ON THE HODGE STRUCTURE OF DEGENERATING HYPERSURFACES IN PROJECTIVE TORIC VARIETIES

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ABSTRACT. We describe the Hodge filtration of degenerating hypersurfaces in projective toric varieties by using the Jacobian rings. The idea is based on original works on hypersurfaces in projective spaces by Griffiths. The theory is extended to the case on hypersurfaces in simplicial projective toric varieties by Dolgachev, Steenbrink, Batyrev and Cox. The purpose of our work is to apply their methods to degenerating families of hypersurfaces in projective toric varieties.

1. INTRODUCTION

Let $\mathbf{P}$ be a nonsingular toric variety, let $\mathbf{A}$ be a simplicial affine toric variety, and let

$$\pi : \mathbf{P} \longrightarrow \mathbf{A}$$

be a proper flat equivariant morphism with connected fibres of dimension n. We consider the associated analytic fibration

$$\pi : \mathbf{P}^{\mathrm{an}} \longrightarrow \mathbf{A}^{\mathrm{an}}$$

with log structure which satisfies the following properties:

(1) the log structure on $\mathbf{A}^{\mathrm{an}}$ is the canonical log structure, which is given to any analytic space associated with a finitely generated monoid,
(2) π is a log smooth vertical morphism of log analytic spaces.

In this case, a general fibre of π is a nonsingular projective toric variety with the trivial log structure. Let

$$\iota : \mathbf{X} \hookrightarrow \mathbf{P}$$

be a π-ample irreducible hypersurface. The log structure on $\mathbf{X}^{\mathrm{an}}$ is given by the pull back of the log structure on $\mathbf{P}^{\mathrm{an}}$. Let Δ be an open subset of $\mathbf{A}^{\mathrm{an}}$ satisfying the following conditions:

(1) $X = (\pi \circ \iota)^{-1}(\Delta)$ is a nonsingular hypersurface in $P = \pi^{-1}(\Delta) \subset \mathbf{P}^{\mathrm{an}}$,
(2) X is log smooth over Δ.

Then the $\mathcal{O}_\Delta$-module $R^q(\pi \circ \iota)_*\omega^p_{X/\Delta}$ is the graded piece of the Hodge filtration of the degenerating hypersurface X over Δ, where $\omega^p_{X/\Delta}$ is the sheaf of log differentials of X over Δ.

For simplicity, we assume that $\mathbf{A}$ is defined by a nondegenerate cone. The homogeneous coordinate ring [2] of $\mathbf{P}$ is the polynomial ring

$$S = \mathbf{C}[y_1, \ldots, y_r, z_1, \ldots, z_s] \simeq \bigoplus_{\mathcal{L} \in \mathrm{Pic}(\mathbf{P})} \Gamma(\mathbf{P}, \mathcal{L}) \simeq \bigoplus_{\mathcal{L} \in \mathrm{Pic}(\mathbf{P})} \Gamma(\mathbf{A}, \pi_*\mathcal{L}),$$

where the variables $z_1, \ldots, z_s$ are corresponding to the invariant prime divisors that dominate $\mathbf{A}$, and the variables $y_1, \ldots, y_r$ are corresponding to the invariant prime divisors that do not dominate $\mathbf{A}$. By the above identification, the homogeneous coordinate ring has a $\Gamma(\mathbf{A}, \mathcal{O}_{\mathbf{A}})$-algebra structure and it is graded by $\mathrm{Pic}(\mathbf{P})$. If $\mathbf{X}$ is defined by a $\mathrm{Pic}(\mathbf{P})$-homogeneous polynomial f, then the Jacobian ring of $\mathbf{X}$ over $\mathbf{A}$ is defined by

$$R = \mathbf{C}[y_1, \ldots, y_r, z_1, \ldots, z_s]/(\frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_s}),$$

and we denote by

$$\mathcal{R} = \bigoplus_{c \in \mathrm{pic}(\mathbf{P})} \mathcal{R}^c$$

the associated sheaf of $\mathrm{Pic}(\mathbf{P})$-graded $\mathcal{O}_{\mathbf{A}^{\mathrm{an}}}$-algebra.

Theorem 1.1. *There is a natural isomorphism of $\mathcal{O}_\Delta$-modules*

$$\mathcal{R}^c|_\Delta \simeq R^{n-p-1}\pi_*\omega^{p+1}_{P/\Delta}(\log X),$$

where $c \in \mathrm{Pic}(\mathbf{P})$ is the class of the invertible sheaf $\omega^n_{\mathbf{P}/\mathbf{A}}((n-p)\mathbf{X})$.

By this identification and the residue map

$$R^q\pi_*\omega^{p+1}_{P/\Delta}(\log X) \xrightarrow{\mathrm{Res}} R^q(\pi \circ \iota)_*\omega^p_{X/\Delta},$$

we can describe the Hodge filtration of the degenerating family X over Δ.

Example 1.2 (The case $\mathbf{A} = \mathrm{Spec}\mathbf{C}$). Then $\mathbf{P}$ is a nonsingular projective toric variety, and in the homogeneous coordinate ring S, the variable y_j does not appear, so the result is contained in [1]. In the case when $\mathbf{P} = \mathbf{P}^n$ and X is a nonsingular hypersurface of degree d, we have

$$R^{(n-p)d-n-1} \simeq H^{n-p-1}(X, \Omega^p_X)_{\mathrm{prim}},$$

and this is the original result by Griffiths [3].

Example 1.3 (The case $\mathbf{A} = \mathbf{A}^1$ and $\mathbf{P}$ is the blowing up of $\mathbf{A}^1 \times \mathbf{P}^n$ along a point). Then the family X over Δ is a semistable degeneration, and it is the case treated in [6].

Finally, we mention that the key of the proof of Theorem 1.1 is the following vanishing theorem on higher direct images of invertible sheaves.

Theorem 1.4. *If $\mathcal{L}$ is a π-ample invertible sheaf on $\mathbf{P}$, then for $p \geq 0$ and $q \geq 1$,*

$$R^q\pi_*(\omega^p_{\mathbf{P}/\mathbf{A}} \otimes \mathcal{L}).$$

2. Logarithmic structure on toric varieties

The purpose of this section is to give logarithmic structure on fibration of toric varieties. A reference on toric varieties is [5], and a reference on logarithmic structure is [4].

Let N be a free $\mathbf{Z}$-module of finite rank, and let M be the dual $\mathbf{Z}$-module of N. The canonical bilinear map is denoted by

$$\langle\ ,\ \rangle : M_{\mathbf{R}} \times N_{\mathbf{R}} \longrightarrow \mathbf{R}.$$

Let σ be a strictly convex rational polyhedral cone in $N_{\mathbf{R}}$, and let $\sigma^\vee$ be the dual cone of σ. Since $M \cap \sigma^\vee$ is a finitely generated submonoid of M, the set

$$\mathbf{A}_\sigma = \mathrm{Hom}(M \cap \sigma^\vee, \mathbf{C})$$

of monoid homomorphisms has a complex analytic space structure, and it is called the affine toric variety associated with σ. In particular, the affine toric variety associated with $\sigma = 0$ is called the torus associated with N and denoted by $\mathbf{T}_N$. For $u \in M \cap \sigma^\vee$, a function on $\mathbf{A}_\sigma$ is defined by

$$\chi_u : \mathbf{A}_\sigma \longrightarrow \mathbf{C}\ ;\ p \mapsto p(u).$$

The monoid homomorphism

$$M \cap \sigma^\vee \longrightarrow \mathit{\Gamma}(\mathbf{A}_\sigma, \mathcal{O}_{\mathbf{A}_\sigma})\ ;\ u \mapsto \chi_u$$

defines a pre-logarithmic structure on $\mathbf{A}_\sigma$, and the associated logarithmic structure

$$\alpha_\sigma : \mathcal{M}_\sigma \longrightarrow \mathcal{O}_{\mathbf{A}_\sigma}.$$

is called the canonical logarithmic structure on $\mathbf{A}_\sigma$.

Let Σ be a finite fan of strictly convex rational polyhedral cones in $N_{\mathbf{R}}$. By glueing the affine toric varieties associated with cones in Σ, the toric variety

$$\mathbf{P}_\Sigma = \bigcup_{\sigma \in \Sigma} \mathbf{A}_\sigma$$

associated with Σ is defined, and the canonical logarithmic structure

$$\alpha_\Sigma : \mathcal{M}_\Sigma \longrightarrow \mathcal{O}_{\mathbf{P}_\Sigma}$$

on $\mathbf{P}_\Sigma$ is defined by the canonical logarithmic structure on $\mathbf{A}_\sigma$ for $\sigma \in \Sigma$.

Next, we introduce invariant closed subvariety of $\mathbf{P}_\Sigma$. For $\tau \in \Sigma$, the $\mathbf{Z}$-module $N/(N \cap \tau_{\mathbf{R}})$ is free and the fan

$$\Sigma_\tau = \{\sigma/\tau = \bar{\sigma} \subset (N/(N \cap \tau_{\mathbf{R}}))_{\mathbf{R}} = N_{\mathbf{R}}/\tau_{\mathbf{R}} \mid \sigma \in \Sigma,\ \tau \prec \sigma\}$$

defines a toric variety $\mathbf{P}_{\Sigma_\tau}$, which is an invariant closed subvariety of $\mathbf{P}_\Sigma$ by

$$\iota_\tau : \mathbf{A}_{\sigma/\tau} = \mathrm{Hom}(M \cap \tau^\perp \cap \sigma^\vee, \mathbf{C}) \hookrightarrow \mathrm{Hom}(M \cap \sigma^\vee, \mathbf{C}) = \mathbf{A}_\sigma$$

for $\tau \prec \sigma \in \Sigma$, where for $p \in \mathbf{A}_{\sigma/\tau}$, the image $\iota_\tau(p)$ is defined by

$$\iota_\tau(p) : M \cap \sigma^\vee \longrightarrow \mathbf{C}\ ;\ u \mapsto \begin{cases} p(u) & (u \in \tau^\perp), \\ 0 & (u \notin \tau^\perp). \end{cases}$$

In the following, we give another logarithmic structure on $\mathbf{P}_\Sigma$, which depends on the fibre space structure. Let

$$\pi : N \longrightarrow N'$$

be a homomorphism of free $\mathbf{Z}$ modules, and let

$$\pi^* : M' \longrightarrow M$$

be the dual homomorphism of π. For a strongly convex rational polyhedral cone σ in $N_{\mathbf{R}}$, the smallest face of $\sigma^\vee$ containing $\pi^*_{\mathbf{R}}(\pi_{\mathbf{R}}(\sigma)^\vee)$ is denoted by F^π_σ. The monoid homomorphism

$$M \cap F^\pi_\sigma \longrightarrow \Gamma(\mathbf{A}_\sigma, \mathcal{O}_{\mathbf{A}_\sigma}) \ ; \ u \mapsto \chi_u$$

defines a pre-logarithmic structure on $\mathbf{A}_\sigma$, and the associated logarithmic structure is denoted by

$$\alpha^\pi_\sigma : \mathcal{M}^\pi_\sigma \longrightarrow \mathcal{O}_{\mathbf{A}_\sigma}.$$

Proposition 2.1. *If σ is a nonsingular cone in $N_{\mathbf{R}}$, then for $\tau \prec \sigma$, the natural inclusion*

$$M \cap F^\pi_\sigma \hookrightarrow M \cap F^\pi_\tau$$

induces an isomorphism

$$\mathcal{M}^\pi_\sigma|_{\mathbf{A}_\tau} \longrightarrow \mathcal{M}^\pi_\tau$$

of logarithmic structure on $\mathbf{A}_\tau$.

By Proposition 2.1, if Σ is a nonsingular fan, then there is a logarithmic structure

$$\alpha^\pi_\Sigma : \mathcal{M}^\pi_\Sigma \longrightarrow \mathcal{O}_{\mathbf{P}_\Sigma}$$

on $\mathbf{P}_\Sigma$ such that it is isomorphic to the logarithmic structure $\mathcal{M}^\pi_\sigma$ on $\mathbf{A}_\sigma$.

3. Differential forms

A reference on differential forms on logarithmic analytic spaces is [4].

In this section, we assume that the cokernel of

$$\pi : N \longrightarrow N'$$

finite. Then the dual homomorphism π^* is injective. Let Σ be a nonsingular fan of $N_{\mathbf{R}}$, and let σ' be a simplicial cone in $N'_{\mathbf{R}}$ containing $\pi_{\mathbf{R}}(|\Sigma|)$. Then we have a dominant equivariant morphism

$$\mathbf{P}_\Sigma \longrightarrow \mathbf{A}_{\sigma'}$$

of toric varieties, and it induces morphisms

$$(\mathbf{P}_\Sigma, \mathcal{M}_\Sigma) \longrightarrow (\mathbf{P}_\Sigma, \mathcal{M}^\pi_\Sigma) \longrightarrow (\mathbf{A}_{\sigma'}, \mathcal{M}_{\sigma'})$$

of logarithmic analytic spaces by

$$\pi^* : M' \cap \sigma'^\vee \longrightarrow M \cap F^\pi_\sigma \subset M \cap \sigma^\vee$$

for $\sigma \in \Sigma$. It is well-known that the morphism

$$(\mathbf{P}_\Sigma, \mathcal{M}_\Sigma) \longrightarrow (\mathbf{A}_{\sigma'}, \mathcal{M}_{\sigma'})$$

is smooth.

Proposition 3.1. *The morphism*

$$(\mathbf{P}_\Sigma, \mathcal{M}^\pi_\Sigma) \longrightarrow (\mathbf{A}_{\sigma'}, \mathcal{M}_{\sigma'})$$

is smooth and vertical.

Remark 3.2. The morphism is vertical if and only if the logarithmic structure $\mathcal{M}_\Sigma^\pi$ is trivial on the inverse image of $\mathbf{T}_{N'}$. The morphism

$$(\mathbf{P}_\Sigma, \mathcal{M}_\Sigma) \longrightarrow (\mathbf{A}_{\sigma'}, \mathcal{M}_{\sigma'})$$

is not necessary vertical.

We denote the sheaf of differential p-forms on $(\mathbf{P}_\Sigma, \mathcal{M}_\Sigma)$ over $(\mathbf{A}_{\sigma'}, \mathcal{M}_{\sigma'})$ by

$$\omega^p_{\Sigma/\sigma'} = \Omega^p_{\mathbf{P}_\Sigma/\mathbf{A}_{\sigma'}}(\log(\mathcal{M}_\Sigma/\mathcal{M}_{\sigma'})),$$

and denote the sheaf of differential p-forms on $(\mathbf{P}_\Sigma, \mathcal{M}^\pi_\Sigma)$ over $(\mathbf{A}_{\sigma'}, \mathcal{M}_{\sigma'})$ by

$$\omega^{\pi,\ p}_{\Sigma/\sigma'} = \Omega^p_{\mathbf{P}_\Sigma/\mathbf{A}_{\sigma'}}(\log(\mathcal{M}^\pi_\Sigma/\mathcal{M}_{\sigma'})).$$

By the smoothness, the sheaves $\omega^p_{\Sigma/\sigma'}$ and $\omega^{\pi,\ p}_{\Sigma/\sigma'}$ are locally free $\mathcal{O}_{\mathbf{P}_\Sigma}$-modules.

The weight filtration on $\omega^p_{\Sigma/\sigma'}$ is defined by

$$W_k\omega^p_{\Sigma/\sigma'} = \mathrm{Image}(\omega^{\pi,\ k}_{\Sigma/\sigma'} \otimes \omega^{p-k}_{\Sigma/\sigma'} \longrightarrow \omega^p_{\Sigma/\sigma'}).$$

We set

$$\Sigma^\pi = \{\sigma \in \Sigma \mid \pi(\sigma) = 0\}.$$

For $\tau \in \Sigma^\pi$, π induces a homomorphism

$$\pi_\tau : N/(N \cap \tau_{\mathbf{R}}) \longrightarrow N',$$

hence we have morphisms

$$(\mathbf{P}_{\Sigma_\tau}, \mathcal{M}_{\Sigma_\tau}) \longrightarrow (\mathbf{P}_{\Sigma_\tau}, \mathcal{M}^{\pi_\tau}_{\Sigma_\tau}) \longrightarrow (\mathbf{A}_{\sigma'}, \mathcal{M}_{\sigma'})$$

of logarithmic analytic spaces. We can consider $\omega^p_{\Sigma/\sigma'}$ as the sheaf $\omega^{\pi,\ p}_{\Sigma/\sigma'}(\log D)$ of differential p-forms on $(\mathbf{P}_\Sigma, \mathcal{M}^\pi_P)$ over $(\mathbf{A}_{\sigma'}, \mathcal{M}_{\sigma'})$ with logarithmic pole along D, where

$$D = \bigcup_{\sigma \in \Sigma^\pi(1)} \mathbf{P}_{\Sigma_\sigma}.$$

Theorem 3.3 (Residue sequence)**.** *There is an exact sequence of $\mathcal{O}_{\mathbf{P}_\Sigma}$-modules*

$$0 \longrightarrow W_{k-1}\omega^p_{\Sigma/\sigma'} \longrightarrow W_k\omega^p_{\Sigma/\sigma'} \longrightarrow \bigoplus_{\tau \in \Sigma^\pi(k)} \iota_{\tau *}\omega^{\pi_\tau,\ p-k}_{\Sigma_\tau/\sigma'} \longrightarrow 0.$$

Theorem 3.4 (Euler sequence)**.** *There is an exact sequence of $\mathcal{O}_{\mathbf{P}_\Sigma}$-modules*

$$0 \longrightarrow \omega^{\pi,\ 1}_{\Sigma/\sigma'} \xrightarrow{\beta} \bigoplus_{\sigma \in \Sigma^\pi(1)} \mathcal{O}_{\mathbf{P}_\Sigma}(-\mathbf{P}_{\Sigma_\sigma}) \longrightarrow \mathcal{O}^{\oplus t}_{\mathbf{P}_\Sigma} \longrightarrow 0,$$

where

$$t = \sharp\Sigma^\pi(1) - \mathrm{rank}N + \mathrm{rank}N'.$$

4. Vanishing theorems

In this section, we assume that Σ is a nonsingular fan, σ' is a simplicial cone, and

$$\pi : \mathbf{P}_\Sigma \longrightarrow \mathbf{A}_{\sigma'}$$

is a proper flat equivariant morphism with connected fibres. Let $\mathcal{L}$ be an invertible sheaf on $\mathbf{P}_\Sigma$.

Theorem 4.1. *If $\mathcal{L}$ is π-nef, then for $q \geq 1$,*

$$R^q\pi_*\mathcal{L} = 0.$$

Theorem 4.2. *If $\mathcal{L}$ is π-ample, then for $q \geq 1$,*

$$R^q\pi_*(W_k\omega^p_{\Sigma/\sigma'} \otimes \mathcal{L}) = 0.$$

Corollary 4.3. *If $\mathcal{L}$ is π-ample, then for $q \geq 1$,*

$$R^q\pi_*(\omega^{\pi,\ p}_{\Sigma/\sigma'} \otimes \mathcal{L}) = 0.$$

Theorem 4.1 is induced by the usual vanishing theorem on toric varieties. Theorem 4.2 is proved by induction on $p-k$, using Theorem 3.3. Since $\omega^p_{\Sigma/\sigma'}$ is a free $\mathcal{O}_{\mathbf{P}_\Sigma}$-module, the case $p-k=0$ is induced by Theorem 4.1. The corollary, which is used in the proof of the main theorem, is the case $k=0$ in Theorem 4.2.

5. Degenerating hypersurfaces

Let $\mathbf{P}_\Sigma$ be a nonsingular toric variety, let $\mathbf{A}_{\sigma'}$ be a simplicial toric variety, let

$$\pi : \mathbf{P}_\Sigma \longrightarrow \mathbf{A}_{\sigma'}$$

be a proper flat equivariant morphism with connected fibres of dimension n. For an irreducible hypersurface

$$\iota : \mathbf{X} \hookrightarrow \mathbf{P}_\Sigma,$$

we defines a logarithmic structure

$$\mathcal{M}_\mathbf{X} = \iota^*\mathcal{M}^\pi_{\mathbf{P}_\Sigma} \longrightarrow \mathcal{O}_\mathbf{X}$$

on $\mathbf{X}$ by the pull-back of the logarithmic structure $\mathcal{M}^\pi_\mathbf{P}$. We assume that there is a nonempty open subset Δ of $\mathbf{A}_{\sigma'}$ such that the $X = (\pi\circ\iota)^{-1}(\Delta)$ is nonsingular and $(X, \mathcal{M}_\mathbf{X}|_X)$ is smooth over $(\Delta, \mathcal{M}_{\mathbf{A}_{\sigma'}}|_\Delta)$. The general fibre of

$$\mathbf{P}_\Sigma \longrightarrow \mathbf{A}_{\sigma'}$$

is a nonsingular complete toric variety $\mathbf{P}_{\Sigma^\pi}$, and

$$\pi\circ\iota : (\pi\circ\iota)^{-1}(\Delta\cap\mathbf{T}_{N'}) \longrightarrow \Delta\cap\mathbf{T}_{N'}$$

is smooth in usual sense, so we call the logarithmic smooth family

$$\pi\circ\iota : (X, \mathcal{M}_X = \mathcal{M}_\mathbf{X}|_X) \longrightarrow (\Delta, \mathcal{M}_\Delta = \mathcal{M}_{\mathbf{A}_{\sigma'}}|_\Delta)$$

a degenerating hypersurface in $\mathbf{P}_{\Sigma^\pi}$. We denote the sheaf of differential p-forms on $(P = \pi^{-1}(\Delta), \mathcal{M}^\pi_{\Sigma/\sigma'}|_P)$ over $(\Delta, \mathcal{M}_\Delta)$ by

$$\omega^p_{P/\Delta} = \omega^{\pi,\ p}_{\Sigma/\sigma'}|_P,$$

and denote the sheaf of differential p-forms on $(X, \mathcal{M}_X)$ over $(\Delta, \mathcal{M}_\Delta)$ by

$$\omega^p_{X/\Delta} = \Omega^p_{X/\Delta}(\log(\mathcal{M}_X/\mathcal{M}_\Delta)).$$

The sheaf of differential p-forms on $(P, \mathcal{M}_P)$ over $(\Delta, \mathcal{M}_\Delta)$ with logarithmic pole along X is defined by

$$\omega^p_{P/\Delta}(\log X) = \mathrm{Ker}(\omega^p_{P/\Delta}(X) \longrightarrow \iota_*(\omega^p_{X/\Sigma} \otimes \mathcal{N}_{X/P})).$$

Proposition 5.1 (Residue sequence)**.** *There is an exact sequence of $\mathcal{O}_P$-modules*

$$0 \longrightarrow \omega^{p+1}_{P/\Delta} \longrightarrow \omega^{p+1}_{P/\Delta}(\log X) \longrightarrow \iota_*\omega^p_{X/\Delta} \longrightarrow 0.$$

6. Jacobian rings

In this section, we use the homogeneous coordinate rings of toric varieties [2]. We assume that $|\Sigma(1)|$ generate $N_{\mathbf{R}}$ over $\mathbf{R}$. Let

$$\begin{cases} D_i = \mathbf{P}_{\Sigma_{\sigma_i}} & (1 \leq i \leq s), \\ E_j = \mathbf{P}_{\Sigma_{\rho_i}} & (1 \leq j \leq r) \end{cases}$$

be the invariant prime divisors on $\mathbf{P}_{\boldsymbol{\Sigma}}$, where

$$\Sigma^\pi(1) = \{\sigma_1, \dots, \sigma_s\} \subset \{\rho_1, \dots, \rho_r, \sigma_1, \dots, \sigma_s\} = \Sigma(1).$$

There is an isomorphism of $\mathrm{Pic}(\mathbf{P}_\Sigma)$-graded $\mathbf{C}$-algebras

$$\mathbf{C}[y_1, \dots, y_r, z_1, \dots, z_s] \simeq \bigoplus_{\mathcal{L} \in \mathrm{Pic}(\mathbf{P}_\Sigma)} H^0(\mathbf{P}_\Sigma, \mathcal{L}),$$

where we set the degree of polynomials by

$$\begin{cases} \deg(z_i) = [D_i] & (1 \leq i \leq s), \\ \deg(y_j) = [E_j] & (1 \leq j \leq r). \end{cases}$$

Let $f \in \mathbf{C}[y_1, \dots, y_r, z_1, \dots, z_s]$ be the polynomial which corresponds to a defining section of the hypersurface $\mathbf{X}$. Then the Jacobian ring of $\mathbf{X}$ over $\mathbf{A}_{\sigma'}$ is defined by

$$R = \mathbf{C}[y_1, \dots, y_r, z_1, \dots, z_s]/(\frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_s}).$$

The Jacobian ring R is $\mathrm{Pic}(\mathbf{P}_\Sigma)$-graded $\mathbf{C}[M' \cap \sigma'^\vee]$-algebra, and we denote the associated sheaf of $\mathcal{O}_{\mathbf{A}_\sigma}$-algebras by

$$\mathcal{R} = \bigoplus_{c \in \mathrm{Pic}(\mathbf{P}_\Sigma)} \mathcal{R}^c.$$

7. Main theorem

Let $\mathbf{P}_\Sigma$ be a nonsingular toric variety, let $\mathbf{A}_{\sigma'}$ be a simplicial toric variety, let

$$\pi : \mathbf{P}_\Sigma \longrightarrow \mathbf{A}_{\sigma'}$$

be a proper flat equivariant morphism with connected fibres of dimension n. We assume that an irreducible hypersurface $\mathbf{X}$ in $\mathbf{P}_\Sigma$ gives a degenerating hypersurface

$$(X, \mathcal{M}_X) \longrightarrow (\Delta, \mathcal{M}_\Delta).$$

Theorem 7.1. *If $\mathbf{X}$ is a π-ample hypersurface then there is a natural isomorphism of $\mathcal{O}_\Delta$-modules*

$$\mathcal{R}^c|_\Delta \simeq R^{n-p-1}\pi_*\omega^{p+1}_{P/\Delta}(\log X),$$

where $c \in \operatorname{Pic}(\mathbf{P}_\Sigma)$ is the class of the invertible sheaf $\omega^n_{\mathbf{P}_\Sigma/\mathbf{A}}((n-p)\mathbf{X})$.

Proof. There is an exact sequence

$$0 \longrightarrow \omega^{p+1}_{P/\Delta}(\log X) \longrightarrow \omega^{p+1}_{P/\Delta}(X) \longrightarrow \iota_*(\omega^{p+2}_{P/\Delta}|_X \otimes \mathcal{N}^{\otimes 2}_{X/P}) \longrightarrow \cdots$$
$$\cdots \longrightarrow \iota_*(\omega^{n}_{P/\Delta}|_X \otimes \mathcal{N}^{\otimes n-p}_{X/P}) \longrightarrow 0.$$

By Corollary 4.3,

$$\begin{cases} R^q\pi_*\omega^{p+1}_{P/\Delta}(X) = 0 & (q \geq 1), \\ R^q(\pi \circ \iota)_*(\omega^{p+i}_{P/\Delta}|_X \otimes \mathcal{N}^{\otimes i}_{X/P}) = 0 & (2 \leq i \leq n-p,\ q \geq 1). \end{cases}$$

So we have

$$\begin{aligned} &R^{n-p-1}\pi_*\omega^{p+1}_{P/\Delta}(\log X) \\ &\qquad \simeq \operatorname{Coker}((\pi \circ \iota)_*(\omega^{n-1}_{P/\Delta}|_X \otimes \mathcal{N}^{\otimes n-p-1}_{X/P}) \longrightarrow (\pi \circ \iota)_*(\omega^{n}_{P/\Delta}|_X \otimes \mathcal{N}^{\otimes n-p}_{X/P})). \end{aligned}$$

There is a commutative diagram

$$\begin{array}{ccc} \pi_*(\bigoplus_{i=1}^s \mathcal{O}_{\mathbf{P}_\Sigma}(D_i) \otimes \omega^n_{P/\Delta}((n-p-1)X)) & \xrightarrow{(\frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_s})} & \pi_*(\omega^n_{P/\Delta}((n-p)X)) \\ \beta^* \downarrow & & \downarrow \\ (\pi \circ \iota)_*(\omega^{n-1}_{P/\Delta}|_X \otimes \mathcal{N}^{\otimes n-p-1}_{X/P}) & \longrightarrow & (\pi \circ \iota)_*(\omega^{n}_{P/\Delta}|_X \otimes \mathcal{N}^{\otimes n-p}_{X/P}), \end{array}$$

where β^* is induced by the dual of β, which appeared in Theorem 3.4. In this diagram, we can check the induced morphism

$$\begin{aligned} &\operatorname{Coker}(\pi_*(\bigoplus_{i=1}^s \mathcal{O}_{\mathbf{P}_\Sigma}(D_i) \otimes \omega^n_{P/\Delta}((n-p-1)X)) \xrightarrow{(\frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_s})} \pi_*(\omega^n_{P/\Delta}((n-p)X))) \\ &\qquad \longrightarrow \operatorname{Coker}((\pi \circ \iota)_*(\omega^{n-1}_{P/\Delta}|_X \otimes \mathcal{N}^{\otimes n-p-1}_{X/P}) \longrightarrow (\pi \circ \iota)_*(\omega^{n}_{P/\Delta}|_X \otimes \mathcal{N}^{\otimes n-p}_{X/P})) \end{aligned}$$

is an isomorphism, so this is the isomorphism

$$\mathcal{R}^c|_\Delta \simeq R^{n-p-1}\pi_*\omega^{p+1}_{P/\Delta}(\log X)$$

in the statement. $\square$

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