ON WEAKENED FANO 3-FOLDS

TATSUHIRO MINAGAWA

1. Introduction

We will work over $\mathbb{C}$ in this talk.

Definition 1.1. Let X be a 3-dimensional smooth projective variety.

(1) We call X a Fano 3-fold when its anti-canonical divisor $-K_X$ is ample.
(2) We call X a weak Fano 3-fold when its anti-canonical divisor $-K_X$ is nef and big.

Definition 1.2. Let X be a smooth weak Fano 3-fold and $(\Delta, 0)$ a germ of the 1-dimensional disk. We call X a weakened Fano 3-fold when

(i) X is not a Fano 3-fold, and
(ii) there exists a small deformation $\mathfrak{f}\colon \mathscr{X} \to (\Delta, 0)$ of X such that the fiber $\mathscr{X}_s = \mathfrak{f}^{-1}(s)$ is a Fano 3-fold for any $s \in (\Delta, 0) \setminus \{0\}$.

Let X be a weakened Fano 3-fold. We remark that $B_2(X) \geq 2$ because X is a weak Fano which is not a Fano 3-fold. Fano 3-folds with $B_2 \geq 2$ are classified by Mori and Mukai (cf. [M-M 1], [M-M 2]). In particular, $2 \leq B_2 \leq 10$.

Example 1.3. $\mathbb{P} \times \mathbb{F}_2$ is a weakened Fano 3-fold with $B_2(X) = 3$ and $(-K)^3 = 48$ which will deform to $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

Let S be a weak del Pezzo surface but a del Pezzo surface. Then it is a weakened del Pezzo surface. Thus $\mathbb{P}^1 \times S$ is a weakened Fano 3-fold. We call such weakened Fano 3-folds "product type".

Theorem 1.4. (H.Sato [Sa]) *There are exacty 15 toric weakened Fano 3-folds X up to isomorphism.*

In this classification, 11 toric weakened Fano 3-folds are of product type. Moreover, we had the following on product type.

Theorem 1.5. *Let X be a weakened Fano 3-fold which will deform to $X_t \simeq \mathbb{P}^1 \times S_t$ where S_t is a del Pezzo surface. Then X is of product type.*

Our main result is the following:

Theorem 1.6. *Weakened Fano 3-folds are classified as in Classified Table.*

Acknowledgement. I would like to thank Doctor Hiroshi Sato for useful discussions and encouraging the author during the preparation of this contents. I also wishes to thank organizers for giving me the opportunity to talk about this contents.

Notations. (1) The i-th Betti number of a manifold X is denoted by $B_i(X)$.
(2) The $\mathbb{P}^1$-bundle $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$ over $\mathbb{P}^1$, a Hirzebruch surface of degree n, is denoted by $\mathbb{F}_n$.

2. Ideas for classification

Definition 2.1. Let X be a smooth weak Fano 3-fold, and $\phi\colon X \to \bar{X}$ a crepant birational projective morphism. We call ϕ primitive when its relative Picard number $\rho(X/\bar{X}) = 1$. Moreover, letting E be the exceptional locus of ϕ, we will define as follows.

(i) ϕ is a crepant primitive birational contraction of type I when $\dim(E) = 1$.
(ii) ϕ is a crepant primitive birational contraction of type II when $\dim(E) = 2$ and $\dim \phi(E) = 0$.
(iii) ϕ is a crepant primitive birational contraction of type III when $\dim(E) = 2$ and $\dim \phi(E) = 1$.

Theorem 2.2. (Cf.[Pa],[Mi1]) *Let X be a smooth weak Fano 3-fold, and $\phi : X \to \bar{X}$ a crepant primitive birational contraction.*

(1) If ϕ is of type I, then any deformation of ϕ, ϕ_t is of type I. In particular ϕ_t is not an isomorphism.
(2) If ϕ is of type II, then any deformation of ϕ, ϕ_t is of type II. In particular ϕ_t is not an isomorphism.
(3) If ϕ is of type III, then any deformation of ϕ, ϕ_t is of type I or type III unless ϕ is of type (III,0,2), that is a contraction which contracts a divisor E to a curve $C \subset \bar{X}$ such that
   (1) $C \simeq \mathbb{P}^1$
   (2) $\phi|_E : E \to C$ is a $\mathbb{P}^1$-bundle structure
   (3) $(-K_{\bar{X}} \cdot C) = 2$

Moreover, If ϕ is of type (III,0,2), then there exists a deformation of ϕ such that ϕ_t is an isomorphism for any $t \in (\Delta, 0) \setminus \{0\}$.

Theorem 2.3. *Let X be a weak Fano 3-fold which is not a Fano 3-fold. X is weakened Fano 3-fold if and only if every primitive crepant contraction is of type (III,0,2).*

Let X be a weakened Fano 3-fold, and $\phi_i : X \to X_i$ be a primitive crepant birational contraction , that is of type (III,0,2). ϕ_i contracts a divisor E_i to a curve $C_i \subset \bar{X}$. Let f_i be a fiber of $\phi_i|_{E_i}$. Then we have an automorphism $r_i : H^2(X, \mathbb{R}) \to H^2(X, \mathbb{R})$ defined by $L \to L + (L \cdot f_i)E_i$ which is a reflection. Let $\mathfrak{f}\colon \mathscr{X} \to (\Delta, 0)$ be a deformation of X to Fano 3-folds and $Nef(X_t)$ be the nef cone of X_t. We may consider $Nef(X_t) \subset H^2(X, \mathbb{R})$.

Theorem 2.4. ([W2])
$$r_i(Nef(X_t)) \subset Nef(X_t)$$

Let $R = \mathbb{R}_{\geq 0}[l]$ be an extremal ray of X (hence an extremal ray of X_t), and $\psi : X \to Y$ the extremal contraction corresponding to R. Let $R_i = \mathbb{R}_{\geq 0}[r_i(l)]$.

Proposition 2.5. *(1) R_i is an extremal ray of X_t.*
(2) Let $\psi_i : X_t \to Y_t$ the extremal contraction corresponding to R_i. If ψ is the blow-up along a smooth curve, then so is ψ_i and $Exc(\psi_i) = r_i(Exc(\psi))$.

Using this proposition and the final column of classified table in [M-M 1], we know the deformation type of X_t.

Classified Table of Weakened Fano 3-folds

$\delta_X := (-K_X)^3$.

X_t : No. of Fano 3-fold in [M-M 1] which is a deformation of X.

e : Number of primitive crepant contractions.

n : Degree of Hirzebruch surface which is the exceptional divisor of a primitive crepant contraction.

Table.1 weakened Fano 3-folds with $B_2 = 2$

No.	δ_X	X_t	X	e	n
1_p	12	6	$Z := \mathbb{P}(\mathcal{O}_{\mathbb{P}^2}(2) \oplus \Omega^1_{\mathbb{P}^2}(2))$ L_Z: the tautological line bundle. 1) $X \in \lvert 2L_Z \rvert$. 2) $X' \in \lvert L_Z \rvert$, X is a double cover of X' whose branch locus is a member of $\lvert -K_{X'} \rvert$.	1	1) 0 2) 2
2	20	12	$E' \cong \mathbb{F}_0$: smooth quadric surface in $\mathbb{P}^3$. $\Gamma \subset E'$: a smooth curve of bi-degree (2,4). X is the blow-up of $\mathbb{P}^3$ along Γ.	1	0
3	28	21	$E' \cong \mathbb{F}_0$: hyperplane in $Q \subset \mathbb{P}^4$. $\Gamma \subset E'$: a smooth curve of bi-degree (1,3). X is the blow-up of $\mathbb{P}^3$ along Γ.	1	0
4_p	48	32	$Z := \mathbb{P}(\mathcal{O}_{\mathbb{P}^2}(2) \oplus \Omega^1_{\mathbb{P}^2}(2))$ L_Z: the tautological line bundle. $X \in \lvert L_Z \rvert$.	1	1

Table.2 weakened Fano 3-folds with $B_2 = 3$

No.	δ_X	X_t	X	e	n
1_p	12	1	$X' := \mathbb{F}_2 \times \mathbb{P}^1$, X is a double cover of X' whose branch locus is a member of $\lvert -K_{X'} \rvert$.	1	0
2	18	3	s: a section with $s^2 = 2$ on $\mathbb{F}_2$. X is a divisor on $\mathbb{F}_2 \times \mathbb{P}^2$ which is a member of $\lvert p_1^* O_{\mathbb{F}_2}(s) \otimes p_2^* O_{\mathbb{P}^2}(2) \rvert$	1	0
3	24	7	$Y := \mathbb{P}^2 \times \mathbb{P}^1$, $E' \in \lvert p_1^* O_{\mathbb{P}^2}(1) \otimes p_2^* O_{\mathbb{P}^1}(1) \rvert$ which is isomorphic to $\mathbb{F}_1$. s: a section with $s^2 = 1$ on E'. $\Gamma \sim 3s$ on E'. X is the blow-up of Y along Γ.	1	1
4	26	9	$\pi : Y = \mathbb{P}(O_{\mathbb{P}^2} \oplus O_{\mathbb{P}^2}(2)) \to \mathbb{P}^2$. $E' \in \lvert \pi^* O_{\mathbb{P}^2}(2) \rvert$ which is isomorphic to $\mathbb{F}_4$. s: a section with $s^2 = 4$ on E'. $\Gamma \sim 3s$ on E'. X is the blow-up of Y along Γ.	1	4
5	26	10	Y : the blow-up of $Q \subset \mathbb{P}^4$ along a conic on it. E': its exceptional divisor which is isomorphic to $\mathbb{F}_0$. $\Gamma \subset E'$: a smooth curve of bi-degree (1,2). X is the blow-up of Y along Γ.	1	0

6	30	13	$Y \subset \mathbb{P}^2 \times \mathbb{P}^2$: a member of $\lvert p_1^* O_{\mathbb{P}^2}(1) \otimes p_2^* O_{\mathbb{P}^2}(1) \rvert$. $\pi : Y \to \mathbb{P}^2$ the restriction of p_2. $E' \subset Y$: a member of $\lvert \pi^* O_{\mathbb{P}^2}(1) \rvert$ which is isomorphic to $\mathbb{F}_1$. s: a section with $s^2 = 1$ on E'. $\Gamma \sim 2s$ on E'. X is the blow-up of Y along Γ.	1	1
			Y: No.4 of Table 1. $\pi : Y \to \mathbb{P}^2$ be the extremal contraction (uniquely determined). $E' \subset Y$: a member of $\lvert \pi^* O_{\mathbb{P}^2}(1) \rvert$ which is isomorphic to $\mathbb{F}_1$. s: a section with $s^2 = 1$ on E'. $\Gamma \sim 2s$ on E'. X is the blow-up of Y along Γ.	2	1,1
7	36	17	$Y := \mathbb{P}^2 \times \mathbb{P}^1$. $E' \subset Y$: a member of $\lvert p_1^* O_{\mathbb{P}^2}(1) \rvert$ which is isomorphic to $\mathbb{F}_0$. $\Gamma \subset E'$: a smooth curve of bi-degree (2,1). X is the blow-up of Y along Γ.	1	0
8	38	19	Y: the blow-up of $Q \subset \mathbb{P}^4$ with center a point $p \in Q$. E' its exceptional divisor. $q \in E'$ a point which does not lies on the conic which parameterize lines in Q which through p. X is the blow-up of Y with center q.	1	1
9	38	20	Y: the blow-up of $Q \subset \mathbb{P}^4$ along a line on it. E': its exceptional divisor which is isomorphic to $\mathbb{F}_1$. s: a section with $s^2 = 1$ on E'. $\Gamma \sim s$ on E'. X is the blow-up of Y along Γ.	1	1
10	44	25	Y: the blow-up of $\mathbb{P}^3$ along a line on it. E': its exceptional divisor which is isomorphic to $\mathbb{F}_0$. $\Gamma \subset E'$: a smooth curve of bi-degree (1,1). X is the blow-up of Y along Γ.	1	0
11_p	48	27	$\mathbb{F}_2 \times \mathbb{P}^1$	1	0
12_p	52	31	Q': the singular quadric 3-fold $xy = z^2$ in $\mathbb{P}^4$. Y: the blow-up of Q' along its singular locus l, which is a crepant resolution of Q'. E': its exceptional divisor which is isomorphic to $\mathbb{F}_0$. Γ: a fiber of $E' \to l$. X is the blow-up of Y along Γ.	1	0

Table.3 weakened Fano 3-folds with $B_2 = 4$

No.	δ_X	X_t	X	e	n
1	24	1	s: a section with $s^2 = 2$ on $\mathbb{F}_2$. X is a divisor on $\mathbb{F}_2 \times \mathbb{P}^1 \times \mathbb{P}^1$ which is a member of $\lvert p_1^* O_{\mathbb{F}_2}(s) \otimes p_2^* O_{\mathbb{P}^1}(1) \otimes p_3^* O_{\mathbb{P}^1}(1) \rvert$	1	0
			t: a fiber of $\mathbb{F}_2 \to \mathbb{P}^1$. $E'_2 \subset \mathbb{F}_2 \times \mathbb{P}^1$: a member of $\lvert p_1^* O_{\mathbb{F}_2}(t) \otimes p_2^* O_{\mathbb{P}^1}(1) \rvert$ which is isomorphic to $\mathbb{F}_2$. h' a section with ${h'}^2 = 2$ on E'_2. $\Gamma \sim 2h'$ on E'. X is the blow-up of $\mathbb{F}_2 \times \mathbb{P}^1$ along Γ.	2	0,2

			$E_2' \subset \mathbb{F}_2 \times \mathbb{P}^1$: a member of $\lvert p_1^* O_{\mathbb{F}_2}(s)\rvert$ which is isomorphic to $\mathbb{F}_0$. $\Gamma \subset E_2'$: a smooth curve of bi-degree (2,2). X is the blow-up of Y along Γ.	2	0,0
2	28	2	$\pi : Y = \mathbb{P}(O_{\mathbb{P}^1\times\mathbb{P}^1} \oplus O_{\mathbb{P}^1\times\mathbb{P}^1}(1,1)) \to \mathbb{P}^1 \times \mathbb{P}^1$. $E' \subset Y$: a member of $\lvert \pi^* O_{\mathbb{P}^1\times\mathbb{P}^1}(1,1)\rvert$ which is isomorphic to $\mathbb{F}_2$. h' a section with ${h'}^2 = 2$ on E_2'. $\Gamma \sim 2h'$ on E'. X is the blow-up of Y along Γ.	1	2
			$p : Y \to Q'$ where Y: No.12 of Table 2 and Q': the singular quadric 3-fold $xy = z^2$ in $\mathbb{P}^4$. $E' \subset Y$: a member of $\lvert p^* O_{Q'}(1)\rvert$ which is isomorphic to $\mathbb{F}_2$. h' a section with ${h'}^2 = 2$ on E_2'. $\Gamma \sim 2h'$ on E'. X is the blow-up of Y along Γ.	2	0,2
3	30	3	s: a section with $s^2 = 1$ on $\mathbb{F}_1$. $Y := \mathbb{P}^1 \times \mathbb{F}_1$. $E' \subset Y$: a member of $\lvert p_2^* O_{\mathbb{F}_1}(s)\rvert$ which is isomorphic to $\mathbb{F}_0$. $\Gamma \subset E'$: a smooth curve of bi-degree (1,2). X is the blow-up of Y along Γ.	1	0
4	32	4	$Z \to Q \subset \mathbb{P}^4$: the blow-up of Q along a conic on it, D_Z': its exceptional divisor, and f_D:its exceptional line. $Y \to Z$: the blow-up of f_D on Z. E': its exceptional divisor, and D_Z: strict transform of D_Z'. $\Gamma := E' \cap D_Z$. X is the blow-up of Y along Γ.	1	1
5	34	6	Y: the blow-up of $\mathbb{P}^3$ along 2 disjoint lines l_1, l_2. E': its exceptional divisor contracted to l_1 which is isomorphic to $\mathbb{F}_0$. $\Gamma \subset E'$: a smooth curve of bi-degree (1,1). X is the blow-up of Y along Γ.	1	0
			Z: the blow-up of $\mathbb{P}^3$ along a line on it. E_1'': its exceptional divisor which is isomorphic to $\mathbb{F}_0$. $\Gamma' \subset E_1''$: a smooth curve of bi-degree (1,1). Y is the blow-up of Z along Γ'. E_2': its exceptional divisor which is isomorphic to $\mathbb{F}_2$, and E_1': strict transform of E_1''. h' a section with ${h'}^2 = 2$ on E_2'. $\Gamma \sim h'$ on E_2'. X is the blow-up of Y along Γ.	2	0,2
6	38	8	s: a section with $s^2 = 2$ on $\mathbb{F}_2$. $Y := \mathbb{P}^1 \times \mathbb{F}_2$. $H \subset Y$: a member of $\lvert p_2^* O_{\mathbb{F}_2}(s)\rvert$ which is isomorphic to $\mathbb{F}_0$. $\Gamma \subset H$: a smooth curve of bi-degree (0,1). X is the blow-up of Y along Γ.	1	0
7	42	10	$\mathbb{P}^1 \times S_7'$ where S_7' is the blow-up of $\mathbb{F}_1$ with center a point p which lies on the (-1)-curve.	1	0

8	46	12	Z: the blow-up of $\mathbb{P}^3$ along a line on it, D'_Z: its exceptional divisor, and f_D: its exceptional line. $Y \to Z$: the blow-up of f_D on Z. E': its exceptional divisor, and D_Z: strict transform of D'_Z. $\Gamma := E' \cap D_Z$. X is the blow-up of Y along Γ.	1	1
9	26	13	s: a section with $s^2 = 2$ on $\mathbb{F}_2$. $Y := \mathbb{P}^1 \times \mathbb{F}_2$. $H \subset Y$: a member of $\lvert p_2^* O_{\mathbb{F}_2}(s)\rvert$ which is isomorphic to $\mathbb{F}_0$. $\Gamma \subset H$: a smooth curve of bi-degree (3,1). X is the blow-up of Y along Γ.	1	0

Table.4 weakened Fano 3-folds with $B_2 = 5$

No.	δ_X	X_t	X	e	n
1	28	1	$Z \to Q \subset \mathbb{P}^4$: the blow-up of Q along a conic on it, D'_Z: its exceptional divisor, and f_1, f_2: 2 exceptional lines. $Y \to Z$: the blow-up of Z along f_1, f_2 . E': the exceptional divisor contracted to f_1, and D_Z: strict transform of D'_Z. $\Gamma := E' \cap D_Z$. X is the blow-up of Y along Γ.	1	1
			$Z \to Q \subset \mathbb{P}^4$: the blow-up of Q along a conic on it, D'_Z: its exceptional divisor, and f_1: its exceptional line. $Y' \to Z$: the blow-up of Z along f_1. E''_1: its exceptional divisor, and D_Z: strict transform of D'_Z. $\Gamma_1 := E''_1 \cap D_Z$. Y is the blow-up of Y' along Γ_1. E'_2: its exceptional divisor, and D: strict transform of D_Z. $\Gamma_2 := E'_2 \cap D$. X is the blow-up of Y along Γ_2.	2	1,1
2	36	2	Z: the blow-up of $\mathbb{P}^3$ along 2 disjoint lines l_1, l_2, D'_1: its exceptional divisor contracted to l_1 and $\Gamma' \subset D'_1$: its exceptional line. Y is the blow-up of Z along Γ'. E': its exceptional divisor, and D_1: strict transform of D'_1. $\Gamma := E' \cap D_1$. X is the blow-up of Y along Γ.	1	1
3	36	3	$\mathbb{P}^1 \times S'_6$ where S'_6 is a weak del Pezzo surface of degree 6, which is not a del Pezzo surface.		0

References

[Ba-Hu] Barth W. and Hulek K., Monads and moduli of vector bundles, Manuscripta Math. 25 (1978)

[Be 1] Beauville A., Variété de Prym et jacobiennes intermédiaires, Ann. Sci. École Norm. Sup. 10 (1977)

[Is 1] Iskovskih V. A., Fano 3-fold I, English translation, Math. USSR Izv. 11 (1977)

[Is 2] Iskovskih V. A., Fano 3-fold II, English translation, Math. USSR Izv. 12 (1978)

[Mi1] Minagawa T., Global smoothing of singular weak Fano 3-folds, J. Math. Soc. Japan 55 (2003)

[Mi2] Minagawa T., On classification of weakened Fano 3-folds with $B_2 = 2$, Proc. of alg. geom. symposium in Kinosaki (2000)

[Mo] Mori S., Threefolds whose canonical bundles are not numerically effective, Ann. of Math. 116 (1982)
[M-M 1] Mori S. and Mukai S., Classification of Fano 3-folds with $B_2 \geq 2$, Manuscripta Math. 36 (1981)
[M-M 2] Mori S. and Mukai S., On Fano 3-folds with $B_2 \geq 2$, Algebraic and Analytic Varieties, Adv. Stud. in Pure Math. 1 (1983)
[Pa] Paoletti, R., The Kähler cone in families of quasi-Fano threefolds, Math. Z. 227 (1998)
[Sa] The classification of smooth toric weakened Fano 3-folds, Manuscripta Math. 109 (2002)
[W1] Wilson, P.M.H., The Kähler cone on Calabi-Yau threefolds, Invent. Math. 107 (1992)
[W2] Wilson, P.M.H., Elliptic ruled surfaces on Calabi-Yau threefolds, Math. Proc. Comb. Phil. Soc. 112 (1992)

Department of Mathematics, Tokyo Institute of Technology , Oh-okayama, Meguro-ku, Tokyo 153-8551, Japan
E-mail address: minagawa@math.titech.ac.jp