

# PROJECTIVE GEOMETRY OF FREUDENTHAL'S VARIETIES OF CERTAIN TYPE

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## 0. INTRODUCTION

H. Freudenthal constructed, in a series of his papers (see [10] and its references), the exceptional Lie algebras of type  $E_8$ ,  $E_7$ ,  $E_6$  and  $F_4$ , with defining various projective varieties. The purpose of our work is to study projective geometry for his varieties of certain type, which are called *varieties of planes* in the symplectic geometry of Freudenthal (see [10, 4.11], [22, 2.3]).

Let  $\mathfrak{g}$  be a graded, simple, finite-dimensional Lie algebra over the complex number field  $\mathbb{C}$  with grades between  $-2$  and  $2$ ,  $\dim \mathfrak{g}_2 = 1$  and  $\mathfrak{g}_1 \neq 0$ , namely a *graded Lie algebra of contact type*:  $\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$  (see §1). We set

$$\mathcal{V} := \{x \in \mathfrak{g}_1 \setminus \{0\} \mid (\text{ad } x)^2 \mathfrak{g}_{-2} = 0\},$$

and define an algebraic set  $V$  in  $\mathbb{P}(\mathfrak{g}_1)$  to be the projectivization of  $\mathcal{V}$ :

$$V := \pi(\mathcal{V}),$$

where  $\pi : \mathfrak{g}_1 \setminus \{0\} \rightarrow \mathbb{P}(\mathfrak{g}_1)$  is the natural projection. Then we call  $V \subseteq \mathbb{P}(\mathfrak{g}_1)$  (with the reduced structure) the *Freudenthal variety* associated to the graded Lie algebra  $\mathfrak{g}$  of contact type, which is a natural generalization of Freudenthal's varieties mentioned above: Note that  $V$  is not necessarily connected in this general setting. We here consider moreover the projectivization of a closed set  $\{x \in \mathfrak{g}_1 \mid (\text{ad } x)^{k+1} \mathfrak{g}_{-2} = 0\}$ , and denote it by  $V_k$ : we have

$$\emptyset = V_0 \subseteq V_1 \subseteq V_2 \subseteq V_3 \subseteq V_4 = \mathbb{P},$$

where we set  $\mathbb{P} := \mathbb{P}(\mathfrak{g}_1)$  for short. Clearly,  $V_3$  is a quartic hypersurface,  $V_2$  is an intersection of cubics and  $V_1 = V$  is an intersection of quadrics, with a few exceptions.

In the literature, several results have been known about the structure of  $\mathfrak{g}_1$  as a  $\mathfrak{g}_0$ -space, case-by-case for each exceptional Lie algebra of types  $E_8$ ,  $E_7$ ,  $E_6$  and  $F_4$ , from the view-point of the invariant theory of prehomogeneous vector spaces (see [13], [15], [18], [21]). By virtue of those results, it can be shown, for example, that the stratification of  $\mathbb{P}$  given by the differences of  $V_k$ 's exactly corresponds to the orbit decomposition of the  $\mathfrak{g}_0$ -space  $\mathfrak{g}_1$  for those exceptional Lie algebras, and also that Freudenthal varieties  $V$  associated to the algebras of type  $E_8$ ,  $E_7$ ,  $E_6$  and  $F_4$  are respectively projectively equivalent to the 27-dimensional  $E_7$ -variety arising from the 56-dimensional irreducible representation, the orthogonal Grassmann variety of isotropic 6-planes in  $\mathbb{C}^{12}$  (namely, the 15-dimensional spinor variety), the Grassmann variety of 3-planes in  $\mathbb{C}^6$  and the symplectic Grassmann variety of isotropic 3-planes in  $\mathbb{C}^6$ , with  $\dim \mathbb{P} = 55, 31, 19$  and  $13$ , respectively (see Appendix): for those homogeneous projective varieties, we refer to [12, §23.3].

In this article we study the Freudenthal varieties  $V$  with the filtration  $\{V_k\}$  of the ambient space  $\mathbb{P}$ , from the view-point of projective geometry, not individually but systematically in

terms of abstract Lie algebras, without depending on the classification of simple Lie algebras as well as on the known results for each case of types  $E_8$ ,  $E_7$ ,  $E_6$  and  $F_4$ .

Before stating the main result, we note that the Lie bracket  $\mathfrak{g}_1 \times \mathfrak{g}_1 \rightarrow \mathfrak{g}_2 \simeq \mathbb{C}$  defines a non-degenerate skew-symmetric form on  $\mathfrak{g}_1$ , so that this form allows us to identify  $\mathfrak{g}_1$  with its dual space, hence  $\mathbb{P}$  with its dual space, and  $\mathfrak{g}_1$  is even-dimensional. Moreover, the quartic form on  $\mathfrak{g}_1$  defining  $V_3$  has a differential which via the symplectic form defines a vector field on  $\mathfrak{g}_1$ , and this vector field defines a 1-dimensional distribution on  $\mathbb{P}$  away from the singular locus of  $V_3$  (see Proposition A1). We denote by  $L_P$  the (closure of the) integral curve of this distribution passing through  $P \in \mathbb{P} \setminus \text{Sing } V_3$ . On the other hand, we have a rational map  $\gamma : \mathbb{P} \dashrightarrow \mathbb{P}$  defined by  $x \mapsto (\text{ad } x)^3 \mathfrak{g}_{-2}$  with base locus  $V_2$ , which turns out to be a Cremona transformation of  $\mathbb{P}$ : It is deduced that  $\gamma^{-1}(V) = V_3 \setminus V_2$ ,  $\gamma^{-1}(\mathbb{P} \setminus V_3) = \mathbb{P} \setminus V_3$ ,  $\gamma^2 = 1$  on  $\mathbb{P} \setminus V_3$ , and  $\gamma$  is explicitly given by the partial differentials of  $q$  (see Proposition A2). Note that our  $\gamma$  is a special case of the Cremona transformations in [7, Theorem 2.8 (ii)].

Our main results are summarized as follows (see Theorems A, B, C, D, E, Corollaries A2, B1, B3 and C):

**Theorem.** *Assume that  $V$  is irreducible. Then we have:*

- (1)  *$V$  is a Legendrian subvariety of  $\mathbb{P}$ , that is, the projectivization of a Lagrangian subvariety of  $\mathfrak{g}_1$ , with  $\dim V = n - 1$ , spans  $\mathbb{P}$ , and is an orbit of the group of inner automorphisms of  $\mathfrak{g}$  with Lie algebra  $\mathfrak{g}_0$ , hence smooth, where  $\dim \mathfrak{g}_1 = 2n$ . In particular, the projective dual  $V^*$  of  $V$  is equal to the union of tangents to  $V$  via the symplectic form.*
- (2)  *$V_2$  is the singular locus of  $V_3$ , and for any  $P \in \mathbb{P} \setminus V_2$ ,  $L_P$  is the line in  $\mathbb{P}$  joining  $P$  and  $\gamma(P)$ . Moreover, we have:*
  - (a) *If  $P \in \mathbb{P} \setminus V_3$ , then  $L_P$  is a unique secant line of  $V$  passing through  $P$ , there is no tangent line to  $V$  passing through  $P$ ,  $L_P \cap V$  consists of harmonic conjugates with respect to  $P$  and  $\gamma(P)$ , and  $L_P \setminus V \subseteq \mathbb{P} \setminus V_3$ . Moreover,  $\gamma$  preserves  $L_P$ , and the automorphism of  $L_P$  induced from  $\gamma$  leaves each point in  $L_P \cap V$  invariant and permutes  $P$  and  $\gamma(P)$ .*
  - (b) *If  $P \in V_3 \setminus V_2$ , then there is no secant line of  $V$  passing through  $P$ ,  $L_P$  is a unique tangent line to  $V$  passing through  $P$ ,  $L_P \cap V = \gamma(P)$ , and  $L_P \setminus V \subseteq V_3 \setminus V_2$ . Moreover,  $L_P$  is contracted by  $\gamma$  to the contact point  $\gamma(P)$ , and conversely the fibre of  $\gamma$  on  $Q \in V$  consists of the points  $P \in V_3 \setminus V_2$  such that  $Q \in L_P$ , or equivalently,  $P$  lies on some tangent to  $V$  at  $Q$ .*

*In particular,  $V$  is a variety with one apparent double point, and  $V_3$  is the union of tangents to  $V$ .*

- (3) *For any  $P \in V_2 \setminus V$ , the family of secants of  $V$  passing through  $P$  is of dimension at least 1, and all of those secants are isotropic with respect to the symplectic form: In particular,  $V_2 \setminus V$  is covered by isotropic secants of  $V$ .*
- (4) *For any  $Q, R \in V$ , the secant line joining  $Q$  and  $R$  is isotropic if and only if the tangents to  $V$  at  $Q$  and at  $R$  are disjoint.*
- (5) *For any  $P \in V_3 \setminus V_2$  and  $Q \in V$ , if the secant line joining  $Q$  and the contact point  $\gamma(P)$  of  $L_P$  is not isotropic, then there is a twisted cubic curve contained in  $V$  to which  $L_P$  and  $L_R$  are tangent at  $\gamma(P)$  and at  $Q$ , respectively, where  $R$  is a point on some tangent to  $V$  at  $Q$  away from  $V_2$ , determined by  $P$  and  $Q$ .*
- (6) *If  $V_2 \neq V$ , then  $V$  is ruled, that is, covered by lines contained in  $V$ .*
- (7) *For any  $P \in V$ , the double projection from  $P$  gives a birational map from  $V$  onto  $\mathbb{P}^{n-1}$ , and by the inverse  $V$  is written as the closure of the image of a cubic Veronese embedding of a certain affine space  $\mathbb{A}^{n-1}$  under some projection to  $\mathbb{P}$ .*

We show also that the three conditions,  $V = \emptyset$ ,  $V_3 = \mathbb{P}$  and  $V_2 = \mathbb{P}$  are equivalent to each other (see Corollary A1), and that if  $V$  is neither empty nor irreducible, then  $\mathfrak{g}_1$  decomposes

naturally into two irreducible  $\mathfrak{g}_0$ -submodules of dimension  $n$  and  $V$  is the (disjoint) union of the projectivizations of those summands (see Corollary B2).

Finally we should mention that S. Mukai announced a theorem [20, (5.8)] on cubic Veronese varieties without proofs. Our work was originated by looking for proofs of the corresponding statements for Freudenthal varieties (Corollaries A2, B1, C and Theorem D): In fact, we see from his list [20, (5.10)] of cubic Veronese varieties (and the list in Appendix) that the notion of our Freudenthal varieties coincides with that of his cubic Veronese varieties. Our result gives a partial explanation for this coincidence (see Theorem D).

This is a joint work with Osami Yasukura. For proofs of the results here, see [FV].

## 1. PRELIMINARIES

For a finite-dimensional, simple Lie algebra  $\mathfrak{g}$  of rank  $\geq 2$ , a graded decomposition of contact type is obtained as follows: Take a Cartan subalgebra  $\mathfrak{h}$  of  $\mathfrak{g}$  and a basis  $\Delta$  of the root system  $R$  with respect to  $\mathfrak{h}$ , and fix an order on  $R$  defined by  $\Delta$ . Denote by  $\rho$  the highest root of  $\mathfrak{g}$ , let  $E_+$  and  $E_-$  be highest and lowest weight vectors, respectively, and set  $H := [E_+, E_-]$ . By multiplying suitable scalars, one may assume that  $(E_+, H, E_-)$  form an  $\mathfrak{sl}_2$ -triple, that is, those vectors have the following standard relations:

$$[H, E_+] = 2E_+, \quad [H, E_-] = -2E_-, \quad [E_+, E_-] = H.$$

Then, the eigenspace decomposition of  $\mathfrak{g}$  with respect to  $\text{ad } H$  gives  $\mathfrak{g}$  a graded decomposition of contact type: In other words, if we set  $\mathfrak{g}_\lambda := \{x \in \mathfrak{g} \mid [H, x] = \lambda x\}$  for  $\lambda \in \mathbb{C}$ , then it follows that  $\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$ ,  $\dim \mathfrak{g}_2 = 1$  and  $\mathfrak{g}_1 \neq 0$ : In fact,  $\mathfrak{g}_1 = 0$  if and only if  $\mathfrak{g} = \mathfrak{sl}_2$ . In terms of root spaces of  $\mathfrak{g}$ , we have

$$\mathfrak{g}_0 = \mathfrak{h} \oplus \bigoplus_{\alpha \in R^+ \setminus (R_\rho \cup \{\rho\})} (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}), \quad \mathfrak{g}_{\pm 1} = \bigoplus_{\alpha \in R_\rho} \mathfrak{g}_{\pm \alpha}, \quad \mathfrak{g}_{\pm 2} = \mathfrak{g}_{\pm \rho} = \mathbb{C}E_{\pm},$$

where  $R^+$  is the set of positive roots and  $R_\rho := \{\alpha \in R^+ \mid \rho - \alpha \in R\}$ : Indeed, let  $\mathfrak{s}_\rho$  be the subalgebra of  $\mathfrak{g}$  spanned by  $E_+$ ,  $H$  and  $E_-$ , which is isomorphic to  $\mathfrak{sl}_2$ . Then the irreducible decomposition of  $\mathfrak{g}$  as an  $\mathfrak{sl}_2$ -module gives the decomposition above (see, for full details, [25]). Conversely, for a graded decomposition  $\mathfrak{g} = \sum \mathfrak{g}_i$  of contact type, taking suitable bases  $E_+$  for  $\mathfrak{g}_2$  and  $E_-$  for  $\mathfrak{g}_{-2}$  with  $H := [E_+, E_-]$ , one may assume that  $(E_+, H, E_-)$  form an  $\mathfrak{sl}_2$ -triple, as before. Then, we see that  $E_+$  and  $E_-$  are some highest and lowest weight vectors, respectively, and each  $\mathfrak{g}_i$  is recovered as an  $(\text{ad } H)$ -eigenspace. Therefore, the graded decompositions of contact type are unique up to automorphism of  $\mathfrak{g}$ , so that the Freudenthal variety  $V$  is essentially unique and determined by  $\mathfrak{g}$  itself (see Appendix).

Now, we define a symmetric product  $\times : \mathfrak{g}_1 \times \mathfrak{g}_1 \rightarrow \mathfrak{g}_0$  by the formula:

$$-2a \times b = [b, [a, E_-]] + [a, [b, E_-]],$$

which induces a symmetric map  $L : \mathfrak{g}_1 \times \mathfrak{g}_1 \rightarrow \text{Hom}(\mathfrak{g}_1, \mathfrak{g}_1)$  and a ternary product  $[\cdot, \cdot, \cdot] : \mathfrak{g}_1 \times \mathfrak{g}_1 \times \mathfrak{g}_1 \rightarrow \mathfrak{g}_1$  by

$$[a, b, c] = L(a, b)c = [a \times b, c].$$

Note that the adjoint action of  $\mathfrak{g}_0$  on  $\mathfrak{g}_1$  is faithful since  $\mathfrak{g}$  is simple (see [25, Lemma 3.2 (1)]): we may assume  $\mathfrak{g}_0 \subseteq \text{Hom}(\mathfrak{g}_1, \mathfrak{g}_1)$ , so that we identify  $L(a, b)$  with  $a \times b$ . We think of  $\mathfrak{g}_1$  as an  $\mathfrak{g}_0$ -module via the adjoint action: For example, we often write  $Dx$  instead of  $(\text{ad } D)x$  and  $[D, x]$  for  $D \in \mathfrak{g}_0$  and  $x \in \mathfrak{g}_1$ . As the skew-symmetric form  $\langle \cdot, \cdot \rangle : \mathfrak{g}_1 \times \mathfrak{g}_1 \rightarrow \mathbb{C}$  and the quartic form on  $\mathfrak{g}_1$  defining  $V_3$  mentioned in Introduction, we use the ones determined by

$$2\langle a, b \rangle E_+ = [a, b], \quad 2q(x)E_+ = (\text{ad } x)^4 E_-.$$

Note that the skew-symmetric form  $\langle, \rangle$  is *non-degenerate* since  $\mathfrak{g}$  is simple (see [25, Lemma 3.2 (2)]).

With the notation above, it follows that

$$\begin{aligned} V &= V_1 = \pi(\{x \in \mathfrak{g}_1 \setminus \{0\} | x \times x = 0\}), \\ V_2 &= \pi(\{x \in \mathfrak{g}_1 \setminus \{0\} | [xxx] = 0\}), \\ V_3 &= \pi(\{x \in \mathfrak{g}_1 \setminus \{0\} | \langle x, [xxx] \rangle = 0\}), \end{aligned}$$

and  $q(x) = \langle x, [xxx] \rangle$ . Note that  $V_0 = \emptyset$  since  $[[x, E_-]E_+] = x$  for any  $x \in \mathfrak{g}_1$ : Indeed, it follows from the Jacobi identity that  $[[x, E_-]E_+] = [[x, E_+], E_-] + [x[E_-, E_+]] = [x, -H] = x$  since  $[x, E_+] \in \mathfrak{g}_3 = 0$ . On the other hand, it follows from Lemma 1 below that  $V \neq \mathbb{P}$ .

**Lemma 1.** *Let  $\mathfrak{g}_{00}$  be the subalgebra of  $\mathfrak{g}_0$  defined by*

$$\mathfrak{g}_{00} := \text{Ker}(\text{ad } E_+|_{\mathfrak{g}_0}) = \text{Ker}(\text{ad } E_-|_{\mathfrak{g}_0}).$$

*Then we have  $\mathfrak{g}_0 = \mathfrak{g}_{00} \oplus \mathbb{C}H$ , and  $\mathfrak{g}_{00}$  is linearly spanned by the elements in  $\mathfrak{g}_0$  of the form  $a \times b$  with  $a, b \in \mathfrak{g}_1$ . In particular,  $\mathfrak{g}_{00} \neq 0$ , and  $x \times x \neq 0$  for some  $x \in \mathfrak{g}_1$ .*

**Lemma 2** (Asano [3]). *For any  $a, b, c \in \mathfrak{g}_1$  and  $D \in \mathfrak{g}_{00}$ , we have*

- (1)  $\langle Da, b \rangle + \langle a, Db \rangle = 0$ .
- (2)  $D(a \times b) = Da \times b + a \times Db$ .
- (3)  $D[abc] = [(Da)bc] + [a(Db)c] + [ab(Dc)]$ .

If we denote by  $G_{00}$  the group of inner automorphisms of  $\mathfrak{g}$  with Lie algebra  $\mathfrak{g}_{00}$ , then Lemma 2 tells that the symplectic form  $\langle, \rangle$ , the symmetric product  $\times$  and the ternary product  $[, , ]$  are equivariant with respect to the action of  $G_{00}$ , so that each  $V_i$  is stable under the action of  $G_{00}$ , that is, a union of some orbits of  $G_{00}$ .

**Lemma 3** (Asano [3]). *We have  $[abc] - [acb] = \langle a, c \rangle b - \langle a, b \rangle c + 2\langle b, c \rangle a$  for any  $a, b, c \in \mathfrak{g}_1$ .*

## 2. A LINE FIELD AND A CREMONA TRANSFORMATION

**Proposition A1.**

- (1) *The quartic form  $q$  on  $\mathfrak{g}_1$  has a differential at  $a \in \mathfrak{g}_1$  as follows:*

$$dq(a) : t_a \mathfrak{g}_1 \rightarrow \mathbb{C}; b \mapsto 4\langle b, [aaa] \rangle,$$

*where  $t_a \mathfrak{g}_1$  is the Zariski tangent space to  $\mathfrak{g}_1$  at  $a$ , naturally identified with  $\mathfrak{g}_1$ .*

- (2) *In particular, the singular locus of  $V_3$  is equal to  $V_2$ .*
- (3) *The vector field on  $\mathfrak{g}_1$  corresponding to  $dq$  via the symplectic form  $\langle, \rangle$  induces a 1-dimensional distribution  $\mathcal{D}$  on  $\mathbb{P}$  away from  $\text{Sing } V_3 = V_2$ , which is given by*

$$\mathcal{D} : \pi(a) \mapsto (\mathbb{C}a + \mathbb{C}[aaa])/\mathbb{C}a,$$

*where  $\pi(a) \in \mathbb{P} \setminus V_2$  and we naturally identify the Zariski tangent space  $t_{\pi(a)} \mathbb{P}$  with the quotient space  $\mathfrak{g}_1/\mathbb{C}a$ .*

**Proposition A2.** *Let*

$$\gamma : \mathbb{P} \dashrightarrow \mathbb{P}$$

*be a rational map induced from the cubic,  $a \mapsto [aaa]$ . Then we have:*

- (1)  $\gamma^{-1}(V) = V_3 \setminus V_2$ .
- (2)  $\gamma^{-1}(\mathbb{P} \setminus V_3) = \mathbb{P} \setminus V_3$ .
- (3)  $\gamma^2 = 1$  on  $\mathbb{P} \setminus V_3$ , hence  $\gamma$  gives an automorphism of  $\mathbb{P} \setminus V_3$ .
- (4)  $\gamma$  is explicitly given by the partial differentials of  $q$ .

In particular,  $\gamma$  is a Cremona transformation of  $\mathbb{P}(\mathfrak{g}_1)$  with order 2 if  $V_2 \neq \mathbb{P}$ .

A secant line of  $V$  is by definition a line in  $\mathbb{P}$  which passes through at least two distinct points of  $V$  and is not contained in  $V$ . We note that for a line  $L$  in  $\mathbb{P}$  if the scheme-theoretic intersection  $L \cap V$  has length more than 2, then  $L \subseteq V$ : Indeed,  $V$  is an intersection of quadric hypersurfaces.

**Theorem A.** *Let  $L_P$  be the closure of the integral curve of  $\mathcal{D}$  through  $P \in \mathbb{P} \setminus V_2$ , where  $\mathcal{D}$  is the 1-dimensional distribution on  $\mathbb{P} \setminus V_2$  induced from the quartic form  $q$ . Then we have:*

- (1) *For any  $P \in \mathbb{P} \setminus V_2$ ,  $L_P$  is the line in  $\mathbb{P}$  joining  $P$  and  $\gamma(P)$ .*
- (2) *If  $P \in \mathbb{P} \setminus V_3$ , then we have:*
  - (a)  *$L_P$  is a secant line of  $V$ , and  $L_P \cap V$  consists of harmonic conjugates with respect to  $P$  and  $\gamma(P)$ .*
  - (b)  *$L_P \setminus V \subseteq \mathbb{P} \setminus V_3$ .*
  - (c)  *$L_P$  is a unique secant line of  $V$  passing through  $P$ .*
  - (d) *There is no tangent line to  $V$  passing through  $P$ .*
  - (e)  *$\gamma(L_P \setminus V) = L_P \setminus V$ , and the automorphism of  $L_P$  induced from  $\gamma$  leaves each point in  $L_P \cap V$  invariant and permutes  $P$  and  $\gamma(P)$ .*
- (3) *If  $P \in V_3 \setminus V_2$ , then we have:*
  - (a)  *$L_P$  is a tangent line to  $V$ , and  $L_P \cap V = \{\gamma(P)\}$ .*
  - (b)  *$L_P \setminus V \subseteq V_3 \setminus V_2$ .*
  - (c) *There is no secant line of  $V$  passing through  $P$ .*
  - (d)  *$L_P$  is a unique tangent line to  $V$  passing through  $P$ .*
  - (e)  *$\gamma(L_P \setminus V) = \gamma(P)$ , and  $\gamma^{-1}(Q) = \{P \in V_3 \setminus V_2 \mid Q \in L_P\} = T_Q V \setminus V_2$  for any  $Q \in V$ , where  $T_Q V$  is the embedded tangent space to  $V$  at  $Q$ .*

**Corollary A1.** *The three conditions,  $V = \emptyset$ ,  $V_3 = \mathbb{P}$  and  $V_2 = \mathbb{P}$  are equivalent to each other.*

*Remark A.* It can be shown that  $V = \emptyset$  if and only if the Lie algebra  $\mathfrak{g}$  is of type  $C$  (see Appendix): In fact, using a theorem of Asano [28, 1.6.Theorem], [4], one can show that if  $q \equiv 0$ , then  $\mathfrak{g} \simeq \mathfrak{sp}_{2n+2}$ , where  $\dim \mathfrak{g}_1 = 2n$ ; The converse is checked by an explicit computation.

Recall that a projective variety  $V \subseteq \mathbb{P}$  is called a *variety with one apparent double point* if for a general point  $P \in \mathbb{P}$  there exists a unique secant line of  $V$  passing through  $P$  (see [23, IX]).

**Corollary A2.** *If  $V \neq \emptyset$ , then  $V$  is a variety with one apparent double point. In particular,  $V$  is non-degenerate in  $\mathbb{P}$ .*

### 3. THE HOMOGENEITY

**Theorem B.** *Let  $G_{00}$  be the group of inner automorphisms of  $\mathfrak{g}$  with Lie algebra  $\mathfrak{g}_{00}$ , where  $\mathfrak{g}_{00}$  is the subalgebra of  $\mathfrak{g}_0$  defined by  $\mathfrak{g}_{00} := \text{Ker}(\text{ad } E_{\pm} | \mathfrak{g}_0)$ . Then we have:*

- (1)  *$G_{00}$  acts transitively on each of irreducible components of  $\mathcal{V}$ . In particular, we have  $t_x \mathcal{V} = \mathfrak{g}_{00}x$  for any  $x \in \mathcal{V}$ , where  $t_x \mathcal{V}$  is the Zariski tangent space to  $\mathcal{V}$  at  $x$ .*
- (2)  *$\mathfrak{g}_{00}x = (\mathfrak{g}_{00}x)^{\perp}$  with  $2 \dim \mathfrak{g}_{00}x = \dim \mathfrak{g}_1$  for any  $x \in \mathcal{V}$ , and  $\mathfrak{g}_1 = \mathfrak{g}_{00}x \oplus \mathfrak{g}_{00}y$  for any  $x, y \in \mathcal{V}$  with  $\langle x, y \rangle \neq 0$ .*

Recall that the *tangent variety* of  $V$ , denoted by  $\text{Tan } V$ , is the union of embedded tangent spaces to  $V$ , and the *projective dual* of  $V$ , denoted by  $V^*$ , is the set of hyperplanes tangent to  $V$  (see, for example, [11, §3]).

**Corollary B1.** *Assume that  $V \neq \emptyset$ . Then we have:*

- (1)  $G_{00}$  acts transitively on each of irreducible components of  $V$ , and  $V$  is smooth, equi-dimensional of dimension  $n - 1$ , where  $\dim \mathfrak{g}_1 = 2n$ .
- (2) Denote by  $L^*$  the set of hyperplanes containing a linear subspace  $L \subseteq \mathbb{P}$ . Then we have  $(T_Q V)^* = T_Q V$  for any  $Q \in V$ , hence

$$\text{Tan } V = V^*,$$

where we identify  $\mathbb{P}$  with its dual space  $\mathbb{P}^\vee := \mathbb{P}(\mathfrak{g}_1^*)$  via the symplectic form  $\langle, \rangle$ .

**Corollary B2.** *If  $V$  is neither empty nor irreducible, then there are irreducible  $\mathfrak{g}_{00}$ -modules  $\mathfrak{s}_1$  and  $\mathfrak{s}_2$  of dimension  $n$  such that  $\mathfrak{g}_1 = \mathfrak{s}_1 \oplus \mathfrak{s}_2$ , and we have*

$$V = \mathbb{P}(\mathfrak{s}_1) \sqcup \mathbb{P}(\mathfrak{s}_2),$$

where  $\dim \mathfrak{g}_1 = 2n$ .

*Remark B1.* It is known that  $V$  is irreducible unless  $\mathfrak{g}$  is of type  $A$  or  $C$  (see Appendix): In fact, if  $\mathfrak{g} = \mathfrak{so}_m$ , then  $V$  is a Segre embedding of  $\mathbb{P}^1 \times Q$  in  $\mathbb{P}^{2m-9}$ , where  $Q$  is a quadric hypersurface in  $\mathbb{P}^{m-5}$ ; if  $\mathfrak{g}$  is of type  $G_2$ , then  $V$  is a cubic Veronese embedding of  $\mathbb{P}^1$  in  $\mathbb{P}^3$ ; for other exceptional Lie algebras  $\mathfrak{g}$ , see Introduction. Conversely, it follows from a direct computation that we are in the case above if  $\mathfrak{g} = \mathfrak{sl}_{n+2}$  with  $n \geq 1$ .

**Corollary B3.** *If  $V \neq \emptyset$  and  $V_2 \neq V$ , then  $V$  is ruled, that is, covered by lines contained in  $V$ .*

*Remark B2.* It can be shown that  $V = V_2$  if and only if  $\mathfrak{g}$  is of type  $G_2$ .

#### 4. ISOTROPIC SECANTS

**Proposition C.** *For  $P = \pi(u) \in \mathbb{P}$ , let  $\Phi_P : \mathbb{P} \dashrightarrow \mathbb{P}$  be a rational map induced from  $L(u, u)$  with base locus  $B_P = \mathbb{P}(\text{Ker } L(u, u))$ . If  $V$  is irreducible and  $P \in V_2 \setminus V$ , then  $\dim \Phi_P(V \setminus B_P) \geq 1$ , hence  $\dim \Phi_P(\mathbb{P} \setminus B_P) \geq 1$  and  $\text{codim } B_P \geq 2$ .*

*Remark C1.* The irreducibility condition for  $V$  is essential in Proposition C: In fact, there is an example of  $u$  satisfying the assumption above such that  $\text{rk } L(u, u) = 1$  in case of  $\mathfrak{g} = \mathfrak{sl}_m$ , where  $V$  is not irreducible (see Remark B3).

*Remark C2.* It follows easily from Proposition 6 that  $\dim \Phi_P(\mathbb{P} \setminus B_P) \geq 1$  if  $P \notin V_2$ , and  $\text{codim } \Phi_P(\mathbb{P} \setminus B_P) \geq 1$  if  $P \in V_3$ , though we do not use these facts in this article.

Recall that the *secant locus*  $\Sigma_P$  as well as the *tangent locus*  $\Theta_P$  of  $V$  with respect to a given point  $P \in \mathbb{P}$  are defined by

$$\Sigma_P := \overline{\{Q \in V \mid \exists R \in V \setminus \{Q\}, P \in Q * R\}}, \quad \Theta_P := \{Q \in V \mid P \in T_Q V\},$$

where we denote by  $Q * R$  the line in  $\mathbb{P}$  joining  $Q$  and  $R$ , and by  $T_Q V$  the embedded tangent space to  $V$  at  $Q$  in  $\mathbb{P}$  (see, for example, [11]).

**Theorem C.** *Assume that  $V$  is irreducible. Then we have:*

- (1) For any  $x, y \in \mathcal{V}$ ,  $\langle x, y \rangle = 0$  if and only if  $\mathfrak{g}_{00}x \cap \mathfrak{g}_{00}y \neq 0$ . In particular, a secant line joining  $Q, R \in V$  is isotropic with respect to the symplectic form if and only if  $T_Q V \cap T_R V \neq \emptyset$ .
- (2)  $V_2 \setminus V$  is covered by isotropic secants of  $V$ . More precisely, for any  $u \in \mathfrak{g}_1$ , we have that  $[uuu] = 0$  and  $u \times u \neq 0$  if and only if  $u = x + y$  for some  $x, y \in \mathcal{V}$  such that  $\langle x, y \rangle = 0$  and  $x \times y \neq 0$ .

(3) If  $P \in V_2 \setminus V$ , then

$$\Phi_P(V \setminus B_P) \subseteq \Sigma_P, \quad \Phi_P(V \cap P^\perp \setminus B_P) \subseteq \Theta_P,$$

where  $\Phi_P : \mathbb{P} \dashrightarrow \mathbb{P}$  is the rational map induced from  $L(u, u)$  with base locus  $B_P = \mathbb{P}(\text{Ker } L(u, u))$  and  $P^\perp = \mathbb{P}(u^\perp)$  with  $P = \pi(u)$ .

(4) We have  $\dim \Sigma_P \geq 1$  for any  $P \in V_2 \setminus V$ .

*Remark C3.* The irreducibility condition for  $V$  is essential in (1) above: In fact, it is easily seen that the conclusion does not hold in case of  $\mathfrak{g} = \mathfrak{sl}_m$ .

**Corollary C.** If  $V$  is irreducible, then  $V_3 = \text{Tan } V$ .

## 5. DOUBLE PROJECTIONS

**Proposition D.** For any  $x, y \in \mathcal{V}$ , let  $\Psi_{xy} : \mathfrak{g}_1 \rightarrow \mathfrak{g}_1$  be a linear map defined by

$$\Psi_{xy}(a) := [axy] + \langle a, x \rangle y.$$

(1) If  $\langle x, y \rangle \neq 0$ , then  $\text{Ker } \Psi_{xy} = \mathfrak{g}_{00}x$  and  $\Psi_{xy}(\mathfrak{g}_1) = \mathfrak{g}_{00}y$ . In particular, a rational map  $\Psi_{PQ} : \mathbb{P} \dashrightarrow \mathbb{P}$  induced from  $\Psi_{xy}$  is a double projection from  $P$  with image  $T_QV$ , that is, a projection with center  $T_PV$  onto  $T_QV$ , hence defines a morphism

$$\Psi_{PQ} : \mathbb{P} \setminus T_PV \rightarrow T_QV,$$

where  $T_PV$  is the embedded tangent space to  $V$  at  $P$  with  $P = \pi(x)$  and  $Q = \pi(y)$ .

(2) Moreover for any  $R \in V$ , the four points  $R, [PQR], \Psi_{PR}(Q)$  and  $\Psi_{QR}(P)$  are collinear, and  $[PQR]$  is the harmonic conjugate of  $R$  with respect to  $\Psi_{PR}(Q), \Psi_{QR}(P)$ , where we set  $[PRR] := \pi([xyz])$  with  $R = \pi(z)$ . In particular, this holds for general  $P, Q, R \in V$  and gives a geometric meaning of our ternary product.

*Remark D1.* In terms of the Lie bracket, we have  $\Psi_{ab}(c) = [b[a[c, E_-]]]$ .

**Theorem D.** For any  $P, Q \in V$ , if the secant line joining  $P$  and  $Q$  is not isotropic, that is,  $T_PV \cap T_QV = \emptyset$ , then we have:

- (1)  $V \setminus P^\perp = (\Psi_{PQ}|_{V \setminus T_PV})^{-1}(T_QV \setminus P^\perp)$ .
- (2) The double projection  $\Psi_{PQ}$  gives an isomorphism  $V \setminus P^\perp \rightarrow T_QV \setminus P^\perp$ . In fact, a rational map  $\Gamma_{QP} : T_QV \dashrightarrow V$  induced from a map  $\Gamma_{yx} : \mathfrak{g}_{00}y \rightarrow \mathcal{V} \cup \{0\}$  defined by

$$\Gamma_{yx}(t) := \langle x, [ttt] \rangle x + 3\langle x, t \rangle [ttx] + 12\langle x, t \rangle^2 t$$

gives the inverse of  $\Psi_{PQ}|_{V \setminus P^\perp}$ , where  $P = \pi(x)$  and  $Q = \pi(y)$ .

- (3) The base locus of  $\Gamma_{QP}$  is  $T_QV \cap P^\perp \cap V_2$ .

In particular, if  $V$  is irreducible, then  $\Psi_{PQ}$  gives a birational map from  $V$  to  $T_QV$ , and  $V$  is the closure of the image of a composition of a cubic Veronese embedding of the affine space  $T_QV \setminus P^\perp$  with some projection to  $\mathbb{P}$ .

*Remark D2.* The morphism  $\Psi_{PQ} : V \setminus T_PV \rightarrow T_QV$  is not necessarily surjective: In fact, if  $\mathfrak{g}$  is of type  $G_2$ , then for any  $P \in V$ ,  $P^\perp$  is the osculating plane to the twisted cubic  $V \subseteq \mathbb{P}^3$  at  $P$ ,  $V \cap P^\perp = \{P\}$ , and  $\Psi_{PQ}(V \setminus T_PV) = T_QV \setminus P^\perp$  for any  $Q \in V$  with  $P \neq Q$ .

## 6. TWISTED CUBIC CURVES

**Proposition E.** *For any  $P \in V_3 \setminus V_2$  and  $Q \in V$ , if the secant line joining  $Q$  and the contact point  $\gamma(P)$  of  $L_P$  is not isotropic, then we have:*

- (1)  $Q \in L_{\Phi_P(Q)}$  and  $\Phi_P^3(Q) = \gamma(P) \in L_P = L_{\Phi_P^2(Q)}$  with  $\Phi_P(Q), \Phi_P^2(Q) \in V_3 \setminus V_2$ .
- (2)  $L_P \cap L_{\Phi_P(Q)} = \emptyset$ , hence  $Q, \Phi_P(Q), \Phi_P^2(Q)$  and  $\Phi_P^3(Q)$  are linearly independent in  $\mathbb{P}$ .

**Theorem E.** *For any  $P \in V_3 \setminus V_2$  and  $Q \in V$  such that the secant line joining  $Q$  and the contact point  $\gamma(P)$  of  $L_P$  is not isotropic, that is,  $T_Q V \cap T_{\gamma(P)} V = \emptyset$ , let  $\mathbb{P}_{PQ}$  be the linear subspace of dimension 3 in  $\mathbb{P}$  spanned by  $Q, \Phi_P(Q), \Phi_P^2(Q)$  (or equivalently  $P$ ) and  $\Phi_P^3(Q) = \gamma(P)$ , that is, spanned by  $L_P$  and  $L_{\Phi_P(Q)}$ , the unique tangent lines to  $V$  passing through  $P$  and  $\Phi_P(Q)$ . Then we have:*

- (1) *The intersection  $V \cap \mathbb{P}_{PQ}$  is a twisted cubic curve in  $\mathbb{P}_{PQ} \simeq \mathbb{P}^3$  given explicitly by the image of  $L_P$  under the cubic map  $\Gamma_{\gamma(P)Q}$ :*

$$V \cap \mathbb{P}_{PQ} = \Gamma_{\gamma(P)Q}(L_P).$$

- (2) *The twisted cubic curve in  $\mathbb{P}_{PQ}$  above has the following properties:*
  - (a)  $L_P$  and  $L_{\Phi_P(Q)}$  are respectively the tangent lines at  $\gamma(P)$  and at  $Q$ , and
  - (b)  $\gamma(P)^\perp \cap \mathbb{P}_{PQ}$  and  $Q^\perp \cap \mathbb{P}_{PQ}$  are respectively the osculating planes at  $\gamma(P)$  and at  $Q$ , which are spanned by  $L_P$  and  $\Phi_P(Q)$  and by  $L_{\Phi_P(Q)}$  and  $\Phi_P^2(Q)$ , respectively.

*Remark E.* Set  $D := L(t, t)$ ,  $E := L(Dx, Dx)$  and  $F := [D, E]$  with  $P = \pi(t)$  and  $Q = \pi(x)$ , and denote by  $\mathfrak{g}_{00PQ}$  the subalgebra of  $\mathfrak{g}_{00}$  generated by  $D, E$  and  $F$ . Then it follows that

$$[F, D] = \frac{4}{3} \langle D^3 x, x \rangle D, \quad [F, E] = -\frac{4}{3} \langle D^3 x, x \rangle E,$$

so that  $\mathfrak{g}_{00PQ}$  is isomorphic to the Lie algebra  $\mathfrak{sl}_2$ . If we denote by  $\mathfrak{g}_{1PQ}$  the subspace of  $\mathfrak{g}_1$  spanned by  $x, Dx, D^2x$  and  $D^3x$ , then we see that  $\mathfrak{g}_{1PQ}$  is an irreducible  $\mathfrak{g}_{00PQ}$ -module of dimension 4 with

$$F(D^k x) = (2k - 3) \frac{2}{3} \langle D^3 x, x \rangle D^k x,$$

and the twisted cubic curve  $V \cap \mathbb{P}_{PQ} = \Gamma_{\gamma(P)Q}(L_P)$  is a unique closed orbit in  $\mathbb{P}_{PQ} = \mathbb{P}(\mathfrak{g}_{1PQ})$  under the natural action of the group of inner automorphisms of  $\mathfrak{g}_{00}$  with Lie algebra  $\mathfrak{g}_{00PQ}$ . Thus, for any  $P \in V_3 \setminus V_2$  and  $Q \in V$  with  $T_{\gamma(P)} V \cap T_Q V = \emptyset$ , a subalgebra  $\mathfrak{g}_{00PQ}$  of  $\mathfrak{g}_{00}$  isomorphic to  $\mathfrak{sl}_2$  and an irreducible  $\mathfrak{g}_{00PQ}$ -submodule  $\mathfrak{g}_{1PQ}$  of  $\mathfrak{g}_1$  with dimension 4 are associated to  $P$  and  $Q$ . If  $\mathfrak{g}$  is of type  $G_2$ , then  $\mathfrak{g}_{00PQ}$  and  $\mathfrak{g}_{1PQ}$  are respectively equal to  $\mathfrak{g}_{00}$  and  $\mathfrak{g}_1$  themselves.

## APPENDIX. A CLASSIFICATION OF FREUDENTHAL VARIETIES

We here give a classification of Freudenthal varieties  $V$  in terms of the root data of  $\mathfrak{g}$ . It would be interesting to compare  $V$  with the adjoint variety associated to  $\mathfrak{g}$  since those varieties are closely related to each other: In fact, for a simple graded Lie algebra  $\mathfrak{g} = \sum \mathfrak{g}_i$  of contact type, denote by  $V$  the Freudenthal variety associated to  $\mathfrak{g}$ , as before, and denote by  $X$  the orbit of the inner automorphism group of  $\mathfrak{g}$  through  $\pi(E_+)$  in  $\mathbb{P}(\mathfrak{g})$ , which is the minimal closed orbit in  $\mathbb{P}(\mathfrak{g})$ , called the *adjoint variety* associated to  $\mathfrak{g}$  (see [16]). Then, according to [17, Theorem B], we have  $V = X \cap \mathbb{P}(\mathfrak{g}_1)$ .



## ADJOINT VARIETIES AND FREUDENTHAL VARIETIES

$\mathfrak{g}$	$X \subseteq \mathbb{P}(\mathfrak{g})$	$\mathfrak{g}_{00}$	$V \subseteq \mathbb{P}(\mathfrak{g}_1)$
$\mathfrak{sl}_m$	$(\mathbb{P}^{m-1} \times \mathbb{P}^{m-1}) \cap (1) \subseteq \mathbb{P}^{m^2-2}$	$\mathfrak{gl}_1 \oplus \mathfrak{sl}_{m-2}$	$\mathbb{P}^{m-3} \sqcup \mathbb{P}^{m-3} \subseteq \mathbb{P}^{2m-5}$
$\mathfrak{so}_m$	$\mathbb{G}_{\text{orthog.}}(2, m) \subseteq \mathbb{P}^{\binom{m}{2}-1}$	$\mathfrak{sl}_2 \oplus \mathfrak{so}_{m-4}$	$\mathbb{P}^1 \times Q^{m-6} \subseteq \mathbb{P}^{2m-9}$
$\mathfrak{sp}_{2m}$	$v_2 \mathbb{P}^{2m-1} \subseteq \mathbb{P}^{\binom{2m+1}{2}-1}$	$\mathfrak{sp}_{2m-2}$	$\emptyset \subseteq \mathbb{P}^{2m-3}$
$\mathfrak{e}_6$	$E_6(\omega_2)^{21} \subseteq \mathbb{P}^{77}$	$\mathfrak{sl}_6$	$\mathbb{G}(3, 6) \subseteq \mathbb{P}^{19}$
$\mathfrak{e}_7$	$E_7(\omega_1)^{33} \subseteq \mathbb{P}^{132}$	$\mathfrak{so}_{12}$	$S_5 = \mathbb{G}_{\text{orthog.}}(6, 12) \subseteq \mathbb{P}^{2^5-1}$
$\mathfrak{e}_8$	$E_8(\omega_8)^{57} \subseteq \mathbb{P}^{247}$	$\mathfrak{e}_7$	$E_7(\omega_6) \subseteq \mathbb{P}^{55}$
$\mathfrak{f}_4$	$F_4(\omega_1)^{15} \subseteq \mathbb{P}^{51}$	$\mathfrak{sp}_6$	$\mathbb{G}_{\text{symp.}}(3, 6) \subseteq \mathbb{P}^{13}$
$\mathfrak{g}_2$	$G_2(\omega_2)^5 \subseteq \mathbb{P}^{13}$	$\mathfrak{sl}_2$	$v_3 \mathbb{P}^1 \subseteq \mathbb{P}^3$

*Notation:* We denote by  $\cap(1)$  cutting by a general hyperplane, and by  $v_d$  the Veronese embedding of degree  $d$ . We denote by  $\mathbb{G}(r, m)$  a Grassmann variety of  $r$ -planes in  $\mathbb{C}^m$ , and denote by  $\mathbb{G}_{\text{orthog.}}(r, m)$  and by  $\mathbb{G}_{\text{symp.}}(r, m)$  respectively an orthogonal and a symplectic Grassmann varieties of isotropic  $r$ -planes in  $\mathbb{C}^m$ . A simple exceptional Lie algebra of Dynkin type  $G$  is denoted by the lowercase of  $G$  in the German character, as in [12], a simple algebraic group of type  $G$  is denoted by just  $G$ , and for a dominant integral weight  $\omega$  of  $G$ , the minimal closed orbit of  $G$  in  $\mathbb{P}(V_\omega)$  is denoted by  $G(\omega)$ , where  $V_\omega$  is the irreducible representation space of  $G$  with highest weight  $\omega$ : For example,  $\mathfrak{g}_2$  in the list is the simple Lie algebra of type  $G_2$ , and  $G_2(\omega_2)$  is the minimal closed orbit of an algebraic group of type  $G_2$  in  $\mathbb{P}(V_{\omega_2})$ , where  $\omega_2$  is the second fundamental dominant weight with the standard notation of Bourbaki [6].

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