambda_1 \leqslant \lambda_2 \ldots$ be the eigenvalues of (21). Then $C(K)$ is given by $\lambda_1^{-1/2}$. Define a Raviart–Thomas FEM space V_e^h over a regular triangulation $\mathbf{T}^h$ of K:

$$V_e^h := \{v_h | v_h \text{ is linear on each element } T \text{ of } \mathbf{T}^h;\ v_h \text{ is continuous on mid-points of interior edges; } \int_{e_i} v^h ds = 0, i = 1,2,3\} \tag{22}$$

The eigenvalue problem in V_e^h is: Find $u_h \in V_e^h$ and $\lambda_h > 0$ such that

$$\sum_{T\in\mathbf{T}^h}\int_T \nabla u_h \cdot \nabla v_h \,\mathrm{d}x = \lambda_h \int_K u_h v_h \,\mathrm{d}x \quad \forall v_h \in V_e^h. \tag{23}$$

From Theorems 2.1 and 3.2, it is known that the first eigenvalue λ_1 of (21) and the first eigenvalue $\lambda_{1,h}$ of (23) have the relation:

$$\lambda_1 \geqslant \lambda_{1,h}/\left(1 + (0.346h)^2\lambda_{1,h}\right).$$

4.2. Variation of $C(K)$ on perturbation of elements

Suppose the three vertices of a triangle element K are $O = (0,0),\ A = (1,0)$, and $B = (a,b)$. We will find the maximum value of $C(K)$ for all possible B such that $|OB| \leqslant 1$ and $|AB| \leqslant 1$. Due to the symmetry of the parameters a and b, we only consider the case that $a \geqslant 1/2$ and $a^2 + b^2 \leqslant 1$; see Fig. 2.

First, we consider the dependency of $C(K)$ on the y-coordinate of B.

Theorem 4.1. *For fixed x-coordinate of B, the constant $C(K)$ is a monotonically increasing on the y-coordinate of vertex B. Therefore, the maximum value of $C(K)$ must be taken when B is on the arc such that $r = 1, \theta \in [0, \pi/3]$.*

Proof. Denote by $\tilde{K}$ the triangle with three vertices $O = (0,0),\ A = (1,0)$, and $\tilde{B} = (a, b+\epsilon)$ $(\epsilon > 0)$. Consider the transform from K to $\tilde{K}$: $(x,y) \in K \to (\tilde{x},\tilde{y}) = (x, (b+\epsilon)/by) \in \tilde{K}$. Let $\alpha = (b+\epsilon)/b(> 1)$. Notice that for v over $\tilde{K}$,

$$\|v\|_{0,\tilde{K}}^2 = \alpha^2\|v\|_{0,K}^2, \quad \|\nabla v\|_{0,\tilde{K}}^2 = \alpha^2\left(\left\|\frac{\partial v}{\partial x}\right\|_{0,K}^2 + \frac{1}{\alpha^2}\left\|\frac{\partial v}{\partial y}\right\|_{0,K}^2\right) \leqslant \alpha^2\|\nabla v\|_{0,K}^2.$$

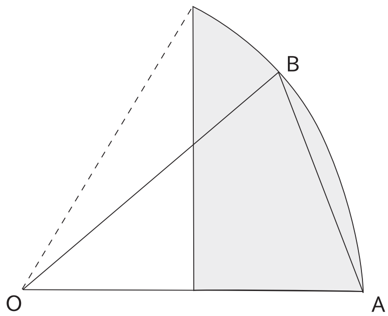

Fig. 2. Possible shapes of OAB.

Thus,

$$C(\tilde{K}) = \sup_{v\in V_e(\tilde{K})} \frac{\|v\|_{0,\tilde{K}}}{\|\nabla v\|_{0,\tilde{K}}} \geqslant \sup_{v\in V_e(K)} \frac{\|v\|_{0,K}}{\|\nabla v\|_{0,K}} = C(K).$$

Therefore, $C(K)$ is monotonically increasing on the y-coordinate of B. □

From Theorem 4.1, we only need to consider the case $B = (\cos\theta, \sin\theta)$, $0 < \theta \leqslant \pi/3$. Next, we consider the perturbation of $C(K)$ along θ direction.

Theorem 4.2. *For $0 < \theta < \pi/3$, let $\tilde{B} = (\cos(\theta+\tau), \sin(\theta+\tau))$ be a perturbation of $B = (\cos\theta, \sin\theta)$. Then, for $\tau < 0$ and $\theta + \tau \geqslant 0$,*

$$C(\tilde{K}) \leqslant \frac{\cos(\theta/2+\tau/2)}{\cos(\theta/2)} C(K).$$

For $\tau > 0$ and $\theta + \tau \leqslant \pi/3$, we have

$$C(\tilde{K}) \leqslant \frac{\sin(\theta/2+\tau/2)}{\sin(\theta/2)} C(K).$$

Proof. The transform that maps (x, y) in $K = OAB$ to $(\tilde{x}, \tilde{y})$ in $\tilde{K} = OA\tilde{B}$ is

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = Q \begin{pmatrix} x \\ y \end{pmatrix}, \quad \text{with } Q = \begin{pmatrix} 1 & (\cos(\theta+\tau) - \cos\theta)/\sin\theta \\ 0 & \sin(\theta+\tau)/\sin\theta \end{pmatrix}.$$

Thus, $(u_x, u_y) = (u_{\tilde{x}}, u_{\tilde{y}})Q$ and

$$\lambda_{min}(QQ^T) \cdot \left(v_{\tilde{x}}^2 + v_{\tilde{y}}^2\right) \leqslant \left(v_x^2 + v_y^2\right) \leqslant \lambda_{max}(QQ^T) \cdot \left(v_{\tilde{x}}^2 + v_{\tilde{y}}^2\right),$$

where $\lambda_{min}(QQ^T)$ and $\lambda_{max}(QQ^T)$ denote the minimum and maximum eigenvalues of QQ^T, respectively. For $\tau < 0$, we have

$$\lambda_{\min}(QQ^T) = \frac{\sin^2(\theta/2+\tau/2)}{\sin^2(\theta/2)}, \quad \lambda_{\max}(QQ^T) = \frac{\cos^2(\theta/2+\tau/2)}{\cos^2(\theta/2)}.$$

Notice that $dx\, dy = \sin\theta/\sin(\theta+\tau)\, d\tilde{x}\, d\tilde{y}$. Thus, for $\tau < 0$,

$$C(\tilde{K}) = \sup_{v\in V_e(\tilde{K})} \frac{\|v\|_{0,\tilde{K}}}{\|\nabla v\|_{0,\tilde{K}}} \leqslant \frac{\cos(\theta/2+\tau/2)}{\cos(\theta/2)} \sup_{v\in V_e(K)} \frac{\|v\|_{0,K}}{\|\nabla v\|_{0,K}}.$$

Similarly, if $\tau > 0$ and $\theta + \tau \leqslant \pi/3$,

$$C(\tilde{K}) \leqslant \frac{\sin(\theta/2+\tau/2)}{\sin(\theta/2)} C(K). \qquad \square$$

4.3. Verification of upper bound for $C(K)$

The approximate computation results imply that $C(K)$ is not monotone for $\theta \in (0, \pi/3]$ and the maximum value is taken at $\theta = \pi/3$. In this sub-section, we prove that $C(K) \leqslant 0.1893$ for all $\theta \in (0, \pi/3]$.

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Define θ_i and the perturbation τ_i as follows:

$$\theta_i = \pi/3 * \begin{cases} i * 0.02 & i = 1, \dots, 48 \\ 0.95 + 0.05(1 - 2^{i-50}), & i = 49, \dots, 59 \\ 1 & i = 60 \end{cases} \quad (24)$$

The perturbation τ_i is selected as follows

$$\tau_1 = \theta_1; \quad \tau_i = \theta_i - \theta_{i-1}, \quad i = 2, \dots, 60. \quad (25)$$

Then, we have, $(0, \pi/3] \subset \cup_{i=1}^{60} (\theta_i - \tau_i, \theta_i]$. Next, we bound $C(K)$ in two steps.

(a) Evaluate the constants for each θ_i defined in (24) by using a non-conforming FEM with a rough bound for C_h as $C_h \leqslant 0.384h$.
(b) For each θ_i, calculate the range of constant $C(K)$ for $(\theta_i - \tau_i, \theta_i]$ by Theorem 4.2.

The computation is executed with a triangulation $h = 1/64$; see Fig. 3 for a sample mesh with $h = 1/8$. To bound the rounding error in the floating number computation, we use INTLAB library developed by Rump [21], a MATLAB toolbox for interval arithmetic computation. The method of Behnke [3] is adopted to give verified bounds for eigenvalues of the generalized matrix eigenvalue problem. Thus the computation bounds for the constants are guaranteed results.

Fig. 4 displays the upper bound of $C(K)$ for each $(\theta_i - \tau_i, \theta_i]$. The x-coordinate is the angle size of $\angle AOB$, which varies in $(0, \pi/3]$, and the y-coordinate is the value or upper bounds of $C(K)$. Verified computation results show that for each $(\theta_i - \tau_i, \theta_i], C(K)$ has an upper bound as $C(K) \leqslant 0.18928$. For $\theta = \pi/3$, by using conforming FEMs, we have $C(K) \geqslant 0.1890$. Also, on $(0, 0.02\pi]$, $C(K)$ has an upper bound to be $C(K) \leqslant 0.1888$.

5. Computation results over an L-shaped domain

In this section, we apply Theorem 2.1 to the eigenvalue problem of Laplacian over an L-shaped domain. Both the uniform mesh and the non-uniform mesh are used for eigenvalue bounds computation. Note that the mesh size h is defined by the longest edge length of a triangulation. The left one in Fig. 5 is a sample uniform mesh with $h = \sqrt{2}/4$. The non-uniform mesh is a geometrically graded one: near the re-entrant corner of domain, the diameter of K, denoted by $h(K)$, is about $h(K) = O(r^{1/3})$, where r is the distance of the element K to the corner.

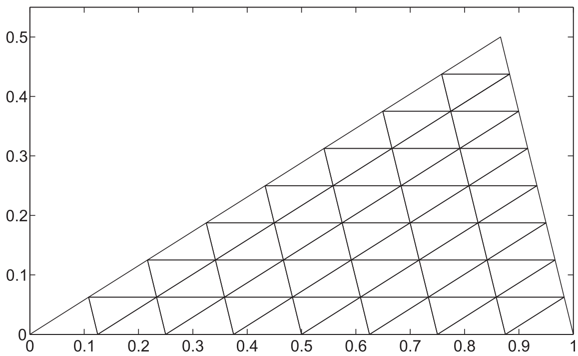

Fig. 3. Triangulation of triangle OAB ($B = (\cos \pi/6, \sin \pi/6)$) with $h = 1/8$.

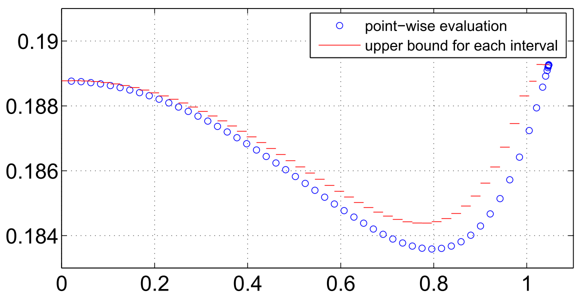

Fig. 4. Point-wise evaluation of $C(K)$ for each θ_i and the upper bound of $C(K)$ for $\theta \in (\theta_i - \tau_i, \theta_i]$.

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The lower bounds for the leading 5 eigenvalues are obtained by using Theorem 2.1, along with the value of C_h taken as $C_h = 0.1893h$.

We also display the lower bounds by using conforming linear FEMs from [15]. For a uniform mesh with $h = \sqrt{2}/4$, [15] shows that the constant C_h for the projection error estimation is $C_h = 0.00348$; for the same non-uniform mesh as in Liu and Oishi [15], $C_h = 0.0337$ is used for the results in Table 3.

By comparing the results of non-conforming FEMs and conforming FEMs in Tables 1–3, we can see that the lower bounds based on non-conforming FEMs give better results. Moreover, to give explicit value of C_h for the conforming FEM, we need to solve an eigenvalue problem with dense matrices, which is quite time-consuming work. But, for the Crouzeix–Raviart FEM, the value of constant $C_h = 0.1829h$ is available only if the mesh size h is known.

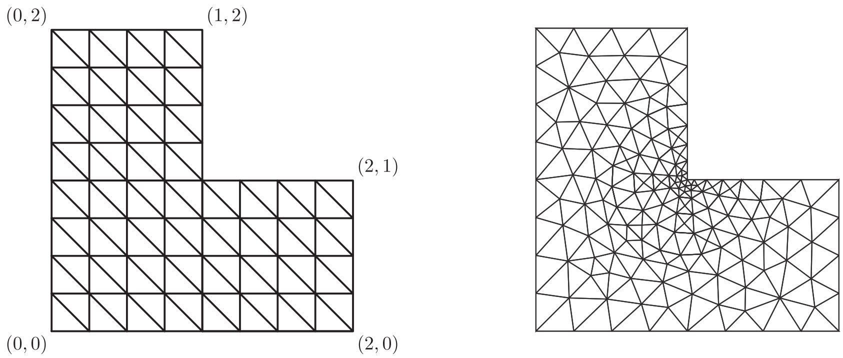

Fig. 5. Uniform and non-uniform meshes for an L-shaped domain.

Table 1
Results of nonconforming FEM (uniform mesh with $h = \sqrt{2}/32,\ C_h = 0.00837$).

λ_i	Lower bound	Approx.	Exact.	ReErr
1	9.6090	9.6155	9.63972	0.0032
2	15.1753	15.1915	15.1973	0.0014
3	19.7067	19.7339	19.7392	0.0016
4	29.4395	29.5003	29.5215	0.0028
5	31.7618	31.8326	31.9126	0.0047

Table 2
Results of conforming FEM (uniform mesh with $h = \sqrt{2}/32,\ C_h = 0.0348$).

λ_i	Lower bound	Exact	Upper bound	ReErr
1	9.5578	9.63972	9.6699	0.008
2	14.949	15.1973	15.225	0.016
3	19.323	19.7392	19.787	0.021
4	28.599	29.5215	29.626	0.031
5	30.859	31.9126	32.058	0.033

Table 3
Results of FEMs on non-uniform mesh (total 3748 elements).

λ_i	Non-conforming FEM ($C_h = 0.0163$)			Conforming FEM ($C_h = 0.0337$)		
	Lower	Approx.	ReErr	Lower	Upper	ReErr
1	9.6033	9.6277	0.0038	9.5284	9.6592	0.0116
2	15.1204	15.1811	0.0051	14.9107	15.2335	0.0189
3	19.6091	19.7113	0.0066	19.2619	19.8041	0.0242
4	29.2378	29.4656	0.0096	28.4643	29.6645	0.0358
5	31.5741	31.8398	0.0106	30.6863	32.0856	0.0384

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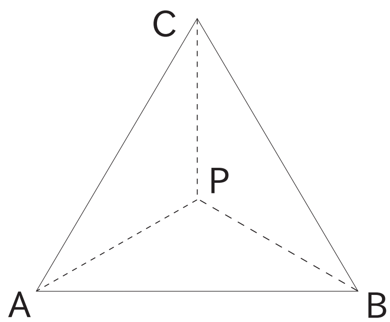

Fig. 6. Triangle K.

6. Summary

In this paper, we propose a universal framework that provides lower eigenvalue bounds for the self-adjoint eigenvalue problems. By adopting the non-conforming finite element, the eigenvalue bounds of Laplacian over domain of general shapes can be easily obtained. A disadvantage about the usage of non-conforming FEM is that, the good property that the projection operator P_h is just a locally defined interpolation operator Π_h cannot be expected for general elliptic differential operators. Thus, one need to pay more efforts to the explicit error estimation of P_h. Another choice is to apply the homotopy method and take the eigenvalue problem of $-\Delta$ as the base problem.

In the future work, we will also apply the framework in Theorem 2.1 to discuss the eigenvalue problems of the Biharmonic operator, Stoke's operator and Maxwell's operators.

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Appendix A

In this appendix, we give two lemmas that have been used in the proof for Theorem 3.2 and Theorem 3.4.

Lemma 6.1. *Let the nodes of a triangle K be A, B and C, and the longest edge length of K be h. Then, the following inequality holds (see Fig. 6)*

$$\min_{P\in K}\ \max(|PA|,|PB|,|PC|) \leqslant \frac{\sqrt{3}}{3}h.$$

Moreover, the equality holds only for a regular triangle with P being the circumcenter of K.

Proof. To give a concise expression, define $f(K,P) := \max(|PA|,|PB|,|PC|)$. The proof is given in two steps.

Step 1: Given a triangle K. Suppose P_0 minimizes $f(K,P)$, i.e.,

$$\min_{P\in K}\ \max(|PA|,|PB|,|PC|) = f(K,P_0). \tag{26}$$

Next, by using the method of reduction to absurdity, we show that[1]

(a) If P_0 is an interior point, then P_0 is the circumcenter of K.
(b) If P_0 is on the boundary of K, then P_0 is the midpoint of the longest edge.

Let us consider the case that P_0 is an interior point. Without loss of generality, we assume $|P_0A| \geqslant |P_0B| \geqslant |P_0C|$. Suppose $|P_0A| > \max(|P_0B|,|P_0C|)$. Move P_0 toward A along the direction perpendicular to BC. Then, we can have $P' \in K$ such that

$$|P_0A| > |P'A| > \max(|P'B|,|P'C|) > \max(|P_0B|,|P_0C|),$$

[1] The point P_0 is called by proximity point of A, B and C in Rademacher and Schoenberg [20], and properties like (a) and (b) of P_0 are stated therein but without a proof.

which implies $f(K,P')$ gives a smaller value than $f(K,P_0)$, leading to a contradiction to the definition of P_0. Similarly, suppose $|P_0A| = |P_0B| > |P_0C|$. Then $f(K,P)$ can be smaller by moving P toward edge AB along the direction perpendicular to AB, which is also a contradiction. Thus $|P_0A| = |P_0B| = |P_0C|$.

For the case $P_0 \in \partial K$, without loss of generality, suppose $P_0 \in AB$. First, we declare $|P_0C| \leqslant \max(|P_0A|, |P_0B|)$. If not so, then $f(K,P_0) = |P_0C|$. Move P_0 toward to C along the direction perpendicular to AB, and a smaller value $f(K,P)$ is possible, which is a contradiction. Next, the equality $|P_0A| = |P_0B|$ holds from analogous argument using perturbation. Notice that $|P_0A| = |P_0B| \geqslant |P_0C|$ implies that K is an obtuse-angle triangle or a right triangle and AB is the longest edge.

Step 2: We show that among all triangles of arbitrary shapes, a regular triangle K gives the maximum value for

$$g(K) := \min_{P \in K} \max(|PA|, |PB|, |PC|)/h. \tag{27}$$

For the triangle that has P_0 on boundary, it is easy see the value of (27) is $1/2$. Next, we only consider the case that P_0 is an interior point of K. Moreover, K should be an acute triangle. Let the triangle K_0 be the one optimizing $g(K)$. If $|AB| > |AC|$, then we can keep the length of BC and move C to C' such that $|AC'| > |AC|$ while $|AC'| \leqslant |AB|$; see Fig. 7. For the new triangle $K' = ABC'$, it is easy to see $g(K') > g(K_0)$, which also leads to a contradiction. Thus, $|AB| = |BC| = |AC|$. For such a K_0, we have $g(K_0) = \frac{\sqrt{3}}{3}h$. □

With analogous argument, we can have a similar result as Lemma 6.1 in the 3D case, for which only the sketch of the proof is given.

Lemma 6.2. *For any tetrahedron K, the vertices of which are denoted by A, B, C, D, let h be the longest edge length. Then,*

$$\min_{P \in K} \max(|PA|, |PB|, |PC|, |PD|) \leqslant \frac{\sqrt{6}}{4} h.$$

Moreover, the equality holds only for a regular tetrahedron K with P selected as the circumcenter of K; see Fig. 8.

Proof. Let us introduce the function $f(K,P) = \max(|PA|, |PB|, |PC|, |PD|)$. The proof is given in two steps.

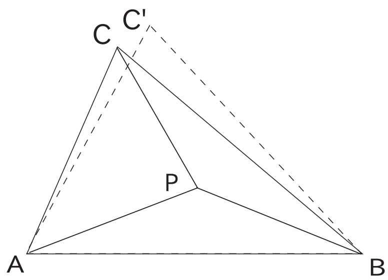

Fig. 7. Perturbation of vertex C for K.

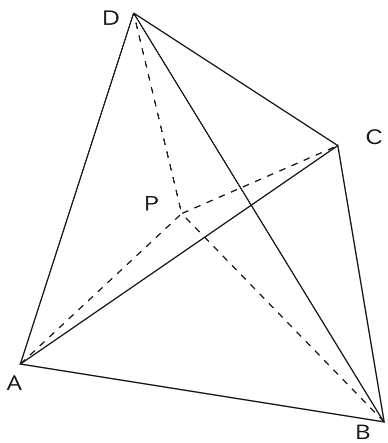

Fig. 8. Tetrahedron element K.

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Step 1: For any tetrahedron K, let P_0 be the point that minimizes $f(K,P)$. Then there are two possibilities of P_0:

(a) P_0 is an interior point of K. (b) P_0 is on one of the surfaces of K.

For case (a), P_0 must be the circumcenter of K, i.e., $|P_0A| = |P_0B| = |P_0C| = |P_0D|$. Otherwise, we can take a perturbation of P_0 to have a smaller value of $f(K,P)$, which will lead to a contradiction with the assumption of P_0. Without loss of generality, assume $|P_0A| \geqslant |P_0B| \geqslant |P_0C| \geqslant |P_0D|$. The perturbation of P_0 can be done by considering three cases.

1. $|P_0A| > \max(|P_0B|, |P_0C|, |P_0D|)$: move P_0 toward BCD along the direction perpendicular to BCD.
2. $|P_0A| = |P_0B| > \max(|P_0C|, |P_0D|)$: move P_0 toward segment CD along the direction perpendicular to AB and CD.
3. $|P_0A| = |P_0B| = |P_0C| > |P_0D|$: move P_0 toward D along the direction perpendicular to ABC.

For case (b), suppose $P_0 \in ABC$. By using the technique of perturbation, we can show that $|P_0D| \leqslant \max(|P_0A|, |P_0B|, |P_0C|)$ and P_0 also gives the minimal value of $\max(|PA|, |PB|, |PC|)$, i.e.,

$$|P_0D| \leqslant \min_{P\in ABC} \max(|PA|, |PB|, |PC|) = \max(|P_0A|, |P_0B|, |P_0C|).$$

Step 2: We show that a regular tetrahedron gives the maximal value of

$$\min_{P\in K} f(K,P). \tag{28}$$

For tetrahedra in case (b), the value of (28) is as maximal as $\frac{\sqrt{3}}{3}h (< \frac{\sqrt{6}}{4}h)$. Therefore, we only need to consider the tetrahedron in case (a). Suppose K_0 maximizes the value of $\min_{P\in K} f(K,P)$ among all kinds of tetrahedra, while K_0 is not a regular tetrahedron. Suppose edge e of K_0 has its length less than h. Then we can make a perturbation of K_0 to make e longer while keeping the length for all other edges. This way, we can obtain a new simplex K' such that $\min_{P\in K'} f(K',P) > \min_{P\in K_0} f(K_0,P)$, which is a contradiction to the assumption of K_0. For a regular tetrahedron K, it is easy to calculate the distance of the circumcenter to each vertex is $\frac{\sqrt{6}}{4}h$. □

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