On Sectional Invariants of Polarized Manifolds

Yoshiaki FUKUMA

1 Introduction.

Let X be a projective variety of $\dim X = n$ over the complex number field, and let L be an ample line bundle on X. Then the pair (X, L) is called a *polarized variety*. Moreover if X is smooth, then (X, L) is called a polarized *manifold*.

When we study polarized varieties, it is useful to use their invariants.

One of the well-known invariants of (X, L) is the sectional genus $g(L)$ of (X, L). Assume that X is smooth and L is very ample. Let $H_1, \dots, H_{n-1}$ be general members of $|L|$. Then $X_{n-1} := H_1 \cap \dots \cap H_{n-1}$ is a smooth projective curve by the Bertini theorem. In this case $g(L) = h^1(\mathcal{O}_{X_{n-1}})$. Furthermore in [3] we define the notion of the i-th sectional geometric genus $g_i(X, L)$ of (X, L) for every integer i with $0 \leq i \leq n$. Here we explain the meaning of this invariant if X is smooth, L is very ample, and i is an integer with $1 \leq i \leq n-1$. Let $H_1, \dots, H_{n-i}$ be general members of $|L|$. We put $X_{n-i} := H_1 \cap \dots \cap H_{n-i}$. Then X_{n-i} is smooth, $\dim X_{n-i} = i$, and $g_i(X, L) = h^i(\mathcal{O}_{X_{n-i}})$.

These induce the notion of the *i-th sectional invariant of* (X, L) (see Definition 2.1 below). The i-th sectional invariants are expected to reflect properties of i-dimensional geometry.

In this paper, we assume that (X, L) is a polarized manifold. First we define another i-th sectional invariants, the *i-th sectional H-arithmetic genus* $\chi_i^H(X, L)$, the *i-th sectional Euler number* $e_i(X, L)$, the *i-th sectional Betti number* $b_i(X, L)$, and the *i-th sectional Hodge number* $h_i^{j,i-j}(X, L)$ *of type* $(j, i-j)$ *of* (X, L) (see Definition 3.1) and we will study some properties of these. Next, we consider the case where $i = 2$. In this case, the second sectional invariants reflect properties of geometry of projective surfaces. So we believe that by using sectional invariants of (X, L), we can propose some problems which are analogous to those of projective surfaces.

The author would like to thank Professors H.Yoshihara and H.Kojima for giving him an opportunity to talk about this topic.

2 Preliminaries.

Notation 2.1 Let (X, L) be a polarized manifold of $\dim X = n$. Assume that $\mathrm{Bs}|L| = \emptyset$. Then by the Bertini theorem, there exists a sequence of subvarieties

$$X_{n-1} \subset \cdots \subset X_1 \subset X$$

such that $\dim X_j = n - j$ and $X_j \in |L_{j-1}|$ for every integer j with $1 \leq j \leq n-1$, where $X_0 := X$, $L_0 := L$, and $L_j := L_{j-1}|_{X_j}$ for $1 \leq j \leq n-1$.

Definition 2.1 Let (X, L) be a polarized manifold of $\dim X = n$. Let $I(Y)$ (or I) be an invariant of a smooth projective variety Y of $\dim Y = i$, where i is an integer with $0 \leq i \leq n$. Then an invariant $F_i(X, L)$ of (X, L) is called the *i-th sectional invariant of* (X, L) *associated with* I if $F_i(X, L) = I(X_{n-i})$ under the assumption that $\mathrm{Bs}|L| = \emptyset$.

Notation 2.2 (1) Let X be a projective variety of $\dim X = n$, let L be an ample line bundle on X, and let t be an indeterminate. Then the Euler-Poincaré characteristic of $L^{\otimes t}$, $\chi(L^{\otimes t})$, is a polynomial in t of degree n (see §1 of Chapter I in [10]), and we put

$$\chi(L^{\otimes t}) = \sum_{j=0}^{n} \chi_j(X, L) \frac{t^{[j]}}{j!},$$

where

$$t^{[j]} = \begin{cases} t(t+1)\cdots(t+j-1) & \text{if } j > 0, \\ 1 & \text{if } j = 0. \end{cases}$$

(2) Let Y be a smooth projective variety of $\dim Y = i$, let $\mathcal{T}_Y$ be the tangent bundle of Y, and let Ω_Y be the dual bundle of $\mathcal{T}_Y$. For every integer j with $0 \leq j \leq i$, we put

$$\begin{aligned} h_{i,j}(c_1(Y), \cdots, c_i(Y)) &:= \chi(\Omega_Y^j) \\ &= \int_Y \mathrm{ch}(\Omega_Y^j)\mathrm{Td}(\mathcal{T}_Y). \end{aligned}$$

(3) Let (X, L) be a polarized manifold of $\dim X = n$. For every integers i and j with $0 \leq i \leq n$ and $0 \leq j \leq i$, we put

$$C_j^i(X, L) := \sum_{l=0}^{j} (-1)^l \binom{n-i+l-1}{l} c_{j-l}(X) L^l,$$

$$w_i^j(X, L) := h_{i,j}(C_1^i(X, L), \cdots, C_i^i(X, L)) L^{n-i}.$$

(4) Let X be a smooth projective variety of $\dim X = n$. For every integers i and j with $0 \leq j \leq i \leq n$, we put

$$H_1(i,j) := \begin{cases} \sum_{s=0}^{i-j-1}(-1)^s h^s(\Omega_X^j) & \text{if } j \neq i, \\ 0 & \text{if } j = i, \end{cases}$$

$$H_2(i,j) := \begin{cases} \sum_{t=0}^{j-1}(-1)^{i-t} h^t(\Omega_X^{i-j}) & \text{if } j \neq 0, \\ 0 & \text{if } j = 0. \end{cases}$$

Definition 2.2 (1) Let X (resp. Y) be an n-dimensional projective manifold, and let L (resp. A) be an ample line bundle on X (resp. Y). Then (X,L) is called a *simple blowing up of* (Y,A) if there exists a birational morphism $\pi : X \to Y$ such that π is a blowing up at a point of Y and $L = \pi^*(A) - E$, where E is the π-exceptional effective reduced divisor.

(2) Let X (resp. Y) be an n-dimensional projective manifold, and let L (resp. A) be an ample line bundle on X (resp. Y). Then we say that (Y,A) is a *reduction of* (X,L) if there exists a birational morphism $\mu : X \to Y$ such that μ is a composite of simple blowing ups and (Y,A) is not obtained by a simple blowing up of any polarized manifold. The morphism μ is called the *reduction map*.

Remark 2.1 Let (X,L) be a polarized manifold and let (M,A) be a reduction of (X,L). Let $\mu : X \to M$ be the reduction map.

(1) Assume that $\mathrm{Bs}|L| = \emptyset$. Then for a general member D of $|L|$, D and $\mu(D) \in |A|$ are smooth.

(2) If (X,L) is not obtained by a simple blowing up of another polarized manifold, then (X,L) is a reduction of itself.

(3) A reduction of (X,L) always exists (see Chapter II, (11.11) in [2]).

3 Definitions of sectional invariants of polarized manifolds and their properties.

First we define the following.

Definition 3.1 (See [3], [7] and [8].) Let (X,L) be a polarized manifold of $\dim X = n$, and let i and j be integers with $0 \leq i \leq n$ and $0 \leq j \leq i$. (Here we use Notation 2.2.)

(1) The *i-th sectional geometric genus $g_i(X,L)$ of (X,L)* is defined as follows:

$$g_i(X,L) = (-1)^i(\chi_{n-i}(X,L) - \chi(\mathcal{O}_X)) + \sum_{j=0}^{n-i}(-1)^{n-i-j}h^{n-j}(\mathcal{O}_X).$$

(2) The *i-th sectional H-arithmetic genus* $\chi_i^H(X,L)$ *of* (X,L) is defined as follows:

$$\chi_i^H(X,L) := \chi_{n-i}(X,L).$$

(3) The *i-th sectional Euler number* $e_i(X,L)$ *of* (X,L) is defined by the following:

$$e_i(X,L) := \sum_{l=0}^{i}(-1)^l\binom{n-i+l-1}{l}c_{i-l}(X)L^{n-i+l}.$$

(4) The *i-th sectional Betti number* $b_i(X,L)$ *of* (X,L) is defined by the following:

$$b_i(X,L) = \begin{cases} e_0(X,L) & \text{if } i=0, \\ (-1)^i\left(e_i(X,L) - \sum_{j=0}^{i-1}2(-1)^j h^j(X,\mathbb{C})\right) & \text{otherwise.} \end{cases}$$

(5) The *i-th sectional Hodge number* $h_i^{j,i-j}(X,L)$ *of* (X,L) is defined by the following:

$$h_i^{j,i-j}(X,L) = (-1)^{i-j}\left\{w_i^j(X,L) - H_1(i,j) - H_2(i,j)\right\}.$$

Remark 3.1 (0) The i-th sectional geometric genus and the i-th sectional H-arithmetic genus can be defined for any polarized *variety*.

(1) Since $\chi_j(X,L) \in \mathbb{Z}$ for every integer j with $0 \le j \le n$, by definition we get that $g_i(X,L) \in \mathbb{Z}$ and $\chi_i^H(X,L) \in \mathbb{Z}$ for every integer i with $0 \le i \le n$.

(2) If $i=0$, then

$$g_0(X,L) = \chi_0^H(X,L) = e_0(X,L) = b_0(X,L) = h_0^{0,0}(X,L) = L^n.$$

(3) If $i=1$, then

$$\begin{aligned} g_1(X,L) &= g(L). \\ \chi_1^H(X,L) &= 1-g(L). \\ e_1(X,L) &= 2-2g(L). \\ b_1(X,L) &= 2g(L). \\ h_1^{1,0}(X,L) &= h_1^{0,1}(X,L) = g(L). \end{aligned}$$

(4) If $i=n$, then

$$\begin{aligned} g_n(X,L) &= h^n(\mathcal{O}_X). \\ \chi_n^H(X,L) &= \chi(\mathcal{O}_X). \\ e_n(X,L) &= e(X). \\ b_n(X,L) &= b_n(X). \\ h_n^{j,n-j}(X,L) &= h^{j,n-j}(X). \\ h_n^{n-j,j}(X,L) &= h^{n-j,j}(X). \end{aligned}$$

Proposition 3.1 *Let (X, L) be a polarized manifold of $\dim X = n$. Assume that X is smooth and $\mathrm{Bs}|L| = \emptyset$. We use Notation 2.1. Then for every integers i and j with $1 \leq i \leq n$ and $0 \leq j \leq i$*

$$\begin{aligned} g_i(X, L) &= h^i(\mathcal{O}_{X_{n-i}}). \\ \chi_i^H(X, L) &= \chi(\mathcal{O}_{X_{n-i}}). \\ e_i(X, L) &= e(X_{n-i}). \\ b_i(X, L) &= b_i(X_{n-i}). \\ h_i^{j,i-j}(X, L) &= h^{j,i-j}(X_{n-i}). \end{aligned}$$

Proof. See [3], [7] and [8]. □

Remark 3.2 By Proposition 3.1 $g_i(X, L)$ (resp. $\chi_i^H(X, L)$, $e_i(X, L)$, $b_i(X, L)$, and $h_i^{j,i-j}(X, L)$) is equal to the geometric genus (resp. the arithmetic genus in the sense of Hirzebruch (see [9]), the Euler number, the i-th Betti number, and the Hodge number of type $(j, i-j)$) of X_{n-i}. (Here we use Notation 2.1.) Namely the i-th sectional geometric genus (resp. the i-th sectional H-arithmetic genus, the i-th sectional Euler number, the i-th sectional Betti number, and the i-th sectional Hodge number of type $(j, i-j)$) of (X, L) is the i-th sectional invariant of (X, L) associated with the geometric genus (resp. the arithmetic genus in the sense of Hirzebruch, the Euler number, the i-th Betti number, and the Hodge number of type $(j, i-j)$).

In general, we can prove the following.

Theorem 3.1 *Let (X, L) be a polarized manifold of $\dim X = n$. For every integers i and j with $0 \leq i \leq n$ and $0 \leq j \leq i$, we get the following.*

(1) $b_i(X, L) = \sum_{k=0}^{i} h_i^{k,i-k}(X, L)$.

(2) *If i is odd, then $b_i(X, L)$ is even.*

(3) $h_i^{j,i-j}(X, L) = h_i^{i-j,j}(X, L)$.

(4) $h_i^{i,0}(X, L) = h_i^{0,i}(X, L) = g_i(X, L)$.

Proof. See [8]. □

Remark 3.3 (1) In Theorem 3.1, we only assume that L is ample (not necessarily base point free).

(2) Let Y be a smooth projective variety of $\dim Y = i$. Then

(2.1) (1) in Theorem 3.1 corresponds to $b_i(Y) = \sum_{j=0}^{i} h^{j,i-j}(Y)$.

(2.2) (2) in Theorem 3.1 corresponds to the following.
If i is odd, then $b_i(Y)$ is even.

(2.3) (3) in Theorem 3.1 corresponds to $h^{j,i-j}(Y) = h^{i-j,j}(Y)$ for every integer j with $0 \leq j \leq i$.

(2.4) (4) in Theorem 3.1 corresponds to $h^{i,0}(Y) = h^{0,i}(Y) = h^i(\mathcal{O}_Y)$.

Proposition 3.2 *Let (X, L) be a polarized manifold, and let (M, A) be a reduction of (X, L). For every integer i with $1 \leq i \leq n$, the following hold.*

(1) $g_i(X, L) = g_i(M, A)$.

(2) $\chi_i^H(X, L) = \chi_i^H(M, A)$.

Proof. See [3] and [7]. □

Proposition 3.3 *Let (X, L) be a polarized manifold of $\dim X = n$. Assume that L is base point free. Then for every integers i and j with $1 \leq i \leq n$ and $0 \leq j \leq i$ the following hold.*

(1) $g_i(X, L) \geq h^i(\mathcal{O}_X)$.

(2) $b_i(X, L) \geq b_i(X)$.

(3) $h_i^{j,i-j}(X, L) \geq h^{j,i-j}(X)$.

Proof. See [3] and [8]. □

By considering Proposition 3.3, we can propose the following conjecture.

Conjecture 3.1 *Let (X, L) be a polarized manifold of $\dim X = n$. Then for every integers i and j with $1 \leq i \leq n$ and $0 \leq j \leq i$ the following hold.*

(1) $g_i(X, L) \geq h^i(\mathcal{O}_X)$.

(2) $b_i(X, L) \geq b_i(X)$.

(3) $h_i^{j,i-j}(X, L) \geq h^{j,i-j}(X)$.

4 The second sectional invariants of polarized manifolds.

By Proposition 3.1 and Remark 3.2, we can expect that the second sectional invariants reflect the "2-dimensional geometry". So it is natural to consider the following. "Can we get results of polarized manifolds which are analogous to theorems related to results of projective surfaces ?"

In this section, we treat this.

First we consider the case where $\mathrm{Bs}|L| = \emptyset$ and we use Notation 2.1.

(A) In this case by Proposition 3.1, the Lefschetz theorem, and the adjunction formula, we get that $g_2(X, L) = h^2(\mathcal{O}_{X_{n-2}})$, $h^1(\mathcal{O}_X) = h^1(\mathcal{O}_{X_{n-2}})$, $b_1(X) = b_1(X_{n-2})$,

$h^{1,0}(X) = h^{1,0}(X_{n-2})$, $h^{0,1}(X) = h^{0,1}(X_{n-2})$, $\chi_2^H(X,L) = \chi(\mathcal{O}_{X_{n-2}})$, $e_2(X,L) = e(X_{n-2})$, $b_2(X,L) = b_2(X_{n-2})$, $h_2^{1,1}(X,L) = h^{1,1}(X_{n-2})$ and $(K_X+(n-2)L)^2L^{n-2} = K^2_{X_{n-2}}$.
(B) Moreover if (X,L) is not a scroll over a smooth surface, then there are the following correspondences between $\kappa(X_{n-2})$ and $\kappa(K_X+(n-2)L)$ (see [7]).

Value of $\kappa(X_{n-2})$.	$\Leftrightarrow$	Value of $\kappa(K_X+(n-2)L)$.
$-\infty$	$\Leftrightarrow^*$	$-\infty$
0	$\Leftrightarrow^{**}$	0
1	$\Leftrightarrow^{**}$	1
2	$\Leftrightarrow^{**}$	≥ 2

(We note that the direction $\Leftarrow$ in $(*)$ and the direction $\Rightarrow$ in $(**)$ need the assumption that (X,L) is not a scroll over a smooth surface.)
(C) Let (X,L) be a polarized manifold which is not a scroll over a smooth surface, let (M,A) be a reduction of (X,L), and we put $M_{n-2} := \mu(X_{n-2})$, where $\mu : X \to M$ is the reduction map. Then M_{n-2} is smooth and $K_{M_{n-2}} = (K_M+(n-2)A)|_{M_{n-2}}$. Assume that $\kappa(X_{n-2}) \geq 0$. (We note that this condition is equivalent to the condition that $\kappa(K_X+(n-2)L) \geq 0$ by above.) Then $\kappa(K_M+(n-2)A) \geq 0$. Hence by the adjunction theory of Beltrametti-Sommese (see [1]), $K_M+(n-2)A$ is nef. In particular, $K_{M_{n-2}}$ is nef. Hence $\mu|_{X_{n-2}} : X_{n-2} \to M_{n-2}$ is the minimalization of X_{n-2}.

From (A), (B), and (C), we infer that there are the following correspondences between invariants of smooth projective surfaces S and invariants of (X,L).

Invariants of S.	$\Leftrightarrow$	Invariants of (X,L).
$h^2(\mathcal{O}_S)$	$\Leftrightarrow$	$g_2(X,L)$
$h^1(\mathcal{O}_S)$	$\Leftrightarrow$	$h^1(\mathcal{O}_X)$
$\chi(\mathcal{O}_S)$	$\Leftrightarrow$	$\chi_2^H(X,L)$
K_S^2	$\Leftrightarrow$	$(K_X+(n-2)L)^2L^{n-2}$
$K_{\widetilde{S}}^2$	$\Leftrightarrow^*$	$(K_M+(n-2)A)^2A^{n-2}$
$\kappa(S)=k$	$\Leftrightarrow^{**}$	$\kappa(K_X+(n-2)L)=k$
$\kappa(S)=2$	$\Leftrightarrow^{***}$	$\kappa(K_X+(n-2)L)\geq 2$
$e(S)$	$\Leftrightarrow$	$e_2(X,L)$
$b_2(S)$	$\Leftrightarrow$	$b_2(X,L)$
$b_1(S)$	$\Leftrightarrow$	$b_1(X)$
$h^{1,1}(S)$	$\Leftrightarrow$	$h_2^{1,1}(X,L)$
$h^{1,0}(S)$	$\Leftrightarrow$	$h^{1,0}(X)$
$h^{0,1}(S)$	$\Leftrightarrow$	$h^{0,1}(X)$

(In $(*)$, we assume that $\kappa(K_X+(n-2)L) \geq 0$ and let $\widetilde{S}$ (resp. (M,A)) be the minimalization of S (resp. a reduction of (X,L)). In $(**)$ $k = -\infty, 0$, or 1, and we assume that (X,L) is not a scroll over a smooth surface. In $(***)$ we assume that (X,L) is not a scroll over a smooth surface.)

By considering these correspondences, we can propose a lot of problems which are analogous to those of smooth projective surfaces. For example here we consider the following four representative theorems of projective surfaces.

Theorem 1 (Noether's formula) *Let S be a smooth projective surface. Then $12\chi(\mathcal{O}_S) = K_S^2 + e(S)$.*

Theorem 2 (Castelnuovo's theorem) *Let S be a smooth projective surface. Assume that $\kappa(S) \geq 0$ (resp. $\kappa(S) = 2$). Then $\chi(\mathcal{O}_S) \geq 0$ (resp. $\chi(\mathcal{O}_S) > 0$).*

Theorem 3 (Noether's inequality) *Let S be a smooth projective surface of general type and let $\widetilde{S}$ be the minimal model of S. Then $K_{\widetilde{S}}^2 \geq 2p_g(\widetilde{S}) - 4$.*

Theorem 4 (Bogomolov-Miyaoka-Yau's inequality) *Let S be a smooth projective surface of general type. Then $9\chi(\mathcal{O}_S) \geq K_S^2$.*

By using the above correspondences, we can give the following conjectures. For $k = 1, \dots 4$, Conjecture k corresponds to Theorem k above.

Conjecture 1 *Let (X, L) be a polarized manifold of $\dim X = n \geq 3$.*
Then $12\chi_2^H(X, L) = (K_X + (n-2)L)^2 L^{n-2} + e_2(X, L)$.

Conjecture 2 *Let (X, L) be a polarized manifold of $\dim X = n \geq 3$. Assume that $\kappa(K_X + (n-2)L) \geq 0$ (resp. ≥ 2). Then $\chi_2^H(X, L) \geq 0$ (resp. > 0).*

Conjecture 3 *Let (X, L) be a polarized manifold of $\dim X = n \geq 3$. Assume that $\kappa(K_X + (n-2)L) \geq 2$. Let (M, A) be a reduction of (X, L). Then $(K_M + (n-2)A)^2 A^{n-2} \geq 2g_2(M, A) - 4$.*

Conjecture 4 *Let (X, L) be a polarized manifold of $\dim X = n \geq 3$. Assume that $\kappa(K_X + (n-2)L) \geq 2$. Then $9\chi_2^H(X, L) \geq (K_X + (n-2)L)^2 L^{n-2}$.*

Here we give some comments about the above conjectures.
(A) We can prove that Conjecture 1 is true. Namely

Theorem 4.1 *Let (X, L) be a polarized manifold of $\dim X = n \geq 3$.*
Then $12\chi_2^H(X, L) = (K_X + (n-2)L)^2 L^{n-2} + e_2(X, L)$.

Proof. See [8]. □
(B) For Conjecture 2 we get the following result.

Theorem 4.2 *Let (X, L) be a polarized manifold of $\dim X = n \geq 3$. Assume that $\kappa(X) \geq 0$. Then $\chi_2^H(X, L) > 0$.*

For the proof, see [4] and [6]. □
(C) We note that in my preprint [7], we obtained some partial results of Conjecture 4. See [7] in detail.
(D) If L is base point free, then Conjecture 2, Conjecture 3, and Conjecture 4 are true.

Similarly, we can propose many problems other than the above. In detail see [5].

Finally we note that we also obtain the following lower bound for $e_2(X, L)$ and $b_2(X, L)$ if the Kodaira dimension of X is non-negative.

Theorem 4.3 *Let (X, L) be a polarized manifold of $\dim X = n \geq 3$. Assume that $\kappa(X) \geq 0$. Then we get the following.*

(1)
$$e_2(X, L) \geq \frac{n-2}{n} K_X L^{n-1} + \frac{(n-1)(n-2)}{n} L^n.$$

(2)
$$b_2(X, L) \geq 0.$$

Proof. See [8]. □

References

[1] Beltrametti, M. C. and Sommese, A. J., *The adjunction theory of complex projective varieties,* de Gruyter Expositions in Math. 16, Walter de Gruyter, Berlin, NewYork, (1995).

[2] Fujita, T., *Classification Theories of Polarized Varieties,* London Math. Soc. Lecture Note Ser. 155, Cambridge University Press, (1990).

[3] Fukuma, Y., *On the sectional geometric genus of quasi-polarized varieties, I,* Comm. Alg. (to appear).

[4] Fukuma, Y., *On the second sectional geometric genus of quasi-polarized manifolds,* Adv. Geom. (to appear).

[5] Fukuma, Y., *Problems on the second sectional invariants of polarized manifolds,* Mem. Fac. Sci. Kochi Univ. Ser.A Math. (to appear).

[6] Fukuma, Y., *A lower bound for the second sectional geometric genus of polarized manifolds,* preprint.

[7] Fukuma, Y., *On the second sectional H-arithmetic genus of polarized manifolds,* preprint.

[8] Fukuma, Y., in preparation.

[9] Hirzebruch, F., *Topological methods in algebraic geometry,* Grundlehren Math. Wiss. 131, Springer-Verlag, New York, (1966).

[10] Kleiman, S. L., *Toward a numerical theory of ampleness,* Ann. of Math. 84 (1966), 293–344.

Yoshiaki Fukuma
Department of Mathematics
Faculty of Science
Kochi University
Akebono-cho, Kochi 780-8520
Japan
E-mail: fukuma@math.kochi-u.ac.jp