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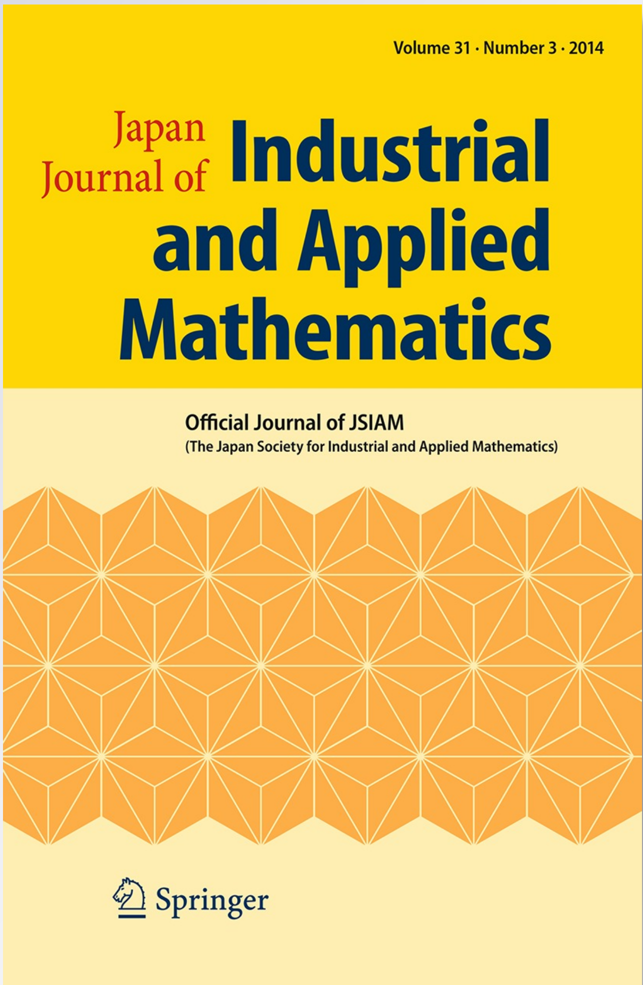

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ORIGINAL PAPER **Area 2**

Verified norm estimation for the inverse of linear elliptic operators using eigenvalue evaluation

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Abstract This paper proposes a verified numerical method of proving the invertibility of linear elliptic operators. This method also provides a verified norm estimation for the inverse operators. This type of estimation is important for verified computations of solutions to elliptic boundary value problems. The proposed method uses a generalized eigenvalue problem to derive the norm estimation. This method has several advantages. Namely, it can be applied to two types of boundary conditions: the Dirichlet type and the Neumann type. It also provides a way of numerically evaluating lower and upper bounds of target eigenvalues. Numerical examples are presented to show that the proposed method provides effective estimations in most cases.

K. Tanaka (✉)
Graduate School of Fundamental Science and Engineering, Waseda University, 3-4-1 Okubo, Shinjuku-ku, Tokyo 169-8555, Japan
e-mail: imahazimari@fuji.waseda.jp

A. Takayasu · S. Oishi
Department of Applied Mathematics, Faculty of Science and Engineering, Waseda University, 3-4-1 Okubo, Shinjuku-ku, Tokyo 169-8555, Japan

X. Liu
Research Institute for Science and Engineering, Waseda University, 3-4-1 Okubo, Shinjuku-ku, Tokyo 169-8555, Japan

Present Address:
X. Liu
Graduate School of Science and Technology, Niigata University, 8050 Ikarashi 2-no-cho, Nishi-ku, Niigata City, Niigata 950-2181, Japan

S. Oishi
JST, CREST, Tokyo, Japan

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1 Introduction

We consider weak solution to the following linear elliptic differential equation:

$$-\Delta u + cu = g \quad \text{in } \Omega, \tag{1}$$

with two types of boundary conditions: the Dirichlet type:

$$u = 0 \quad \text{on } \partial\Omega \tag{2}$$

and the Neumann type:

$$\frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega. \tag{3}$$

Here, we assume that $c \in L^{\infty}(\Omega)$ and $g \in L^{2}(\Omega)$, where $\Omega \subset \mathbb{R}^{2}$ is a bounded convex polygonal domain. We then define the operator $\mathcal{L} : V \to V^{*}$ by

$$\langle \mathcal{L}u, v \rangle := (\nabla u, \nabla v)_{L^2} + (cu, v)_{L^2}, \quad \forall v \in V, \tag{4}$$

where V is $H^{1}(\Omega)$ or $H_0^1(\Omega)$, V^{*} is the dual space of V, and $\langle F, v \rangle := F(v)$ for all $F \in V^{*}$ and $v \in V$. Note that the definition of functional spaces $L^{\infty}(\Omega)$, $L^{2}(\Omega)$, $H^{1}(\Omega)$, and $H_0^1(\Omega)$ are introduced in Sect. 2.1. Now, we can write the weak form of (1) as follows:

$$\text{Find } u \in V \text{ s.t. } \langle \mathcal{L}u, v \rangle = (g, v)_{L^2}, \quad \forall v \in V.$$

In this paper, we shall present a sufficient condition for the invertibility of $\mathcal{L}$. We also provide a verified numerical method of estimating an upper bound of the operator norm $\|\mathcal{L}^{-1}\|_{V^*,V}$. This method can be applied to verified computations of solutions to the semilinear elliptic boundary value problem as follows:

$$-\Delta u = f(u) \quad \text{in } \Omega, \tag{5}$$

with the boundary condition (2) or (3). Several works show verified numerical method for weak solutions to (5), e.g., [1,2,4], etc. They first construct an approximate solution of (5) in a finite dimensional subspace of V. Then, the method tries to ensure that the exact solution of (5) exists in a neighborhood of the approximate solution. In an affirmative case, it also proves the local uniqueness of the exact solution. The estimation of $\|\mathcal{L}^{-1}\|_{V^*,V}$ is important for this type of verified numerical method. The details of this application are discussed in Sect. 5.

There are several previous works with respect to a verified norm estimation for the inverse of linear elliptic operators, e.g., [1,2], etc. Plum [1] has proposed a homotopy based method. The norm estimation can be regarded as a generalized eigenvalue problem for a linear operator, the details of which are discussed in Sect. 3. Plum's method evaluates the eigenvalues using the homotopy method to give the norm estimation. The homotopy method gives the eigenvalue evaluation using some homotopic steps with additional starting functions and relatively small matrix eigenvalue evaluation. The details are referred to [3]. Moreover, Plum's method has the feature that the inner product in V is changed depending on $c(x)$. Our method also changes the inner product in V depending on $c(x)$ and evaluates target eigenvalues to give the norm estimation. However, we do not use the homotopy method but directly evaluate the eigenvalues using Theorem 4 in Sect. 4. This theorem is a generalization of the theorem in [6] which is proved by Liu and Oishi. Their theorem gives lower bounds of eigenvalues of the Laplace operator. It is well known that an upper bound of the target eigenvalue is easily obtained using the Rayleigh–Ritz method. A lower bound, however, is not obtained easily. Theorem 4 gives the lower bound directly.

Nakao et al. [2] have proposed another verified numerical method of the norm estimation. Their method is different from our method at the following points. Their method can be applied to the elliptic operator as follows:

$$\langle \mathcal{L}u, v\rangle := (\nabla u, \nabla v)_{L^2} + (b\cdot\nabla u, v)_{L^2} + (cu, v)_{L^2}\,, \quad \forall v \in H_0^1(\Omega)\,,$$

where $b \in (L^\infty(\Omega))^2$. In addition, the inner product used in their method is always the same regardless of $c(x)$. However, since the domain of $\mathcal{L}$ in their method is $H_0^1(\Omega)$, a direct application of this method to the Neumann type seems to be not straightforward. In Sect. 6, we compare our method with their method in the case of the Dirichlet type. Our method tends to give better estimations especially when mesh sizes are large, e.g., in Tables 1 and 2 in Sect. 6.

This paper is constructed as follows: in Sect. 2, the definition of functional spaces is introduced and a projection error constant is discussed. In Sect. 3, a framework of proving the invertibility of $\mathcal{L}$ and a norm estimation for $\mathcal{L}^{-1}$ are proposed. In Sect. 4, a method of verified evaluation of target eigenvalues is proposed. In Sect. 5, an application to a semilinear elliptic boundary value problem is discussed. In Sect. 6, numerical examples of computing the norm of inverse operator are discussed. Section 3 and 4 are the main sections in this paper.

2 Preliminary

In this section, we introduce the definition of functional spaces and discuss a projection error constant C_M.

2.1 Definition of functional spaces

Throughout this paper, let $L^2(\Omega)$ be the functional space of Lebesgue-measurable square-integrable functions with the L^2 inner product and the L^2 norm:

$$(u, v)_{L^2} := \int_\Omega u(x)\, v(x)\, dx, \quad \|u\|_{L^2} := \sqrt{(u, u)_{L^2}}.$$

Let $L^\infty(\Omega)$ be the functional space which is essentially bounded on Ω with the norm:

$$\|u\|_{L^\infty} := \operatorname{ess\,sup}_{x\in\Omega} |u(x)|.$$

We denote the kth-order L^2 Sobolev space on Ω by $H^k(\Omega)$ $(k \in \mathbb{N})$ with the inner product and the norm:

$$(u, v)_{H^k} := \sum_{|\alpha|\le k} \int_\Omega \partial^\alpha u(x)\, \partial^\alpha v(x)\, dx, \quad \|u\|_{H^k} := \sqrt{(u, u)_{H^k}}.$$

We also define one kind of subspace of $H^1(\Omega)$:

$$H_0^1(\Omega) := \left\{ u \in H^1(\Omega) \ : \ u = 0 \quad \text{on } \partial\Omega \text{ in the trace sense} \right\}.$$

In this paper, V will be taken as $H^1(\Omega)$ or $H_0^1(\Omega)$. In the case of $V = H^1(\Omega)$, we endow V with another special inner product and the norm:

$$(u, v)_\sigma := (\nabla u, \nabla v)_{L^2} + \sigma (u, v)_{L^2}, \quad \|u\|_\sigma := \sqrt{(u, u)_\sigma}, \tag{6}$$

where σ is a positive number. In the case of $V = H_0^1(\Omega)$, we endow V with the same form inner product and the norm (6), however σ is a nonnegative number. Note that the concrete value of σ is chosen in Sect. 3 depending on $c(x)$ of equation (1).

Next, we subdivide Ω into a mesh triangulation T_h which is composed of triangular elements, where h is the mesh size defined by

$$h := \max_{K\in T_h} (\text{the second longest side of element } K).$$

Let N_j $(j = 1, 2, \ldots, m)$ be the node point of T_h. We use piecewise linear base functions ϕ_i $(i = 1, 2, \ldots, m)$ which satisfies

$$\phi_i(N_j) = \begin{cases} 1 & (i = j) \\ 0 & (i \neq j) \end{cases}.$$

Let $K \subset \mathbb{R}^2$ be a triangle element and p_k $(k = 1, 2, 3)$ be the vertex of K. Using this notation, we define the FEM space $V_h := \operatorname{span}\{\phi_1, \phi_2, \ldots, \phi_m\} \cap V$. Recall that V can be taken as $H^1(\Omega)$ or $H_0^1(\Omega)$. The FEM space V_h varies after the selection of V.

2.2 Projection error constant

In this subsection, we introduce an orthogonal projection $P_h : V \to V_h$ along with a projection error constant C_M. We first define a subspace $W \subset H^2(\Omega)$ as follows:

in the case of $V = H^1(\Omega)$, $W := \{u \in H^2(\Omega) : \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega\}$; in the case of $V = H_0^1(\Omega)$, $W := \{u \in H^2(\Omega) : u = 0 \text{ on } \partial\Omega\}$.

Let $P_h : V\ (\supset W) \to V_h$ be the orthogonal projection defined by

$$(P_h u - u, v_h)_\sigma = 0, \quad \forall v_h \in V_h.$$

Let C_M be a positive number satisfying

$$\|u - P_h u\|_\sigma \le C_M \|-\Delta u + \sigma u\|_{L^2}, \quad \forall u \in W. \tag{7}$$

In order to calculate C_M, we use the following theorem and lemma:

Theorem 1 (Kobayashi [5]) *Let us define* $V^2(K) := \{\phi \in H^2(K) : \phi(p_k) = 0\}$. *There exist the following constants*

$$c_1(K) := \sup_{\phi \in V^2(K)\setminus\{0\}} \frac{\|\phi\|_{L^2}}{|\phi|_{H^2}},$$
$$c_2(K) := \sup_{\phi \in V^2(K)\setminus\{0\}} \frac{\|\nabla\phi\|_{L^2}}{|\phi|_{H^2}},$$

of which upper bounds are computable as

$$c_1(K) \le \sqrt{\frac{A^2B^2 + B^2C^2 + C^2A^2}{83} - \frac{1}{24}\left(\frac{A^2B^2C^2}{A^2 + B^2 + C^2} + S^2\right)},$$
$$c_2(K) \le \sqrt{\frac{A^2B^2C^2}{16S^2} - \frac{A^2 + B^2 + C^2}{30} - \frac{S^2}{5}\left(\frac{1}{A^2} + \frac{1}{B^2} + \frac{1}{C^2}\right)},$$

where A, B, C *are the sides of* K *and* S *is the measure of* K.

Another formula to give an explicit upper bound of the constant $c_2(K)$ can be found in [7].

Lemma 1 *For any* $\sigma \ge 0$, *it follows that*

$$\|-\Delta u\|_{L^2} \le \|-\Delta u + \sigma u\|_{L^2}, \quad \forall u \in W. \tag{8}$$

Proof We consider the eigenvalue problem as follows:

$$\text{Find } \psi \in V \text{ and } \tau \in \mathbb{R} \text{ s.t. } (\nabla\psi, \nabla v) = \tau(\psi, v), \quad \forall v \in V \tag{9}$$

Since the eigenfunctions ψ_i $(i = 1, 2, 3, \ldots)$ of (9) form an orthonormal basis of $L^2(\Omega)$, any $u \in W(\subset L^2(\Omega))$ can be expressed by $u = \sum_{i=1}^{\infty} a_i\psi_i$, where $a_i := (u, \psi_i)_{L^2}$ [8]. Let τ_i be the eigenvalue of (9) corresponding to ψ_i. Because all eigenfunctions ψ_i belong to W in the sense of distributions, we can obtain

$$
\begin{aligned}
\|-\Delta u+\sigma u\|_{L^2}^2 &= \left((-\Delta+\sigma)\sum_{i=1}^{\infty} a_i\psi_i, (-\Delta+\sigma)\sum_{i=1}^{\infty} a_i\psi_i\right)_{L^2} \\
&= \sum_{i=1}^{\infty} a_i^2 \left\{(\Delta\psi_i, \Delta\psi_i)_{L^2} + 2\sigma(-\Delta\psi_i, \psi_i)_{L^2} + \sigma^2(\psi_i, \psi_i)_{L^2}\right\} \\
&= \sum_{i=1}^{\infty} a_i^2 \left(\tau_i^2 + 2\sigma\tau_i + \sigma^2\right)(\psi_i, \psi_i)_{L^2} \\
&= \sum_{i=1}^{\infty} a_i^2 (\tau_i+\sigma)^2 . \qquad (10)
\end{aligned}
$$

Setting $\sigma = 0$, we also obtain

$$\|-\Delta u\|_{L^2}^2 = \sum_{i=1}^{\infty} a_i^2 \tau_i^2. \qquad (11)$$

From (10) and (11), the inequality (8) is proved. □

Using Theorem 1 and Lemma 1, we can get the following result.

Theorem 2 *Define two constants by* $C_1 := \max_{K\in T_h} c_1(K)$ *and* $C_2 := \max_{K\in T_h} c_2(K)$. *Then, the constant defined by*

$$C_M := \sqrt{C_2^2 + \sigma C_1^2}$$

satisfies (7).

Proof We define the interpolation $\Pi_1 : H^2(\Omega) \to V_h$ by

$$\Pi_1 u(N_i) = u(N_i), \quad i = 1, 2, \ldots, m.$$

Since $\Omega \subset \mathbb{R}^2$ is a polygonal domain, it follows that

$$|u|_{H^2} = \|\Delta u\|_{L^2}, \quad \forall u \in W,$$

see [9]. Therefore, we have

$$
\begin{aligned}
\|u - P_h u\|_\sigma^2 &\le \|u - \Pi_1 u\|_\sigma^2 \\
&= \|\nabla(u - \Pi_1 u)\|_{L^2}^2 + \sigma\|u - \Pi_1 u\|_{L^2}^2 \\
&\le \left(C_2^2 + \sigma C_1^2\right)\|-\Delta u\|_{L^2}^2 . \qquad (12)
\end{aligned}
$$

From (12) and Lemma 1,

$$\|u - P_h u\|_\sigma^2 \le \left(C_2^2 + \sigma C_1^2\right)\|-\Delta u + \sigma u\|_{L^2}^2, \quad \forall u \in W$$

is obtained. Hence, we can choose $C_M = \sqrt{C_2^2 + \sigma C_1^2}$. □

The inequality in the following lemma is needed to evaluate target eigenvalues in Sect. 4.

Lemma 2 *For any $u \in W$, it follows that*

$$\|u - P_h u\|_{L^2} \leq C_M \|u - P_h u\|_\sigma . \tag{13}$$

Proof For any $g \in L^2(\Omega)$, the problem:

$$\text{Find } \varphi \in V \text{ s.t. } (\varphi, v)_\sigma = (g, v)_{L^2}, \quad \forall v \in V$$

has a unique solution $\varphi_g \in W$. For any $u \in W$, we take $g = u - P_h u$. Using Aubin-Nitsche's trick and (7), it follows that,

$$\begin{aligned}
\|g\|_{L^2}^2 &= (g, g)_{L^2} \\
&= \left(-\Delta\varphi_g + \sigma\varphi_g, g\right)_{L^2} \\
&= \left(\varphi_g, g\right)_\sigma \\
&= \left(\varphi_g - P_h\varphi_g, g\right)_\sigma \\
&\leq \left\|\varphi_g - P_h\varphi_g\right\|_\sigma \|g\|_\sigma \\
&\leq C_M \left\|-\Delta\varphi_g + \sigma\varphi_g\right\|_{L^2} \|g\|_\sigma \\
&= C_M \|g\|_{L^2} \|g\|_\sigma .
\end{aligned}$$

This simply implies (13). □

3 Invertibility and inverse norm estimation for the elliptic operator

In this section, we propose a framework of proving the invertibility of $\mathcal{L}$ and a norm estimation for $\mathcal{L}^{-1}$. The effective framework in the case of $V = H_0^1(\Omega)$ is proposed in [1]. We adapt this framework to the case of $V = H^1(\Omega)$. First, we choose a number σ in order to determine the inner product $(\cdot, \cdot)_\sigma$ and the norm $\|\cdot\|_\sigma$ in V. In the case of $V = H^1(\Omega)$, σ is a positive number satisfying

$$\sigma > c(x) \ \ (\text{a.e. } x \in \Omega). \tag{14}$$

In the case of $V = H_0^1(\Omega)$, σ is a nonnegative number satisfying (14). Due to (14), we can define the function $a(x) := \sqrt{\sigma - c(x)}$ $(\in L^\infty(\Omega))$. We then define the operator $A : V\left(\subset L^2(\Omega)\right) \to V^*$ by

$$\langle Au, v\rangle := \left(a^2 u, v\right)_{L^2}, \ \forall v \in V. \tag{15}$$

Note that the operator A is compact because the embedding from $L^2(\Omega)$ to V^* is compact. We also define the unitary operator $\Phi : V \to V^*$ by

$$\langle \Phi u, v \rangle := (u, v)_\sigma, \quad \forall v \in V. \tag{16}$$

The boundedness of Φ and Φ^{-1} implies that Φ is a Fredholm operator. For any operator T, let $\sigma(T)$ be the spectrum of T and $\sigma_p(T)$ be the point spectrum of T. The following theorem gives an estimation of the operator norm $\|\mathcal{L}^{-1}\|_{V^*,V}$.

Theorem 3 *Suppose that* $0 \notin \sigma_p\left(\Phi^{-1}\mathcal{L}\right)$*, then the inverse of* $\mathcal{L}$ *defined by* (4) *exists and satisfies*

$$\left\|\mathcal{L}^{-1}\right\|_{V^*,V} \le |\mu_0|^{-1},$$

where $\mu_0 := \min\left\{|\mu| \;:\; \mu \in \sigma_p\left(\Phi^{-1}\mathcal{L}\right) \cup \{1\}\right\}$.

Proof We prove the invertibility of $\mathcal{L}$ by ensuring that $\mathcal{L}$ is a Fredholm operator. The method adapted to this proof is proposed by Oishi [10]. From (4), (15), and (16), it becomes clear that

$$\mathcal{L} = \Phi - A. \tag{17}$$

Because of (17) and the compactness of A, $\mathcal{L}$ is a Fredholm operator and we have

$$\begin{aligned}
\sigma\left(\Phi^{-1}\mathcal{L}\right) &= 1 - \sigma\left(\Phi^{-1}A\right) \\
&= 1 - \left\{\sigma_p\left(\Phi^{-1}\mathcal{L}\right) \cup \{0\}\right\} \\
&= \left\{1 - \sigma_p\left(\Phi^{-1}A\right)\right\} \cup \{1\} \\
&= \sigma_p\left(\Phi^{-1}\mathcal{L}\right) \cup \{1\}.
\end{aligned}$$

Next, since $\mathcal{L}$ is a self-adjoint operator, it follows that, for any $u \in V$,

$$\|\mathcal{L}u\|_{V^*}^2 = \left\|\Phi^{-1}\mathcal{L}u\right\|_\sigma^2 = \int_{-\infty}^{\infty} \mu^2 d\left(E_\mu u, u\right)_\sigma = \mu_0^2 \int_{-\infty}^{\infty} d\left(E_\mu u, u\right)_\sigma = \mu_0^2 \|u\|_\sigma^2,$$

where E_μ is the resolution of the identity of $\Phi^{-1}\mathcal{L}$. Therefore, we have

$$\sup_{u \in V \backslash \{0\}} \frac{\|u\|_\sigma}{\|\mathcal{L}u\|_{V^*}} \le |\mu_0|^{-1} < \infty.$$

Thus, $\mathcal{L}$ is one to one so that there exists $\mathcal{L}^{-1}$. Hence,

$$\left\|\mathcal{L}^{-1}\right\|_{V^*,V} := \sup_{g \in V^* \backslash \{0\}} \frac{\left\|\mathcal{L}^{-1}g\right\|_\sigma}{\|g\|_{V^*}} = \sup_{u \in V \backslash \{0\}} \frac{\|u\|_\sigma}{\|\mathcal{L}u\|_{V^*}} \le |\mu_0|^{-1}$$

is obtained. □

We consider the eigenvalue problem $\Phi^{-1}\mathcal{L}u = \mu u$ in V, i.e.,

$$(\nabla u, \nabla v)_{L^2} + (cu, v)_{L^2} = \mu\,(u, v)_\sigma\,, \quad \forall v \in V. \tag{18}$$

By setting $a(x) = \sqrt{\sigma - c(x)}$ and $\lambda = (1-\mu)^{-1}$, we transform (18) into

$$(u, v)_\sigma = \lambda\left(a^2 u, v\right)_{L^2}, \quad \forall v \in V. \tag{19}$$

According to Theorem 3, the eigenvalue λ of (19) minimizing $|\mu| = \left|1-\lambda^{-1}\right|$ gives an upper bound of $\left\|\mathcal{L}^{-1}\right\|_{V^*,V}$.

4 Verified evaluation of eigenvalues

In this section, we propose a method of verified evaluation for the eigenvalue problem (19). This method is a generalization of the method in [6] which is proposed by Liu and Oishi. Theorem 4 in this section and their theorem are the same in the case of $\sigma = 0$ and $a(x) = 1$ for all $x \in \Omega$. As a preliminary, let (λ_k, u_k) be kth eigenpair of (19) and E_k^0 be the space spanned by the eigenfunctions $\{u_i\}_{i=1}^k$. We define $E_k := \left\{v \in E_k^0 \;:\; \|av\|_{L^2} = 1\right\}$. Then, let λ_k^h be the kth eigenvalue of the problem as follows:

$$\text{Find } u_h \in V_h \text{ and } \lambda^h \in \mathbb{R} \text{ s.t. } (u_h, v_h)_\sigma = \lambda^h\left(a^2 u_h, v_h\right)_{L^2}, \quad \forall v_h \in V_h.$$

Each λ_k^h is computable with verification by using a method such as [11,12], etc. Using the min-max principle, we easily obtain an upper bound of λ_k as follows:

$$\lambda_k = \min_{H_k \subset V}\left(\max_{v \in H_k \setminus \{0\}} \frac{\|v\|_\sigma^2}{\|av\|_{L^2}^2}\right) \le \lambda_k^h,$$

where H_k denotes any k-dimensional subspace of V. This bound is well known as the Rayleigh-Ritz bound. However, a lower bound of λ_k is not obtained easily. The main purpose of this section is proposing how to estimate the lower bound. The following theorem gives the lower bound.

Theorem 4 *It follows that*

$$\frac{\lambda_k^h}{\lambda_k^h \|a\|_\infty^2 C_M^2 + 1} \le \lambda_k. \tag{20}$$

Proof In the case of $\lambda_k \|a\|_\infty^2 C_M^2 \ge 1$, it becomes clear that

$$\lambda_k \ge \frac{1}{\|a\|_\infty^2 C_M^2} \ge \frac{\lambda_k^h}{\lambda_k^h \|a\|_\infty^2 C_M^2 + 1}. \tag{21}$$

Therefore, we only have to consider the case of $\lambda_k \|a\|_\infty^2 C_M^2 < 1$. Since all eigenfunctions u of (19) are in W (defined in Sect. 2.2), from (7), it follows that

$$\begin{aligned} \|v - P_h v\|_\sigma &\le C_M \|-\Delta v + \sigma v\|_{L^2} \\ &\le C_M \left\|\lambda_k a^2 v\right\|_{L^2} \\ &\le \lambda_k C_M \|a\|_\infty \|av\|_{L^2} \\ &= C_M \|a\|_\infty \lambda_k, \end{aligned} \tag{22}$$

for any $v \in E_k$. Then, from Lemma 2 and (22),

$$\|aP_h v - av\|_{L^2} \le \|a\|_\infty C_M \|v - P_h v\|_\sigma \le \lambda_k \|a\|_\infty^2 C_M^2 < 1, \quad \forall v \in E_k$$

is obtained. Using the min-max principle, it follows that

$$\begin{aligned} \lambda_k^h &\le \max_{v \in E_k \setminus \{0\}} \frac{\|P_h v\|_\sigma^2}{\|aP_h v\|_{L^2}^2} \\ &= \max_{v \in E_k \setminus \{0\}} \frac{\|v\|_\sigma^2 - \|v - P_h v\|_\sigma^2}{\|av + aP_h v - av\|_{L^2}^2} \\ &= \max_{v \in E_k \setminus \{0\}} \frac{\|v\|_\sigma^2 - \|v - P_h v\|_\sigma^2}{\|av\|_{L^2}^2 + 2\,(av, aP_h v - av)_{L^2} + \|aP_h v - av\|_{L^2}^2} \\ &\le \max_{v \in E_k \setminus \{0\}} \frac{\lambda_k - \|v - P_h v\|_\sigma^2}{1 + 2\,(av, aP_h v - av)_{L^2} + \|aP_h v - av\|_{L^2}^2} \\ &\le \max_{v \in E_k \setminus \{0\}} \frac{\lambda_k - \|v - P_h v\|_\sigma^2}{1 - 2\,\|aP_h v - av\|_{L^2} + \|aP_h v - av\|_{L^2}^2} \\ &= \max_{v \in E_k \setminus \{0\}} \frac{\lambda_k - \|v - P_h v\|_\sigma^2}{\left(1 - \|aP_h v - av\|_{L^2}\right)^2} \\ &\le \max_{v \in E_k \setminus \{0\}} \frac{\lambda_k - \|v - P_h v\|_\sigma^2}{(1 - \|a\|_\infty C_M \|v - P_h v\|_\sigma)^2}. \end{aligned} \tag{23}$$

Then, we define $g(t) := \left(\lambda_k - t^2\right) / (1 - C_M \|a\|_\infty t)^2$. The function $g(t)$ is monotonically increasing in the domain $t \le C_M \|a\|_\infty \lambda_k$ and $t < (C_M \|a\|_\infty)^{-1}$. The assumption of this theorem, i.e., $C_M \|a\|_\infty \lambda_k < (C_M \|a\|_\infty)^{-1}$ implies that $g(t)$ is monotonically increasing only under the condition $t \le C_M \|a\|_\infty \lambda_k$. Hence, from (22) and (23),

$$\lambda_k^h \le \frac{\lambda_k - \lambda_k^2 \|a\|_\infty^2 C_M^2}{\left(1 - \lambda_k \|a\|_\infty^2 C_M^2\right)^2} = \frac{\lambda_k}{1 - \lambda_k \|a\|_\infty^2 C_M^2} \tag{24}$$

is obtained. The equivalent transformation of (24) gives (20). □

Remark 1 This theorem also can be proved by using the max-min principle; see [13] for a simple case.

5 Applications to semilinear elliptic problems

In this section, we discuss an application of the invertibility of $\mathcal{L}$ and a verified estimation of the operator norm $\|\mathcal{L}^{-1}\|_{V^*,V}$. These can be applied to verified computations of solutions to the semilinear elliptic boundary value problem (5) with the boundary condition (2) or (3). We assume that the nonlinear operator f from V to $L^2(\Omega)$ is Fréchet differentiable and bounded. Now, we can write the weak form of (5) as follows:

$$\text{Find } u \in V \text{ s.t. } (\nabla u, \nabla v)_{L^2} = (f(u), v)_{L^2}, \quad \forall v \in V. \tag{25}$$

Next, we define the nonlinear operator $\mathcal{F} : V \to V^*$ by

$$\langle \mathcal{F}(u), v\rangle := (\nabla u, \nabla v)_{L^2} - (f(u), v)_{L^2}, \quad \forall v \in V.$$

Using this notation, the weak form (25) can be transformed into the following problem:

$$\text{Find } u \in V \text{ s.t. } \mathcal{F}(u) = 0 \text{ in } V^*. \tag{26}$$

Moreover, we construct the approximate solution $u_h \in V_h$ to (26) by solving

$$(\nabla u_h, \nabla v_h)_{L^2} = (f(u_h), v_h)_{L^2}, \quad \forall v_h \in V_h.$$

We denote a numerical approximation of u_h by $\hat{u}_h \in V_h$, where $\hat{u}_h \in V_h$ may have rounding error inside due to floating-point number computation. Several works show verified numerical methods for weak solutions to (5), e.g., [1,2,4], etc. They need the invertibility of the linearized operator $\mathcal{F}'[\hat{u}_h]$ and a verified estimation of the operator norm $\|\mathcal{F}'[\hat{u}_h]^{-1}\|_{V^*,V}$, where $\mathcal{F}'[\hat{u}_h]$ is the Fréchet derivative of $\mathcal{F}$ at $\hat{u}_h$. The Fréchet derivative $\mathcal{F}'[\hat{u}_h] : V \to V^*$ satisfies

$$\left\langle \mathcal{F}'\left[\hat{u}_h\right] u, v\right\rangle = (\nabla u, \nabla v)_{L^2} + \left(-f'\left[\hat{u}_h\right] u, v\right)_{L^2}, \ \forall v \in V,$$

where $f'[\hat{u}_h] : V \to L^2(\Omega)$ is the Fréchet derivative of f at $\hat{u}_h$. If we have

$$-f'\left[\hat{u}_h\right] u = cu, \quad \forall u \in V \tag{27}$$

for some $c \in L^\infty(\Omega)$, $\mathcal{F}'[\hat{u}_h]$ is a specialized operator of $\mathcal{L}$ defined by (4). For example, in the case of $f(u) = u^2$, the Fréchet derivative of f at $\hat{u}_h \in L^\infty(\Omega)$ satisfies the condition (27). In another case, e.g., $f(u) = b \cdot \nabla u$ with $b \in (L^\infty(\Omega))^2$, the condition (27) is not satisfied. Under the assumption (27), we can give an estimation of $\|\mathcal{F}'[\hat{u}_h]^{-1}\|_{V^*,V}$ using the method in the previous sections.

6 Numerical Examples

In this section, we present three numerical examples of computing $\|\mathcal{F}'[\hat{u}_h]^{-1}\|_{V^*,V}$ and compare our results with the results based on the method in [2] in the Dirichlet type. All computations are carried out on a Windows 7, Intel(R) Core(TM) i7 860 CPU 2.80 GHz with 16.0 GB RAM by using MATLAB 2012a with INTLAB, a toolbox for verified numerical computations [14]. Therefore, the accuracy of numerical values in the following tables is verified in the sense that rounding error is strictly estimated. In the following example, we set $\Omega = (0,1) \times (0,1)$. The domain Ω is subdivided into a triangular mesh with isosceles right triangles. We use piecewise linear base functions as mentioned in Sect. 2.1.

6.1 Example 1

We consider the following semilinear problem with the Dirichlet boundary condition:

$$\begin{cases} -\Delta u = u^2 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{28}$$

This is well known as the Emden equation. The approximate solution $\hat{u}_h$ in consideration is a convex one such that $\max \hat{u}_h \approx 29$. In this case, we set $V = H_0^1(\Omega)$, $\sigma = 0$, and $c = -2\hat{u}_h$. Therefore, the linearized operator $\mathcal{F}'\left[\hat{u}_h\right] : V \to V^*$ satisfies

$$\left\langle \mathcal{F}'\left[\hat{u}_h\right] u, v \right\rangle = (\nabla u, \nabla v)_{L^2} + \left(-2\hat{u}_h u, v\right)_{L^2}, \ \forall v \in V.$$

In addition, we have $\|a\|_\infty^2 \approx 58$ for all mesh sizes. In Tables 1, 2, and 3 λ^* is an interval that encloses the eigenvalue λ of (19) minimizing $|\mu| = |1 - \lambda^{-1}|$.

Table 1 shows estimations of $\|\mathcal{F}'[\hat{u}_h]^{-1}\|_{V^*,V}$ by both methods. Each of them seems to converge on the same value as a mesh is refined. Our method gives a little better estimation than the method in [2] for all mesh sizes especially when mesh sizes are large. In the case of $n = 3$, however, only our method can give the estimation.

Table 1 Verified estimation of $\|\mathcal{F}'[\hat{u}_h]^{-1}\|_{V^*,V}$ for (28)

Mesh size: 2^{-n}	C_M	$\lambda^* \|a\|_\infty^2 C_M^2$	λ^*	$\left\|\mathcal{F}'\left[\hat{u}_h\right]^{-1}\right\|_{V^*,V}$	Method [2]
$n = 3$	6.1450E-2	3.8171E-1	$1.{}_{1727}^{6206}$	9.3749	Failed
$n = 4$	3.0725E-2	8.8968E-2	$1.{}_{4553}^{5850}$	3.4798	6.4603
$n = 5$	1.5363E-2	2.1846E-2	$1.5{}_{418}^{757}$	2.9074	3.2816
$n = 6$	7.6812E-3	5.4369E-3	$1.5{}_{646}^{733}$	2.7858	2.8690
$n = 7$	3.8406E-3	1.3577E-3	$1.57{}_{04}^{27}$	2.7565	2.7750

Table 2 Verified estimation of $\|\mathcal{F}'[\hat{u}_h]^{-1}\|_{V^*,V}$ for (29)

Mesh size: 2^{-n}	C_M	$\lambda^* \|a\|_\infty^2 C_M^2$	λ^*	$\left\|\mathcal{F}'\left[\hat{u}_h\right]^{-1}\right\|_{V^*,V}$	Method [2]
$n = 3$	6.5031E-2	4.3809E-1	$1.^{5780}_{0971}$	16.220	Failed
$n = 4$	3.2516E-2	1.0223E-1	$1.^{5429}_{3996}$	3.8594	8.2292
$n = 5$	1.6258E-2	2.5110E-2	$1.^{5337}_{4959}$	3.0916	3.4882
$n = 6$	8.1289E-3	6.2495E-3	1.5^{314}_{216}	2.9347	2.9790
$n = 7$	4.0645E-3	1.5607E-3	1.5^{308}_{282}	2.8973	2.8676

6.2 Example 2

We consider the following semilinear problem with the Dirichlet boundary condition:

$$\begin{cases} -\Delta u = u^2 - u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{29}$$

The approximate solution $\hat{u}_h$ in consideration is a convex one such that $\max \hat{u}_h \approx 30$. In this case, we set $V = H_0^1(\Omega)$, $\sigma = 1$, and $c = -2\hat{u}_h + 1$. Therefore, the linearized operator $\mathcal{F}'[\hat{u}_h] : V \to V^*$ satisfies

$$\left\langle \mathcal{F}'\left[\hat{u}_h\right] u, v \right\rangle = (\nabla u, \nabla v)_{L^2} + \left(\left(-2\hat{u}_h + 1\right) u, v\right)_{L^2}, \ \forall v \in V.$$

In addition, we have $\|a\|_\infty^2 \approx 61$ for all mesh sizes. Note that our method and the method in [2] endow V with different inner products in this case. Our method endows V with the inner product $(\cdot,\cdot)_{\sigma=1} := (\nabla\cdot, \nabla\cdot)_{L^2} + (\cdot,\cdot)_{L^2}$. Their method endows V with the inner product $(\cdot,\cdot)_{\sigma=0} = (\nabla\cdot, \nabla\cdot)_{L^2}$. Since we have

$$\sup_{u\in V} \frac{\|u\|_{\sigma=0}}{\|\mathcal{L}u\|_{V^*}} < \sup_{u\in V} \frac{\|u\|_{\sigma=1}}{\|\mathcal{L}u\|_{V^*}},$$

the estimation values of each method converge on different values.

In Table 2, estimation results for (29) derived by our method and the method in [2] are summarized. In the cases of $n \le 6$, our method gives a better estimation than their method. On the other hand, in the case of $n = 7$, their method gives a better estimation. This is because our method and their method endow V with different inner products as mentioned above.

6.3 Example 3

We consider the following semilinear problem with the Neumann boundary condition:

$$\begin{cases} -\Delta u = u^3 - u & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{30}$$

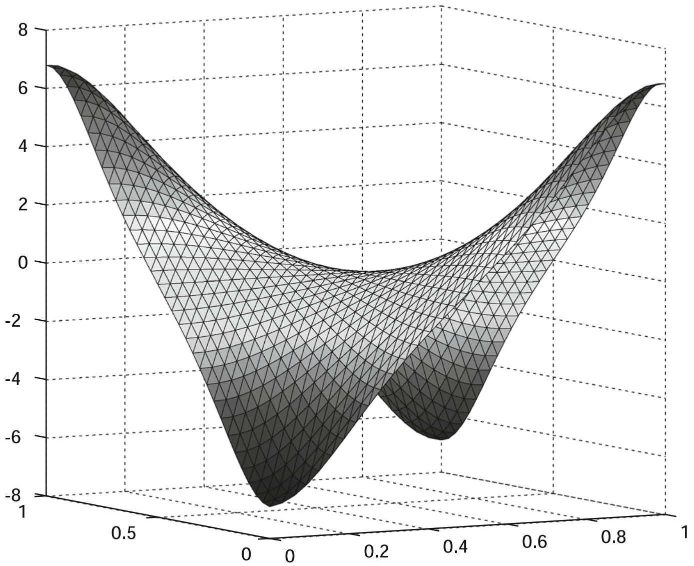

Fig. 1 The approximate solution $\hat{u}_h$ to (30)

Table 3 Verified estimation of $\|\mathcal{F}'[\hat{u}_h]^{-1}\|_{V^*,V}$ for (30)

Mesh size: 2^{-n}	C_M	$\lambda^* \|a\|_\infty^2 C_M^2$	λ^*	$\left\|\mathcal{F}'\left[\hat{u}_h\right]^{-1}\right\|_{V^*,V}$
$n = 3$	6.5031E-2	1.2663	$^{1.7964}_{0.7926}$	Failed
$n = 4$	3.2516E-2	2.6447E-1	$1.^{6834}_{3313}$	5.0818
$n = 5$	1.6258E-2	6.2054E-2	$1.^{6458}_{5496}$	2.9946
$n = 6$	8.1289E-3	1.5214E-2	1.6^{353}_{108}	2.6776
$n = 7$	4.0650E-3	3.7822E-3	1.6^{326}_{266}	2.6064

The approximate solution $\hat{u}_h$ in consideration (see Fig. 1) is characterized by the following properties:

$$\max_{x\in\Omega} \hat{u}_h \approx 6.8 \text{ is taken at the two points } x = (1, 0)\,,\ (0, 1)\,,$$
$$\min_{x\in\Omega} \hat{u}_h \approx -6.8 \text{ is taken at the two points } x = (0, 0)\,,\ (1, 1)\,.$$

In this case, we set $c = 1 - 3\hat{u}_h^2$ and $\sigma = 1$. Therefore, the linearized operator $\mathcal{F}'[\hat{u}_h] : V \to V^*$ satisfies

$$\left\langle \mathcal{F}'\left[\hat{u}_h\right] u, v \right\rangle = (\nabla u, \nabla v)_{L^2} + \left(\left(1 - 3\hat{u}_h^2\right) u, v\right)_{L^2}, \ \forall v \in V.$$

In addition, we have $\|a\|_\infty^2 \approx 140$ for all mesh sizes.

Table 3 shows estimation results for (30). In the cases of $n \geq 4$, our method can give an estimation of $\|\mathcal{F}'[\hat{u}_h]^{-1}\|_{V^*,V}$. In the case of $n = 3$, however, our method fails in the estimation. This is because the interval λ^* encloses 1 so that the point spectrum of $\Phi^{-1}\mathcal{F}'[\hat{u}_h]$ may enclose 0 (see Theorem 3).

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References

1. Plum, M.: Computer-assisted proofs for semilinear elliptic boundary value problems. Jpn. J. Ind. Appl. Math. **26**, 419–442 (2009)
2. Nakao, M.T., Hashimoto, K., Watanabe, Y.: Numerical method to verify the invertibility of linear elliptic operators with applications to nonlinear problems. Computing **75**(1), 1–14 (2005)
3. Breuer, B., Horak, J., McKenna, P.J., Plum, M.: A computer-assisted existence and multiplicity proof for travelling waves in a nonlinearly supported beam. J. Differ. Equ. **224**, 60–97 (2006)
4. Takayasu, A., Liu, X., Oishi, S.: Verified computations to semilinear elliptic boundary value problems on arbitrary polygonal domains. NOLTA IEICE **4**(1), 34–61 (2013)
5. Kobayashi, K.: On the interpolation constants over triangular elements. RIMS Kōkyūroku **1733**, 58–77 (2011). (in Japanese)
6. Liu, X., Oishi, S.: Verified eigenvalue evaluation for the Laplacian over polygonal domains of arbitrary shape. SIAM J. Numer. Anal. **51**(3), 1634–1654 (2013)
7. Kikuchi, F., Liu, X.: Estimation of interpolation error constants for the P0 and P1 triangular finite elements. Comput. Methods Appl. Mech. Eng. **196**, 3750–3758 (2007)
8. Dautray, R., Lions, J.: Mathematical Analysis and Numerical Methods for Science and Technology, vol. 3. Springer, Heidelberg (2000)
9. Grisvard, P.: Elliptic problems in nonsmooth domains. Classics in Applied Mathematics, vol. 69. SIAM (2011)
10. Oishi, S.: Numerical verification of existence and inclusion of solutions for nonlinear operator equations. J. Comput. Appl. Math. **60**, 171–185 (1995)
11. Behnke, H.: The calculation of guaranteed bounds for eigenvalues using complementary variational principles. Computing **47**, 11–27 (1991)
12. Rump, S.M.: Verified bounds for singular values, in particular for the spectral norm of a matrix and its inverse. BIT Numer. Math. **51**, 367–384 (2011)
13. Liu, X., Oishi, S.: Guaranteed high-precision estimation for P0 interpolation constants on triangular finite elements. Jpn. J. Ind. Appl. Math. **30**(3), 635–652 (2013)
14. Rump, S.M.: INTLAB-INTerval LABoratry. In: Csendes, T. (ed.) Developments in Reliable Computing, pp. 77–104. Kluwer, Dordrecht (1999)