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Xuefeng Liu & Shin'ichi Oishi

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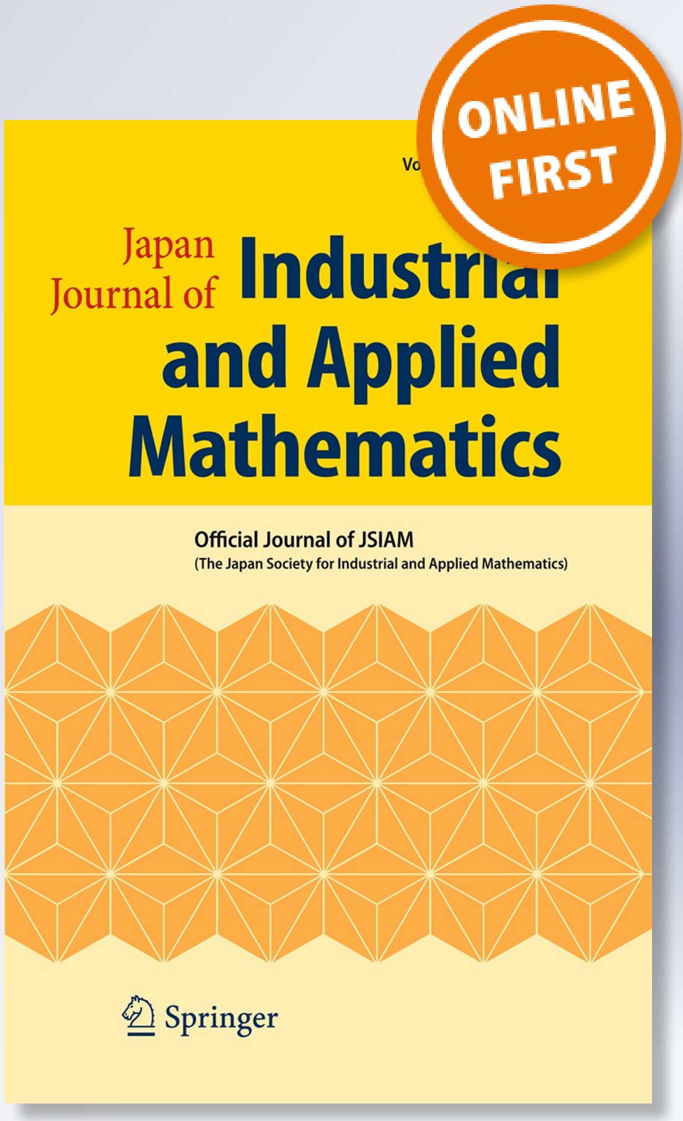

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ORIGINAL PAPER

Area 1

Guaranteed high-precision estimation for P_0 interpolation constants on triangular finite elements

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Abstract We consider an explicit estimation for error constants from two basic constant interpolations on triangular finite elements. The problem of estimating the interpolation constants is related to the eigenvalue problems of the Laplacian with certain boundary conditions. By adopting the Lehmann–Goerisch theorem and finite element spaces with a variable mesh size and polynomial degree, we succeed in bounding the leading eigenvalues of the Laplacian and the error constants with high precision. An online demo for the constant estimation is also available at http://www.xfliu.org/onlinelab/.

Keywords Interpolation error constants · Eigenvalue problem · Finite element method · Lehmann–Goerisch theorem · hp-FEM

Mathematics Subject Classification 35P15 · 65N30 · 65N25 · 65N15

1 Introduction

The finite element method (FEM) has received considerable attention from both mathematicians and engineers in the past decades. This is due to its sound mathematical

X. Liu (✉)
Research Institute for Science and Engineering, Waseda University,
3-4-1 Okubo.Shinjuku-ku, Tokyo 169-8555, Japan
e-mail: xfliu.math@gmail.com

S. Oishi
Faculty of Science and Engineering, Waseda University,
3-4-1 Okubo, Shinjuku-ku, Tokyo 169-8555, Japan
e-mail: oishi@waseda.jp

S. Oishi
CREST/JST, Kawaguchi, Japan

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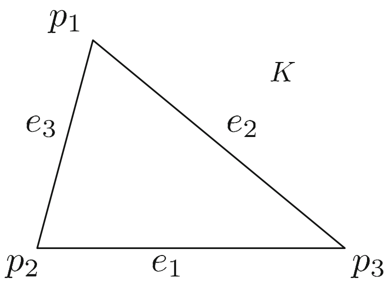
Fig. 1 Triangle K



foundations, such as a priori and a posteriori error estimations, as well as its practicality in solving problems in industry.

In the classical error analysis of FEM, besides the discretization parameter h, various interpolation constants appear in the final error estimation. Usually, the boundedness of such constants is enough for a qualitative error analysis. However, in many cases, such as adaptive FEM and the associated computations, such constants must be evaluated as precisely as possible [1,20]. In this paper, we propose an algorithm that gives a high-precision bound for the interpolation constants appearing in two constant interpolations over a triangular finite element. The algorithm has the potential to be extended to interpolations of higher order.

Let K be any triangle with vertices p_1, p_2, and p_3. The edges of K are denoted by e_1, e_2, and e_3 (see Fig. 1). Let $L_2(K)$ and $H^1(K)$ be the standard Lebesgue and Sobolev function spaces over K. We consider two constant interpolations $\Pi_1^{(0)}$ and $\Pi_2^{(0)}$:

$\Pi_1^{(0)}$: For u in $L_2(K)$, $\Pi_1^{(0)} u$ is a constant such that

$$\Pi_1^{(0)} u = \int_K u dx / |K| \quad (|K| : \text{area of K}) ;$$

$\Pi_2^{(0)}$: For u in $H^1(K)$, $\Pi_2^{(0)} u$ is a constant such that

$$\Pi_2^{(0)} u = \int_{e_1} u ds / |e_1| \quad (|e_1| : \text{length of } e_1).$$

The following interpolation error estimation is well-known. For $u \in H^1(K)$,

$$\|u - \Pi_1^{(0)} u\|_0 \le C_1(K)|u|_1, \quad \|u - \Pi_2^{(0)} u\|_0 \le C_2(K)|u|_1, \tag{1}$$

where $\|\cdot\|_0$ and $|\cdot|_1$ are the L_2 norm and H^1 semi-norm, respectively; $C_1(K)$ and $C_2(K)$ are constants depending only on the shape of triangle K. The boundedness of constants C_1 and C_2 is assured by the embedding theorem of Sobolev spaces; see, e.g., [3]. In particular, the constant $C_1(K)$, as one kind of Poincaré constant, has been rigorously investigated. The constant $C_2(K)$ was proposed by Babuška and Aziz [3] to discuss the maximum inner angle condition for the P_1 interpolation. When K is a unit isosceles right triangle and e_1 is the leg of the right angle, the constant $C_2(K)$ is called the Babuška–Aziz constant [13,18].

There are several formulas that give upper bounds for these constants over an arbitrary triangle. In [18], an upper bound is given as follows:

$$C_1(K) \le \frac{L}{\pi}\sqrt{1+|\cos\theta|}, \quad C_2(K) \le 0.493L\sqrt{1+|\cos\theta|},$$

where L is the second-longest edge length and θ is the maximum inner angle. Another formula for a bound on C_1 was given by Laugesen and Siudeja [16]:

$$C_1(K) \le \mathrm{Diam}(K)/j_{1,1} \quad (\mathrm{Diam}(K): \text{ the diameter of } K),$$

where $j_{1,1} \approx 3.8317$ denotes the first positive root of the Bessel function J_1. In [15], Kobayashi reports a new formula for bounding several interpolation constants, although a full paper is not yet available. Kobayashi's result for C_1 is

$$C_1(K) \le \sqrt{\frac{|e_1|^2+|e_2|^2+|e_3|^2}{28} - \frac{|K|^4}{|e_1|^2\,|e_2|^2\,|e_3|^2}},$$

where $|e_i|$ is the length of edge e_i.

Recently, there have been several results on the concrete bounds for these constants. For the constant $C_2(K)$, Nakao and Yamamoto [20,21] developed a slightly complex algorithm to verify the existence of a solution for a related boundary value problem, and then enclosed the Babuška–Aziz constant in the interval [0.492, 0.493]. In [13], Kikuchi and Liu proved that the Babuška–Aziz constant is equal to the maximum positive zero point of $1/c + \tan 1/c = 0$; their numerical computations show that this value is $0.4929124517549\cdots$. However, the reflection technique used in [13] cannot be extended to triangles of general shape. In [18], Liu and Kikuchi develop an algorithm based on the finite element method to find bounds for the constants $C_1(K)$ and $C_2(K)$ on a general triangle. Although a refined mesh will improve these bounds, the huge matrices produced by the dense mesh are not easy to process.

The problem of estimating the interpolation error constants C_1 and C_2 is equivalent to solving the eigenvalue problem of the Laplacian with certain boundary conditions. In this paper, by assembling several known theorems and methods, we provide a new framework to give high-precision bounds for the eigenvalues related to the interpolation constants:

- Firstly, in Sect.3 and Sect.4, by adopting linear finite element methods, we obtain rough bounds for the leading eigenvalues of the Laplacian, with particular emphasis on the a priori error estimation of the projection operator associated with non-homogeneous boundary conditions. Specifically, a condition required to obtain the rough eigenvalue bounds in [19] is removed here by means of a new proof; see Theorem 4 and Remark 2.
- Secondly, in Sect.5, based on the rough bounds for the leading eigenvalues, we use the mixed FEM and the conforming FEM of high degree, along with the Lehmann–

Goerisch theorem, to obtain high-precision bounds for the interpolation constants. Only very light computation is needed.

The features of our proposed algorithm can be summarized as follows.

- An FEM that employs elements of variable mesh size (h) and polynomial degree (p) (hp-FEM) (see, e.g., [12]) is adopted to rapidly approximate the eigenfunction. The high-precision upper bound for the eigenvalues can be obtained directly from the Rayleigh–Ritz method for a suitable choice of (p, h).
- Although it is not easy to give an explicit error estimation for the eigenvalue approximation using hp-FEM, adopting the Lehmann–Goerisch theorem overcomes this difficulty, and makes it possible to give accurate lower bounds for the eigenvalues.

The paper is constructed as follows: In Sect.2, we provide some preliminary background and introduce finite element spaces. In Sect.3, an explicit *a priori* error estimate is derived for the projection operator. In Sect.4, we present the theorem that gives rough bounds for the eigenvalues related to the interpolation constants. In Sect.5, by applying the Lehmann–Goerisch theorem, we propose an algorithm for high-precision bounds for the interpolation constants. The final section gives some computational results.

2 Preliminaries

This discussion assumes the framework of Sobolev spaces. The space $L_2(K)$ contains the real square integrable functions over K, and $H^n(K)$ $(n = 1, 2, \ldots)$ are the n-th order Sobolev spaces of a function in $L_2(K)$ up to its nth derivative. We denote the L_2 norm of $v \in L_2(K)$ as $\|v\|_0$, and denote by $|v|_k$ and $\|v\|_k$ the semi-norm and norm of $H^k(K)$, respectively. Integration $(\cdot, \cdot)$ is the inner product in $L_2(K)$ or $(L_2(K))^2$. The subspace $H_0^1(K)$ of $H^1(K)$ contains all functions of $H^1(K)$ that vanish on the boundary of K.

Note that $\int_K (u - \Pi_1^{(0)} u)\, dx = 0$ and $\int_{e_1} (u - \Pi_2^{(0)} u) ds = 0$. The optimal constants in (1) are characterized by the infimums of the Rayleigh quotient R over spaces V_1 and V_2,

$$C_i^{-2}(K) = \inf_{v \in V_i} R(v) \quad (i = 1, 2), \quad R(v) := \frac{|v|_1^2}{\|v\|_0^2}, \tag{2}$$

where

$$V_1 := \left\{ v \in H^1(K) | \int_K v dx = 0 \right\}, \quad V_2 := \left\{ v \in H^1(K) | \int_{e_1} v ds = 0 \right\}. \tag{3}$$

Therefore, the determination of $C_1(K)$ and $C_2(K)$ is equivalent to solving the following eigenvalue problem:

$$\text{Find } u \in V \text{ and } \lambda \in \mathbf{R} \text{ s.t. } (\nabla u, \nabla v) = \lambda (u, v) \quad \forall v \in V, \tag{4}$$

where V is taken as $V := V_1$ for $C_1(K)$ and $V := V_2$ for $C_2(K)$. The eigenpair (u, λ) of the above problem, after scaling u, satisfies the following differential equations:

$$-\Delta u = \lambda u \text{ in } K; \quad \begin{cases} \partial u/\partial n = c, & \text{on } e_1 \\ \partial u/\partial n = 0, & \text{on } e_2, e_3 \end{cases}, \tag{5}$$

where $c = 0$ for $V := V_1$ and $c = 1$ for $V := V_2$ in the weak formula (4).

Remark 1 The eigenvalue problem for $i = 1$ is the classical Laplacian problem with the Neumann condition. As the eigenvalue problem in the case $i = 2$ is not so well known, we describe in detail how to obtain the boundary condition of the eigenfunctions. By taking $v \in H_0^1(K)(\subset V_2)$, we have $-\Delta u = \lambda u$ in the sense of a distribution. Further, for any $v \in H^1(K)$, we have $(I - \Pi_2^{(0)})v \in V_2$. Thus,

$$(\nabla u, \nabla (I - \Pi_2^{(0)})v) = (\lambda u, (I - \Pi_2^{(0)})v).$$

By applying Green's theorem to the left-hand side of the above equation, we have

$$\int_{e_1 \cup e_2 \cup e_3} \frac{\partial u}{\partial n} (I - \Pi_2^{(0)}) v ds = 0.$$

By selecting $v \in H^1(K)$ that vanishes on e_1 and e_2, the arbitrariness of v on e_3 forces $\partial u/\partial n = 0$ on e_3. Similarly, we have $\partial u/\partial n = 0$ on e_2. Thus,

$$0 = \int_{e_1} \frac{\partial u}{\partial n} (I - \Pi_2^{(0)}) v ds = \int_{e_1} (I - \Pi_2^{(0)}) \frac{\partial u}{\partial n} v ds \quad \forall v \in H^1(K).$$

Due to the arbitrariness of v on e_1, we see $(I - \Pi_2^{(0)})\partial u/\partial n = 0$ on e_1, i.e., $\partial u/\partial n$ is a constant on e_1. Moreover, the integration of $-\Delta u = \lambda u$ on K gives

$$\int_K \lambda u \, dx = \int_K -\Delta u dx = \int_{e_1 \cup e_2 \cup e_3} \frac{\partial u}{\partial n} \, ds = \frac{\partial u}{\partial n} |e_1|.$$

Consider the boundary value problem:

$$\text{Given } f \in L_2(K), \text{ find } u \in V \text{ s.t. } (\nabla u, \nabla v) = (f, v) \quad \forall v \in V.$$

The existence and uniqueness of the solution is assured by the Lax–Milgram theorem. The operator that maps $f \in L_2(K)$ to the solution $u \in H^1(K)$ is a self-adjoint compact operator. Thus, from the spectral theorem, we know that (4) has a spectrum of infinitely many eigenvalues (see, e.g., [9])

$$0 < \lambda_1 \le \lambda_2 \le \lambda_3 \cdots .$$

It is easy to see that, for each $i = 1, 2$, the interpolation constants $C_i(K)$ are simply the inverse of the square root of the first eigenvalue of (4) or (5).

The estimation for the first eigenvalue of (4), based on FEM, has already been proposed in [18], where only a rough bound of C_1 and C_2 is available. As we will see in Sect. 5, the lower bound of the second eigenvalue can help to improve the bound for the first eigenvalue. Therefore, we extend the technique in [18] to consider the estimation of the leading eigenvalues of (4).

We now introduce several finite element spaces that will be used. Let $\mathcal{T}^h$ be the triangulation of the polygonal domain Ω. In this paper, Ω is nothing but a triangle K. We denote by $P_n(T)$ the set of polynomials over element T whose order is not greater than n.

– L_h^d $(d > 0)$: The C^0 Lagrange finite element space of order d.

$$L_h^d := \{\, v \in H^1(\Omega) \mid v|_T \in P_d(T), T \in \mathcal{T}^h\}$$

– X_h^d $(d \geq 0)$: The piecewise polynomial functions of order up to d.

$$X_h^d := \{\, v \in L_2(\Omega) \mid v|_T \in P_d(T), T \in \mathcal{T}^h\}$$

– Σ_h^d $(d \geq 0)$: The Brezzi–Douglas–Marini finite element of order d ([8]).

$$\Sigma_h^d := \{\, p_h \in (L_2(\Omega))^2 \ \mid \ p_h|_T \in (P_d(T))^2, T \in \mathcal{T}^h; \\ p_h \cdot n_e \text{ is continuous across interior edges } e.\}$$

Recall that $\mathrm{div}(\Sigma_h^d) = X_h^{d-1}$ (refer to Chapter IV.1 of [7]).

The Lagrange element of order 1, i.e., L_h^1, contains all the continuous piecewise linear functions whose degree of freedom is the function value on each vertex of the mesh.

The eigenvalue problem (4) can be solved approximately by applying the Lagrange element L_h^d: For $i = 1, 2$, find $u_h \in L_h^d \cap V_i$ and $\lambda^h \in \mathbb{R}$ s.t.

$$(\nabla u_h, \nabla v_h) = \lambda^h (u_h, v_h) \quad \forall v_h \in L_h^d \cap V_i. \tag{6}$$

The above problem is just a generalized matrix eigenvalue problem, the solution of which can be exactly enclosed by an interval vector using interval arithmetic computation (see, e.g., [5]).

The Lagrange element space L_h^d with high order d can accurately approximate the exact eigenvalues, although it will be more difficult to give an explicit error estimation. In our proposed framework, we will first apply the Lagrange element L_h^1 to obtain a rough lower bound for the eigenvalues. Second, we solve the eigenvalue problem (6) in L_h^d for an appropriate d and mesh size h. With the help of Lehmann–Goerisch's theorem, a high-precision bound on these eigenvalues is then achieved. The finite element spaces Σ_h^d and X_h^d will play an important role in applying Lehmann–Goerisch's theorem.

3 Explicit error estimation for the projection to L_h^1

To give an explicit bound for the eigenvalue based on L_h^1, we introduce a projection $P_{h,i} : V_i \to L_h^1 \cap V_i$ $(i = 1, 2)$ along with an explicit error estimation. For the purpose of simple notation, we denote $L_h^1 \cap V_i$ by $V_{h,i}$ $(i = 1, 2)$.

For any $u \in V_i$ $(i = 1, 2)$, the projection $P_{h,i}u \in V_{h,i}$ is defined as

$$(\nabla(u - P_{h,i}u), \nabla v_h) = 0 \quad \forall v_h \in V_{h,i}. \tag{7}$$

Note that, for each i, the projection $P_{h,i}$ does not change if the range of the test function v_h changes from $V_{h,i}$ to V_h.

For each $i = 1, 2$, assume $u_i \in V_i$ and $u_{h,i} \in V_{h,i}$ are the solutions of the following two boundary problems, respectively: for $f \in L_2(K)$

$$(\nabla u_i, \nabla v) = (f, v), \quad \forall v \in V_i, \tag{8}$$

$$(\nabla u_{h,i}, \nabla v_h) = (f, v_h), \quad \forall v_h \in V_{h,i}. \tag{9}$$

The existence and uniqueness of the solutions for (8) and (9) are easily seen using the Lax–Milgram theorem.

The explicit error estimation for $(u_1 - P_{h,1}u_1)$ is well known, as quoted in Theorem 1. The error estimation for $(u_2 - P_{h,2}u_2)$ can be found in [18], although the argument is somewhat lengthy. In Theorem 2, we give the proof in a concise manner.

P_1 Interpolation function and error estimation

Before discussing the projection error estimation, let us introduce the P_1 interpolation operator $\Pi_h^{(1)}$ over the triangulation $\mathcal{T}^h$. This has an important role in constructing an explicit *a priori* error estimation for the projection $P_{h,i}$.

For $u \in H^2(\Omega)$, $\Pi_h^{(1)}u \in L_h^1$ is the Lagrange interpolation of u, that is,

$$(\Pi_h^{(1)}u)\,(p_i) = u(p_i), \text{ for each node } p_i \text{ of triangulation } \mathcal{T}^h. \tag{10}$$

The computable interpolation error estimation for $\Pi_h^{(1)}$ has been comprehensively investigated (e.g., [14,15,18]):

$$|u - \Pi_h^{(1)}u|_1 \le C_3 h |u|_2 \quad \text{for } u \in H^2(\Omega), \tag{11}$$

where h is the mesh size to be specified. The constant C_3 is defined as follows:

$$C_3 := \max_{T \in \mathcal{T}^h} C_3(T)/h; \quad C_3(T) := \sup_{\substack{v \in H^2(T), v=0 \\ \text{on vertices of } T}} \frac{|v|_{1,T}}{|v|_{2,T}}. \tag{12}$$

For each element T of $\mathcal{T}^h$, letting L be the second-longest edge length, θ the maximum angle, and αL the smallest edge length (see Fig. 2), the result in Liu and Kikuchi [18] gives

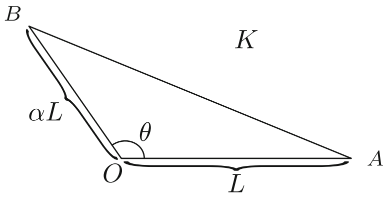


Fig. 2 Configuration of triangle element K by (α, θ, L)

$$C_3(T) \le 0.493L \frac{1+\alpha^2+\sqrt{1+2\alpha^2\cos 2\theta+\alpha^4}}{\sqrt{2\left(1+\alpha^2-\sqrt{1+2\alpha^2\cos 2\theta+\alpha^4}\right)}}.$$

When $\mathcal{T}^h$ is the uniform mesh with isosceles right triangle elements, we can take:

$$h = \text{leg length of right triangle element}, \; C_3 = 0.493.$$

Now, we continue with the projection error estimation. First, we consider the case of $P_{h,1}$. The following theorem can be found in classical textbooks, see, e.g., [24].

Theorem 1 *Let $u \in V_1$ and $u_h = P_{h,1}u \in V_{h,1}$ be the solutions of (8) and (9) in the case $i = 1$, respectively. The error estimation is given as*

$$|u - P_{h,1}u|_1 \le M_{h,1}\|f\|_0, \quad \|u - P_{h,1}u\|_0 \le M_{h,1}|u - P_{h,1}u|_1 \le M_{h,1}^2\|f\|_0. \tag{13}$$

where $M_{h,1} = C_3 h$.

For the case $i = 2$, we summarize one result of [18] to give a concise proof.

Theorem 2 *Let $u \in V_2$ and $u_h = P_{h,2}u \in V_{h,2}$ be the solutions of (8) and (9) in the case $i = 2$, respectively. Then, the error estimation is given as*

$$|u - P_{h,2}u|_1 \le M_{h,2}\|f\|_0, \quad \|u - P_{h,2}u\|_0 \le M_{h,2}|u - P_{h,2}u|_1 \le M_{h,2}^2\|f\|_0, \tag{14}$$

where $M_{h,2} := \left(2 + \sqrt{2}/2\right) C_3 h$.

Proof First, we show that $|u|_2 \le \left(2 + \sqrt{2}/2\right)\|f\|_0$. We assume that the vertex p_1 of K is (x_1, y_1). As suggested by [20], we introduce a quadratic function f_1

$$f_1(x, y) := |e_1|[(x - x_1)^2 + (y - y_1)^2]/(4|K|).$$

It is easy to see that $\partial f_1/\partial n \equiv 1$ on e_1 and $\partial f_1/\partial n \equiv 0$ on e_2 and e_3. Because the solution $u \in V_2$ has the property that

$$\partial u/\partial n \equiv (f, 1)/|e_1| \quad \text{on } e_1, \quad \partial u/\partial n = 0 \quad \text{on } e_2, e_3.$$

We see that $\hat{u} := u - |e_1|^{-1}(f, 1)\, f_1$ satisfies the homogeneous Neumann boundary condition and $|\hat{u}|_2 = \|\Delta \hat{u}\|_0$ (c.f. Theorem 4.3.1.4 of [11]). Note that $\|\Delta f_1\|_0 = \sqrt{2}|f_1|_2 = |e_1|/\sqrt{|K|}$. Thus, using the triangle and Schwarz's inequalities,

$$|u|_2 \leq \|\Delta u\|_0 + |e_1|^{-1}|(f, 1)|(|f_1|_2 + \|\Delta f_1\|_0) \\ \leq \left(2 + \sqrt{2}/2\right) \|f\|_0.$$

Second, we give the error estimation by introducing a new minimizing principle. Define another projection $Q : H^1(K) \to V_{h,2}$ such that

$$Qv = v - |e_1|^{-1} \int\limits_{e_1} v ds.$$

Therefore, $(\nabla u - \nabla u_h, \nabla(Qv_h)) = 0$ for any $v_h \in L^1_h$. Further, noticing that $\nabla v_h = \nabla(Qv_h)$, we have $(\nabla u - \nabla u_h, \nabla v_h) = 0$ for any $v_h \in L^h_1$. Hence,

$$|u - u_h|_1 = \min_{v_h \in L^1_h} |u - v_h|_1 \leq |u - \Pi^{(1)}_h u|_1 \leq C_3 h |u|_2 \leq M_{h,2} \|f\|_0.$$

The estimation for the L_2 norm can be obtained by applying Nitsche's technique.

4 Rough eigenvalue bounds based on L^1_h

This section is devoted to obtaining an estimation for the approximate eigenvalues of (6).

The discussion will be shaped in a parallel way. Let (V, V_h, P_h) be $(V_1, V_{h,1}, P_{h,1})$ or $(V_2, V_{h,2}, P_{h,2})$. Denote by $\lambda_1 \leq \lambda_2 \leq \cdots$ the stationary values of the Rayleigh quotient R over V. Suppose $\dim(V_h) = n$. The finite stationary values of R over V_h are denoted by $\lambda^h_1 \leq \lambda^h_2 \cdots \leq \lambda^h_n$. The associated eigenfunctions $\{u_i\}$ for $\{\lambda_i\}$ are normalized to form an orthonormal set in $L_2(\Omega)$, that is, $(u_i, u_j) = \delta_{ij}$, where δ_{ij} is Kronecker's delta.

The qualitative estimation for λ^h_k is well known (see, e.g., Strang and Fix [24]),

$$\lambda_k \leq \lambda^h_k \leq \lambda_k \left(1 + C \sup_{v \in E_k, \|v\|=1} |v - P_h v|^2_1\right), \tag{15}$$

where E_k is the space spanned by the eigenfunctions $\{u_i\}^k_{i=1}$. Recall the a priori error estimation for P_h in Sect. 2,

$$\|u - P_h u\|_0 \leq M_h |u - P_h u|_1, \tag{16}$$

where $M_h = M_{h,i}$ for $P_{h,i}$ $(i = 1, 2)$. There are other qualitative estimations for the eigenvalues, but most of them remain as asymptotic bounds, and the concrete eigenvalue bounds are difficult to obtain (see the survey paper on these theories [4]).

Using the explicit values of M_h, we will derive the eigenvalue bound formula in [19], but with a weakened prerequisite condition. First, we give a lemma for the eigenvalue approximation based on the max–min principle.

Lemma 3 *Let* $v_1^h, \ldots, v_{k-1}^h$ *be any functions of* V_h *and* $V_{k-1}^h := span\{v_1^h, \cdots, v_{k-1}^h\}$. *Let* $V_{k-1}^{h,\perp}$ *be the complementary space of* V_{k-1}^h *in* V. *Define* $\tilde{\lambda}_k$ *over* $L_h^1 \cap V_{k-1}^{h,\perp}$ *as*

$$\tilde{\lambda}_k = \min_{v_h \in V_h \cap V_{k-1}^{h,\perp}} \frac{(\nabla v_h, \nabla v_h)}{(v_h, v_h)}.$$

Thus, we have a lower bound for λ_k,

$$\lambda_k \geq \tilde{\lambda}_k / \left(1 + M_h^2 \tilde{\lambda}_k\right), \tag{17}$$

where M_h *is as in* (16).

Proof From the max–min principle, we have

$$\lambda_k = \max_{W \subset V, \dim(W) \leq k-1} \min_{v \in W^\perp} \frac{(\nabla v, \nabla v)}{(v, v)}.$$

Thus, by choosing $W := V_{k-1}^h$, a lower bound for λ_k is given by

$$\lambda_k \geq \min_{v \in V_{k-1}^{h,\perp}} \frac{(\nabla v, \nabla v)}{(v, v)}. \tag{18}$$

For any $v \in V_{k-1}^{h,\perp}$, $P_h v \in V_h$. Let w_h be any function in $V_{k-1}^h (\subset V_h)$. Then, $(\nabla v, \nabla w_h) = 0$. Thus, $(\nabla P_h v, \nabla w_h) = (\nabla v, \nabla w_h) = 0$, which implies that $P_h v \in V_h \cap V_{k-1}^{h,\perp}$. Considering (14) and the definition of $\tilde{\lambda}_k$,

$$\|v\|_0 \leq \|P_h v\|_0 + \|v - P_h v\|_0 \leq \tilde{\lambda}_k^{-1/2} \|\nabla P_h v\|_0 + M_h \|\nabla (v - P_h v)\|_0.$$

Thus,

$$\|v\|_0^2 \leq \left(\tilde{\lambda}_k^{-1} + M_h^2\right) (\|\nabla P_h v\|_0^2 + \|\nabla (v - P_h v)\|_0^2) = \left(\tilde{\lambda}_k^{-1} + M_h^2\right) \|\nabla v\|_0^2.$$

Hence, for any $v \in V_{k-1}^{h,\perp}$, we have

$$\|\nabla v\|_0^2 / \|v\|_0^2 \geq \tilde{\lambda}_k / \left(1 + M_h^2 \tilde{\lambda}_k\right).$$

Using (18), we can draw the conclusion in (17).

Take the subspace V_{k-1}^h in Lemma 3 to be the space spanned by the eigenfunctions corresponding to $\lambda_1^h, \ldots, \lambda_{k-1}^h$, such that $\tilde{\lambda}_k = \lambda_k^h$. We summarize the bounds for the eigenvalues in the following theorem.

Theorem 4 *For each $i = 1, 2$, let λ_k and λ_k^h be the kth eigenvalues of (4) and (6), respectively. Then, the lower and upper bounds of λ^k are given by*

$$\lambda_k^h / \left(1 + M_h^2 \lambda_k^h\right) \le \lambda_k \le \lambda_k^h \quad (1 \le k \le n). \tag{19}$$

Remark 2 The estimation (19) in Theorem 4 was also obtained by Liu and Oishi [19], who considered the eigenvalue problem of the Laplacian with a singularity. In [19], the min–max principle is the starting point for the proof of (19). However, the method in [19] introduces the additional condition $\lambda_k M_h^2 < 1$ to make the estimation hold, which is in fact not necessary considering the proof of Lemma 3.

5 High-precision bounds based on Lehmann–Goerisch's theorem

As we will see in Sect. 6, the eigenvalue bounds based on the L_h^1 finite element method are somewhat rough. To have high-precision bounds, a natural idea is to refine the mesh and deal with large scale matrices in the computation. Another approach is to consider utilizing L_h^d with some high degree d. However, it is hard to obtain an explicit error estimation if d is large. To overcome this difficulty, we introduce the Lehmann–Goerisch theorem [6,10,17], which uses the a priori rough lower bound for certain eigenvalues to sharpen the bounds for others.

In this section, let V represent the function space V_1 or V_2. The high-precision bound for $C_1(K)$ and $C_2(K)$ can be obtained with the same algorithm.

First, let us quote the Lehmann–Goerisch theorem as follows.

Theorem 5 (Lehmann–Goerisch's theorem) Assumptions and notation.

A1 *D is a real vector space. M and N are symmetric bilinear forms on D; $M(f, f) > 0$ for all $f \in D$, $f \neq 0$.*

A2 *There exist sequences $\{\lambda_i\}_{i \in \mathbf{N}}$ and $\{\phi_i\}_{i \in \mathbf{N}}$ such that $\lambda_i \in \mathbf{R}$, $\phi_i \in D$, $M(\phi_i, \phi_k) = \delta_{ik}$ for $i, k \in \mathbf{N}$,*

$$M(f, \phi_i) = \lambda_i N(f, \phi_i) \quad \textit{for all } f \in D, i \in \mathbf{N}. \tag{20}$$

$$N(f, f) = \sum_{i=1}^{\infty} (N(f, \phi_i))^2 \quad \textit{for all } f \in D. \tag{21}$$

A3 *X is a real vector space; $G : D \to X$ is a linear operator; b is a symmetric bilinear form on X. $b(f, f) \ge 0$ for all $f \in X$ and $b(Gf, Gg) = M(f, g)$ for all $f, g \in D$.*

A4 *$n \in \mathbf{N}$, $v_i \in D$ for $i = 1, \ldots, n$. $w_i \in X$ satisfies*

$$b(Gf, w_i) = N(f, v_i) \quad \textit{for all } f \in D, i = 1, \ldots, n\,; \tag{22}$$

A5 $\rho \in \mathbf{R}$, $\rho > 0$. *Define matrices as*

$$A_0 := (M(v_i, v_k))_{i,k=1,\dots,n}\,, \quad A_1 := (N(v_i, v_k))_{i,k=1,\dots,n}\,,$$
$$A_2 := (b(w_i, w_k))_{i,k=1,\dots,n}\,,$$
$$A^L = A_0 - \rho A_1, \quad B^L = A_0 - 2\rho A_1 + \rho^2 A_2;$$

B^L is positive definite. For $i = 1, \dots, n$, the ith smallest eigenvalue of the eigenvalue problem $A^L z = \mu B^L z$ is denoted by μ_i.

Assertion: *For all i, $1 \le i \le n$, such that $\mu_i < 0$, the interval $[\rho - \rho/(1-\mu_i), \rho)$ contains at least i eigenvalues of (20).*

Remark 3 The method based on Lehmann–Goerisch's theorem can provide lower bounds for the eigenvalues of (20). Suppose the eigenvalues of (20) are ordered by $\lambda_1 \le \lambda_2 \le \dots$. If ρ is a lower bound of the eigenvalue λ_{N+1}, i.e.,

$$\rho \le \lambda_{N+1}, \tag{23}$$

then we have a lower bound for λ_{N+1-i} in the case $\mu_i < 0$,

$$\rho - \rho/(1-\mu_i) \le \lambda_{N+1-i} \quad (1 \le i \le N).$$

Remark 4 Generally, it is not easy to provide the a priori lower bound (23). The *homotopy method* developed by Plum [22] is one approach for finding such a bound. This method considers a family of eigenvalue problems that connects the objective eigenvalue problem to one with an explicit spectrum, which is called the "*base problem*." The complex manipulation of eigenvalue problems and requirement of a *base problem* limit its use in dealing with problems over a general domain.

To obtain high-precision bounds for $C_1(K)$ and $C_2(K)$, we will apply the Lehmann–Goerisch theorem to sharpen the bounds of the eigenvalues corresponding to the Laplacian. We choose the function spaces and operator G as follows:

$$D := V, \quad X := (L_2(K))^2\,, \quad G := \nabla,$$

and assume the bilinear forms to be

$$M(u, v) := (\nabla u, \nabla v), \quad N(u, v) := (u, v), \quad \text{for } u, v \in H^1(T),$$
$$b(u, v) := (u, v) \quad \text{for } u, v \in (L_2(K))^2\,.$$

As we are only concerned with the lower bound of the minimal eigenvalue of problem (4), we seek ρ satisfying $\lambda_1 < \rho \le \lambda_2$. If the first N eigenvalues are the same, then we seek ρ such that $\lambda_N < \rho \le \lambda_{N+1}$. The lower bound (19) based on the L_h^1 finite element provides a good choice of ρ,

$$\rho := \lambda_{N+1}^h/(1 + M_h^2 \lambda_{N+1}^h) \le \lambda_{N+1}.$$

The function v_i in the Lehmann–Goerisch theorem is chosen as the approximation, denoted by $u_{i,h}$, to the eigenfunction u_i from $L_h^d \cap V$ $(1 \le i \le N)$. The function $w_i \in (L_2(K))^2$ corresponding to v_i, which is not unique, will be selected from Σ_h^d.

Define by $\Sigma_{h,c}^d$ the subset of Σ_h^d,

$$\Sigma_{h,c}^d := \{p_h \in \Sigma_h^d \mid p_h \cdot n = 0 \text{ on } e_2, e_3;\ p_h \cdot n = c \text{ on } e_1\}.$$

For a fixed approximate eigenvector $u_h \in L_h^d \cap V$, define $c_0 := \int_T u_h dx/|e_1|$. Notice that $c_0 = 0$ for $u_h \in L_h^d \cap V_1$. We consider the problem of finding $(p_h, \rho_h) \in \Sigma_{h,c_0}^{d+1} \times X_h^d$ with

$$\begin{cases} (p_h, q_h) + (\rho_h, \operatorname{div} q_h) = 0 & \forall q_h \in \Sigma_{h,c_0}^{d+1} \\ (\operatorname{div} p_h, f_h) + (u_h, f_h) = 0 & \forall f_h \in X_h^d \end{cases} \tag{24}$$

The system (24) admits a unique solution (p_h, ρ_h) (c.f., e.g., [2], §IV.1 Prop. 1.1 of [7]). Moreover, the matrices that present the map from u_h to p_h and ρ_h, which is just the operator G in *A3* of Theorem 5, can be exactly enclosed by solving (24) using interval arithmetic. However, such a matrix representation involves the full matrix, and an effective algorithm is needed; for a detailed discussion, see [19].

The second equation in (24) implies that $\operatorname{div} p_h + u_h = 0$. Thus, combining the boundary condition of $p_h \in \Sigma_{h,c_0}^{d+1}$, it is easy to see that the solution p_h satisfies

$$(p_h, \nabla f) = (u_h, f) \quad \text{for all } f \in V. \tag{25}$$

Therefore, for each $v_i := u_{h,i}$, the w_i corresponding to v_i can be taken as the solution p_h of (24) with $u_h := u_{u,i}$.

Remark 5 As (24) is obtained by applying the Lagrange multiplier to the minimization problem,

$$\min_{p_h \in \Sigma_{h,c_0}^{d+1},\ \operatorname{div} p_h + u_h = 0} \|p_h\|_0^2, \tag{26}$$

the minimizer p_h for (26) (also the solution of (24)) gives the smallest value of $\|p_h\|^2$. If ρ can be selected as $\lambda_1 < \rho \le \lambda_2$, and the matrices in condition *A5* of Lehmann–Goerisch's theorem are of dimension 1, it is easy to see that p_h gives an optimal lower bound for the eigenvalue λ_1 over all candidates of w_i in Σ_{h,c_0}^{d+1}.

Let us summarize the algorithm that gives high-precision bounds for the constants $C(K) = C_1(K)$ or $C(K) = C_2(K)$.

Algorithm 1: high-precision bound for P_0 interpolation constant

S1. Solve the eigenvalue problem (6) in L_h^1 and denote the eigenvalues by $\lambda_{1,h}^{(1)} \le \lambda_{2,h}^{(1)} \le \ldots$. With a proper mesh size and index N, we can obtain an estimation as follows:

$$\lambda_{N,h}^{(1)} < \rho := \lambda_{N+1,h}^{(1)}/(1 + M_h^2 \lambda_{N+1,h}^{(1)}) \le \lambda_{N+1} \tag{27}$$

S2. For a proper degree $d > 1$, solve the eigenvalue problem (6) in L_h^d and denote the eigenvalues by $\lambda_{1,h}^{(d)} \le \lambda_{2,h}^{(d)} \le \ldots$. Let $u_{h,i}$ be the eigenfunction corresponding to $\lambda_{i,h}^{(d)}$.

S3. For each $u_{h,i}$, $i = 1, \cdots, N$, let $w_{h,i}$ be the solution $p_h \in \Sigma_h^{d+1}$ of (24) with $u_h := u_{h,i}$.

S4. Define A_0, A_1, and A_2 with functions $u_{h,i}, w_{h,i}, i = 1, \ldots, N$.

$$A_0 = \big((u_{h,i}, u_{h,j})\big)_{i,j=1,\ldots,N}, \quad A_1 = \big((\nabla u_{h,i}, \nabla u_{h,j})\big)_{i,j=1,\cdots,N},$$
$$A_2 = \big((w_{h,i}, w_{h,j})\big)_{i,j=1,\ldots,N},$$
$$A^L = A_1 - \rho A_0, \quad B^L = A_1 - 2\rho A_0 + \rho^2 A_2.$$

(Notice that the selection of ρ in (27) satisfies $\lambda_{N,h}^{(d)} \le \lambda_{N,h}^{(1)} < \rho$. Thus, $-A^L$ is positive definite.)

S5. Let the eigenvalues of $A^L = \mu B^L$ be $\mu_1 \le \cdots \le \mu_N$. If B^L is positive definite, then $\mu_N < 0$ and the lower bound of λ_1 in (4) is given by

$$\rho - \rho/(1 - \mu_N).$$

S5. The high-precision bound for $C(K)$ is given as

$$(\lambda_{1,h}^{(d)})^{-1/2} \le C(K) \le (\rho - \rho/(1 - \mu_N))^{-1/2}.$$

6 Computational results and applications

We apply Algorithm 1 to bound the P_0 interpolation constants $C_1(K)$ and $C_2(K)$ over a triangle K of different shapes. The triangle K has vertices $(0, 0)$, $(1, 0)$, and (a, b), where (a, b) will be assigned different values. A sample triangulation of K is displayed in Fig. 3. Let $L(T)$ be the second edge length of $T \in \mathcal{T}^h$; then, the mesh size h is defined by the maximal $L(T)$ over all the elements.

To verify the estimation of the constants, we apply interval arithmetic in the numerical computations. The method of [5], along with the INTLAB toolbox for MATLAB [23], is adopted to solve the matrix eigenvalue problem with verified bounds.

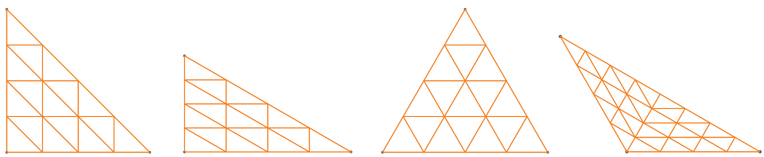


Fig. 3 Triangulation of K (from *left* to *right*, $h = 0.25, 0.25, 0.25, 0.22$)

First, we give a rough estimate for the leading 3 eigenvalues of (4) based on L_h^1. The lower and upper bounds of the eigenvalues, in the form of intervals, are displayed in Tables 1 and 2. If λ_N and λ_{N+1} are not close to each other, then the lower bound of λ_{N+1}, which is underlined in Tables 1 and 2, is chosen as ρ for the purpose of giving high-precision bounds.

Second, we apply the Lehmann–Goerisch theorem to sharpen the eigenvalue bounds; see results in Tables 3 and 4. The high-precision bounds are presented in compact form. For example, 0.4929124517^{97}_{54} denotes the interval [0.492912451754, 0.492912451797] that encloses the constant $C_2(K)$ for the unit isosceles right triangle K. We can see, even with a sparse mesh, that high-order finite element spaces can give dramatically improved bounds.

To see the dependency of the precision on the polynomial degree (p) and mesh size (h), we estimate the value of $C_2(K)$ with different pairs of (p, h). The vertices of K are fixed to $(0, 0)$, $(1, 0)$, and $(-1/2, \sqrt{3}/2)$. For each $p = 1, 2, 3$, the mesh size is taken to be $h = 0.433, 0.217, 0.108$, and 0.054, whereas for $p = 4, 5, 6$, only the first three mesh sizes are considered. In this example, the floating point computations are performed with fixed rounding-to-nearest mode. The width of the constant bounds, i.e., the distance between the lower and the upper bounds, is denoted by 'Err,' and their values along with the degrees of freedom (DOF) of L_h^p are displayed in Fig. 4. We can see that, for suitable pairs (p, h), 'Err' converges to zero very quickly as DOF increases.

Table 1 Eigenvalue bound based on L_h^1 element ($C_1(K)$)

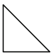
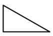
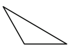
(a, b)	Shape	h	$M_{h,1}$	λ_1	λ_2	λ_3
$(0, 1)$		0.063	0.0308	[9.80, 9.91]	[19.55, 19.93]	[38.52, 39.99]
$(0, \sqrt{3}/3)$		0.063	0.0308	[13.04, 13.21]	[38.48, 39.95]	[50.74, 53.31]
$(1/2, \sqrt{3}/2)$		0.063	0.0654	[16.38, 17.63]	[16.38, 17.63]	[43.42, 53.31]
$(-1/2, \sqrt{3}/2)$		0.054	0.0231	[7.13, 7.17]	[17.43, 17.61]	[37.00, 37.76]

Table 2 Eigenvalue bound based on L_h^1 element ($C_2(K)$)

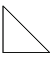
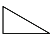
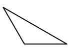
(a, b)	Shape	h	$M_{h,2}$	λ_1	λ_2	λ_3
$(0, 1)$		0.063	0.0846	[4.00, 4.12]	[17.44, 19.93]	[20.71, 24.32]
$(0, \sqrt{3}/3)$		0.063	0.0834	[6.67, 7.00]	[27.99, 34.76]	[31.25, 39.95]
$(1/2, \sqrt{3}/2)$		0.063	0.177	[5.67, 6.90]	[11.35, 17.63]	[17.03, 36.50]
$(-1/2, \sqrt{3}/2)$		0.054	0.0626	[2.85, 2.89]	[14.89, 15.82]	[29.63, 33.52]

Table 3 High-precision bound for constant $C_1(K)$ $(d = 5)$

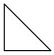
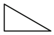
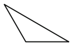
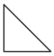
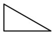
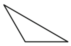
(a, b)	Shape	h	λ_1	$C_1(K)$
(0, 1)		0.25	9.869604^{41}_{39}	0.318309886^{24}_{18}
$(0, \sqrt{3}/3)$		0.25	13.1594725^{36}_{23}	0.275664447^{83}_{70}
$(1/2, \sqrt{3}/2)$		0.25	17.5459633^{81}_{27}	0.238732414^{993}_{633}
$(-1/2, \sqrt{3}/2)$		0.22	7.1553^{53}_{26}	0.37383^{96}_{83}

Table 4 High-precision bound for constant $C_2(K)$ $(d = 5)$

(a, b)	Shape	h	λ_1	$C_2(K)$
(0, 1)		0.25	4.11585836^{57}_{49}	0.4929124517^{97}_{54}
$(0, \sqrt{3}/3)$		0.25	6.98559906^{70}_{18}	0.378353862^{72}_{57}
$(1/2, \sqrt{3}/2)$		0.25	6.892786^{705}_{695}	0.380892639^{68}_{43}
$(-1/2, \sqrt{3}/2)$		0.22	2.88855^{609}_{497}	0.588382^{42}_{29}

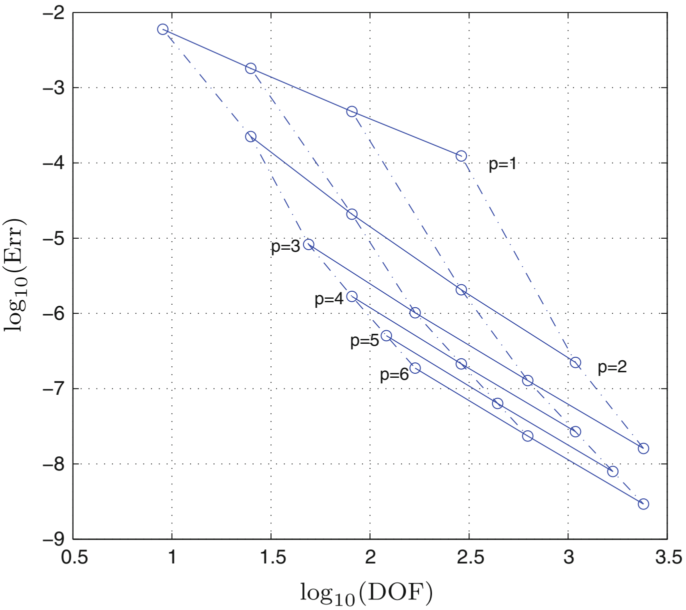


Fig. 4 Dependency of the precision on h and p

7 Conclusion and future work

In this paper, we proposed a new algorithm to obtain high-precision bounds for the P_0 interpolation error constants. The method developed in this paper can be extended to give sharp bounds for the general P_n interpolation error constant. For this purpose, there are two difficulties we must overcome. First, to give the error estimation for a high-order interpolation, e.g., $|u - \Pi_h^{(1)} u|_1 \leq C_3 h |u|_2$, the corresponding eigenvalue problem of weak form may have second-order derivatives, making it difficult to obtain even rough bounds for the eigenvalues.

Second, it is not easy to find w_i that correspond to v_i in the *A4* condition of Lehmann–Goerisch's theorem, as there will be very complex boundary conditions. On the advice of a referee, the helpful "spectral shift" technique will be applied to overcome this difficulty in future work.

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