GALOIS GROUPS OF PROJECTIONS

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1. Forward

This paper contains the text of a talk given at the "Symposium on Algebraic Geometry" held at Niigata, February 4-6, 2004. Full details can be found in [13]. We are very grateful to Professor Hisao Yoshihara for inviting us to the Symposium at Niigata.

2. Galois and monodromy groups

Let X and Y be irreducible complex algebraic varieties; $\pi : X \to Y$ a finite morphism of degree d ($\dim X = \dim Y$). The Galois group of π is by definition

$$Gal(\hat{K}/k(Y)) \subseteq S_d$$

where $K(Y)$ is the field of rational functions on Y, $\hat{K}$ is the Galois closure of the field extension $K(X)/K(Y)$, and S_d is the symmetric group on d letters. The Galois group acts on the roots of the minimal polynomial of a primitive element of $K(X)/K(Y)$, and it is well defined up to conjugation as a subgroup of S_d.

One can also define the monodromy group G_π of π as the image of the "path-lifting" representation:

$$\rho : \pi_1(U, y) \to \mathrm{Aut}(\pi^{-1}(y)) \cong S_d$$

where U is a Zariski open subset of Y over which π is étale.

It is not difficult to see the monodromy group is isomorphic to the Galois group [7], and so in particular does not depend on the choice of the open set U.

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3. Main results

Zariski [17] proved that, if $g > 6$ and X is a general smooth projective curve of genus g, then the Galois group of any covering $X \to \mathbb{P}^1$ is not solvable.

Zariski thought of this result as the geometric analogue of Abel's Theorem on the non solvability by radicals of a general algebraic equation of degree $n \geq 5$. In modern terminology, Zariski's Theorem says a general curve X of genus $g > 6$ cannot be rationally uniformized by radicals. In his paper, Zariski goes further and asks whether a general X can be *algebraically* uniformized by radicals, that is, whether there exists a finite map $Z \to X$ where Z is a curve that can be rationally uniformized by radicals. As far as we know this problem is still open. We will give in a forthcoming paper [14] an example of a genus 7 curve that can be uniformized by radicals algebraically but not rationally.

Zariski's result has been since greatly strengthened, and it is now known that, for a general curve X of genus $g > 3$, the monodromy group of a degree d indecomposable covering $X \to \mathbb{P}^1$ is either S_d -in which case we say the covering is *uniform* - or A_d [6].

As far as existence of uniform coverings (Galois groups S_d), it is classical that a general curve X of genus g admits a degree d uniform covering $X \to \mathbb{P}^1$ if $d \geq \frac{g+2}{2}$ (every curve if $d \geq g + 2$).

The corresponding results for the alternating groups A_d are more recent. Magaard and Volklein [9] have proven that a general curve X of genus g admits a degree d covering $X \to \mathbb{P}^1$ with monodromy group A_d if $d \geq 2g + 1$. Artebani and Pirola [2] have shown every X curve of genus g admits such a covering if $d \geq 12g + 4$.

In this paper we will be interested in coverings of $\mathbb{P}^1$ obtained by projecting a curve embedded in projective space from a codimension two linear subspace, thus for example we project a plane curve from a point, or a space curve from a line.

We use the following notation: $C \subset \mathbb{P}^r$ will be a reduced, irreducible, nondegenerate curve of degree d; $L \subset \mathbb{P}^r$ an $(r-2)$-plane;

We say L is uniform if the projection $\pi_L : C \dashrightarrow \mathbb{P}^1$ is uniform. We mention a few results in this context:

- Miura and Yoshihara [11, 12, 15] have classified the Galois groups of projections of smooth plane curves of degree ≤ 5. In particular, they showed there are only finitely many non-uniform points for these curves.
- Cukierman [4] has proven that, if C is a general plane curve of degree d, f or every $x \in \mathbb{P}^2 \setminus C$ the projection $\pi_x : C \to \mathbb{P}^1$ is uniform.

- the Uniform Position Principle can be stated as follows: for any nondegenerate curve $C \subset \mathbb{P}^r$, the general $(r-2)$-plane L is uniform.

It is therefore natural to ask for a bound for the dimension of non-uniform subspaces in the grassmannian $\mathbb{G}(r-2, \mathbb{P}^r)$. In this respect we have

Theorem 3.1. *For any nondegenerate curve $C \subset \mathbb{P}^r$, The locus of non-uniform subspaces has codimension at least two in $\mathbb{G}(r-2, \mathbb{P}^r)$.*

For smooth curves the theorem is easily proven (cf. the proof of Theorem 3.2 below); for curves with arbitrary singularities the proof is more involved and can be found in [13].

Theorem 3.1 is sharp: in $\mathbb{P}^2$ there are plane curve with non uniform points (e.g. the Galois coverings of Miura and Yoshihara). When $r > 2$, one can construct "trivial" examples of codimension two families of non-uniform subspaces as follows. Suppose $x \in \mathbb{P}^r$ is a **non-birational** point, that is, projection from x

$$\pi_x : C \dashrightarrow \mathbb{P}^{r-1}$$

is not birational onto its image. Then the Schubert cycle of (r–2)-planes through x:

$$\sigma(x) = \{L \in \mathbb{G}(r-2, \mathbb{P}^r) : \ x \in L\}$$

is contained in the non-uniform locus, because $x \in L$ not birational and $r > 2 \Rightarrow L$ decomposable (π_L factors through π_x) $\Rightarrow L$ non-uniform.

For smooth curves in $\mathbb{P}^3$ we have improved Theorem 3.1 classifying curves with a locus of non-uniform lines of codimension two. The result is:

Theorem 3.2. *Suppose: $C \subset \mathbb{P}^3$ is a smooth irreducible nondegenerate curve; $\Sigma \subset \mathbb{G}(1, \mathbb{P}^3)$ is an irreducible surface; the general line L in Σ is non-uniform and does not meet C. Then one of the following possibilities holds:*

(1) *$\Sigma = \sigma(x)$, with $x \in \mathbb{P}^3 \setminus C$ non-birational;*
(2) *C is a twisted cubic curve, and $L = H_1 \cap H_2$ where H_1, H_2 are osculating planes for C;*
(3) *C is a rational curve of degree 4 (resp. 6), and $L = H_1 \cap H_2$ where H_1, H_2 bitangent (resp. tritangent) planes for C.*

Remarks

- The theorem holds as well for curves with mild singularities (for example for curves with only nodes and simple cusps).

- Every rational curve of degree 3 or 4 admits a surface of non-uniform lines as in the theorem.
- The rational sextic curve studied by in [3] admits a surface of non-uniform lines, but this is not the case for a general rational sextic.

Sketch of the proof of Theorem 3.2
The codomain of projection from the line L is the line $\mathbb{P}^1_L$ of planes through L in the dual projective space. A standard argument shows that, since the projection is not uniform, $\mathbb{P}^1_L$ contains at least two planes that are either multitangent or osculating planes for C, that is, are in the singular locus C^*_{sing} of the dual surface C^*. Since C^*_{sing} is one dimensional, we see $\dim \Sigma \leq 2$, and so obtain a proof of Theorem 3.1 in case C is smooth - more generally, in case C^* is reduced.
Case one: $\mathbb{P}^1_L$ contains at least three singular points of C^*.
Since L varies in a surface, it follows C^*_{sing} has a planar component Z of degree ≥ 3 and $\mathbb{P}^1_L$ belongs to the plane of Z. Dualizing we see $\Sigma = \sigma(x)$, and conclusion (1) holds.
Case two: $\mathbb{P}^1_L$ contains exactly two singular points and no smooth point of C^*.
Then the projection π_L has only two branch points, hence $C \cong \mathbb{P}^1$ and $\pi_L(z) = z^d$.
Since L varies in a surface, we find infinitely many osculating planes H with

$$C \cap H = dP.$$

But then $d = 3$, and conclusion (2) holds.
Case three: $\mathbb{P}^1_L$ contains exactly two singular points and at least one smooth point of C^*.
Then the monodromy group G_L contains a transposition, while $G_L \neq S_d$. It follows that G_L is not primitive, hence π_L factors nontrivially as

$$C \xrightarrow{\alpha} Y \xrightarrow{\beta} \mathbb{P}^1.$$

The morphism β has exactly two branch points, otherwise $\mathbb{P}^1_L$ would meet C^*_{sing} in more than 2 points. Thus $Y \cong \mathbb{P}^1$ and $\beta(z) = z^e$ for some $e \geq 2$. Since C has a finite number of hyperosculating and multiosculating planes, we have $e = \deg \beta = 2$.
We may assume $\Sigma \neq \sigma(x)$, otherwise conclusion (1) holds. Then a general plane H contains some line $L \in \Sigma$. We claim this implies C is rational. Since the projection from L factors as $C \xrightarrow{\alpha} \mathbb{P}^1 \xrightarrow{\beta} \mathbb{P}^1_L$, we have

$$C \cap H = E_1 + E_2$$

where the divisors E_1 and E_2 are fibres of α, and therefore are lineraly equivalent. Since H is a general plane, the Uniform Position Principle now implies $A_1 \sim A_2$ for *every* pair of divisors A_1, A_2 of degree $a = \deg\alpha$ with $A_1 + A_2 = C \cap H$.

Given 2 distinct points P and Q in $C \cap H$, we choose F_1 and F_2 of degree $a-1$ such that

$$P + Q + F_1 + F_2 = C \cap H.$$

Then

$$P + F_1 \sim Q + F_2 \quad \text{e} \quad P + F_2 \sim Q + F_1$$

hence

$$2P \sim 2Q.$$

It follows the Jacobian contains infinitely many 2-torsion points and C must be rational.

Thus C is rational, and the general $L \in \Sigma$ is contained in 2 multitangent planes H such that $C \cap H = 2E$. Since L varies in a surface, we must have a curve of divisors $2E$ in the linear system defining the embedding $C \hookrightarrow \mathbb{P}^3$.

We set:

- $\mathbb{P}_b = \mathbb{P}H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(b))$ = effective divisors of degree b on $\mathbb{P}^1$;
- $\mathbb{P}(V) \cong \mathbb{P}^{3*} \subset \mathbb{P}_{2a}$ = linear series defining $C \hookrightarrow \mathbb{P}^3 = \mathbb{P}(V^*)$;
- $q : \mathbb{P}_a \longrightarrow \mathbb{P}_{2a}$ the map $E \mapsto 2E$;
- $Y \subset \mathbb{P}_a$ a curve of divisors E with $q(Y) \subset \mathbb{P}(V)$.

Now the conclusion follows from:

Proposition 3.3. *Suppose*

(1) $Y \subset \mathbb{P}_a$ *is a reduced, irreducible, curve of degree* e*;*
(2) $q(Y)$ *spans a* 3*-plane* $\mathbb{P}(V) \subset \mathbb{P}_{2a}$*;*
(3) *the linear series* $\mathbb{P}(V)$ *is base point free and defines an embedding of* $\mathbb{P}^1$ *in* $\mathbb{P}^3$.

Then either $a = e = 2$ *and* Y *is a smooth conic or* $a = e = 3$ *and* Y *is a twisted cubic curve.*

Examples

(1) Suppose $a = 2$. Then $q(\mathbb{P}_2)$ is a Veronese surface in $\mathbb{P}_4$, and every 3-plane $\mathbb{P}(V)$ in $\mathbb{P}_4$ cuts a curve $q(Y)$ on the Veronese. It follows every rational quartic admits a surface of non uniform (even decomposable) lines.

(2) Case $a = 3$: the intersection of $q(\mathbb{P}_3)$ with a general 3-plane $\mathbb{P}(V)$ is zero dimensional, hence the general rational sextic does not have a surface of lines as in conclusion (3) of Theorem 3.2. On the other hand, let $Y \subset \mathbb{P}_3$ be the twisted cubic:

$$f(t) = [x + tx^3 + t^2 - t^3x^2]$$

The image $q(Y) \subset \mathbb{P}_6$ spans a $\mathbb{P}^3$. We obtain the rational sextic studied by Barth and Moore, which has a surface of non-uniform lines as in conclusion (3) of Theorem 3.2..

(3) The normal rational quartic $Y \subset \mathbb{P}_4$:

$$f(t) = [1 - 2xt - 2x^2t^2 - 4x^3t^3 + 4x^4t^4]$$

has image $q(Y) \subset \mathbb{P}_8$ spanning just a $\mathbb{P}^4$:

$$[1 - 4tx + 28t^4x^4 - 32t^7x^7 + 16t^8x^8].$$

and only a $\mathbb{P}^3$ in characteristic 7 !! Thus proposition is false in $char. = 7$.

Sketch of the proof of Proposition 3.3 Given $s \in \mathbb{P}^1$, let $H(s) \subset \mathbb{P}_a$ denote the hyperplane of divisors containing s. The curve Y is not in $H(s)$ for any s. We claim:
For a general $s \in \mathbb{P}^1$, the intersection $H(s) \cap Y$ consists of $e = \deg Y$ distinct points.
Otherwise: for every s the hyperplane $H(s)$ is tangent to Y:

$$\Gamma = \{H(s) : s \in \mathbb{P}^1\} \subset Y^* \subset \mathbb{P}_a^*$$

Biduality theorem (roughly) $\Rightarrow$ $Y \subset \Gamma^* = \Delta$ where Δ is the hypersurface defined by the discriminant. Thus every divisor in Y contains at least one point of multiplicity ≥ 2, and so the linear series $\mathbb{P}(V)$ generated by $q(Y)$ contains infinitely many divisors with a point of multiplicity ≥ 4, absurd.
Consider the projection of C from a generic tangent line: for $s \in \mathbb{P}^1$ general, the tangent line $T_{f(s)}C$ meets C only at the point of tangency $f(s)$ (Kaji's theorem [8]) and with multiplicity 2. Therefore projection from $T_{f(s)}C$ is a morphism of degree $2a - 2$

$$\pi : C = \mathbb{P}^1 \to \mathbb{P}^1.$$

Furthermore, if E is a divisor in $Y \cap H(s)$, the linear series defining π contains the divisor

$$2E - 2s = 2s_1 + \cdots + 2s_{a-1}$$

and so $s_1 + \cdots + s_{a-1}$ is contained in the ramification divisor R_π of π. We conclude, since $Y \cap H(s)$ consists of e distinct divisors, that $\deg R_\pi \geq e(a - 1)$.

We then have by Riemann-Hurwitz

$$-2 = -2(2a-2) + \deg(R_\pi) \geq -2(2a-2) + e(a-1)$$

and so

$$e \leq 4 - \frac{2}{a-1}.$$

Since $e > 1$, we must have either $e = 3$ and $a \geq 3$ or $e = 2$ and $a \geq 2$. One can now finish the proof using results due to Hopf and Eisenbud [5].

In case $C \subset \mathbb{P}^r$ is an arbitrary reduced, irreducible, nondegenerate curve, essentially the same arguments prove the following theorem [13]:

Theorem 3.4. *Suppose*

- $C \subset \mathbb{P}^r$ *reduced, irreducible, nondegenerate curve.*
- $\Sigma \subset \mathbb{G}(r-2, \mathbb{P}^3)$ *irreducible of dimension* $r-1$.
- *Generic subspace of* Σ *is decomposable and does not meet* C.

Then either C *is rational or the projection*

$$\{(L, H) \in \Sigma \times \mathbb{P}^{r*} : \ L \subset H\} \rightarrow \mathbb{P}^{r*}$$

is not dominant.

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