Infinitesimal deformation of Galois covering space and its application to Galois closure curves

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Abstract. In the article [1] the author developed a general framework of the infinitesimal deformations of finite branched Galois covering spaces of complex dimension one. By using the framework, he discussed the correspondence between the infinitesimal deformations of a branched covering map and those of the Galois closure curve of the map in the article. In particular, he computed Kodaira-Spencer maps for families of Galois closure curves constructed on base spaces which consist of branched covering maps. In the presentation he explained the outline of the research.

Galois closure curve

We discuss compact complex manifolds of dimension one, in other words, we discuss curves. Let X, Y be two curves and

$$Y \xrightarrow{\alpha} X \tag{1}$$

a holomorphic branched covering map from the curve Y to the curve X. The branched covering map α induces the following finite extension of fields via pull-back of functions

$$k(X) \overset{\alpha^*}{\hookrightarrow} k(Y), \tag{2}$$

where $k(X)$ (resp. $k(Y)$) denotes the field of all rational functions on X (resp. Y). For a given branched covering map α, the extension (2) may not necessarily be a Galois extension. In the case that the extension (2) is not a Galois extension, we can find uniquely up to isomorphism a curve Z_α and a holomorphic branched covering map

$$Z_\alpha \xrightarrow{\beta} Y \tag{3}$$

which give the Galois closure

$$k(X) \overset{\alpha^*}{\hookrightarrow} k(Y) \overset{\beta^*}{\hookrightarrow} k(Z_\alpha) \tag{4}$$

of the extension (2). The curve Z_α above is called the *Galois closure curve* of the extension (2). Let G denote the Galois group of the extension $k(Z_\alpha)/k(X)$ and $H \subset G$ that of $k(Z_\alpha)/k(Y)$. Then the group G acts on the curve Z_α and the curves X, Y can be written as

$$X = Z_\alpha/G, \quad Y = Z_\alpha/H. \tag{5}$$

The projection $Z_\alpha \xrightarrow{\pi} X$ (resp. $Z_\alpha \xrightarrow{\beta} Y$) is a Galois covering space with Galois group G (resp. H)

We have interests in describing how the moduli of the complex structure of the Galois closure curve Z_α varies as we deform the branched covering map (1). In the article [1] the author developed a general framework of infinitesimal deformation of a Galois covering space of curves and, by using it, he discussed the correspondence between the infinitesimal deformations of branched covering maps and the infinitesimal deformations of Galois closure curves.

Automorphism group

We denote the group of all holomorphic automorphisms of X (resp. Y) by $\mathrm{Aut}(X)$ (resp. $\mathrm{Aut}(Y)$). The groups $\mathrm{Aut}(X), \mathrm{Aut}(Y)$ have finite dimensional complex Lie group structures. An element $(\sigma, \xi) \in \mathrm{Aut}(X) \times \mathrm{Aut}(Y)$ acts on α via compositions of maps

$$\alpha \to \sigma \circ \alpha \circ \xi^{-1}. \tag{6}$$

Since the extension of fields $k(X) \hookrightarrow k(Y)$ induced by an element α and that induced by an element $\sigma \circ \alpha \circ \xi^{-1}$ are equivalent as field extensions, they define the same Galois closure curve.

Infinitesimal deformation of maps

For the purpose to discuss the correspondence between infinitesimal deformation of maps and that of Galois closure curves, we sketch the infinitesimal deformations of maps and the infinitesimal actions of the automorphism groups.

Definition 1. For a holomorphic map α, we call a section of the pull-back bundle $\alpha^*(TX)$

$$s \in H^0(Y, \mathcal{O}(\alpha^*(TX))) \tag{7}$$

an *infinitesimal deformation* of the map α.

The actions of the automorphism groups $\mathrm{Aut}(X), \mathrm{Aut}(Y)$ on the map α induce vector space homomorphisms from the Lie algebras of those groups

$$\mathrm{Lie}(\mathrm{Aut}(X)) \cong H^0(X, \mathcal{O}(TX)), \quad \mathrm{Lie}(\mathrm{Aut}(Y)) \cong H^0(Y, \mathcal{O}(TY)) \tag{8}$$

to the vector space $H^0(Y, \alpha^*(TX))$. Each homomorphism is written as follows: The former is the pull-back of the holomorphic sections

$$\mathcal{P} : \ H^0(X, \mathcal{O}(TX)) \xrightarrow{\text{pull-back}} H^0(Y, \mathcal{O}(\alpha^*(TX))). \tag{9}$$

And the latter is the homomorphism of cohomology groups

$$\mathcal{Q} : \ H^0(Y, \mathcal{O}(TY)) \xrightarrow{d\alpha} H^0(Y, \mathcal{O}(\alpha^*(TX))), \tag{10}$$

induced by the homomorphism of sheaves

$$\mathcal{O}(TY) \xrightarrow{d\alpha} \mathcal{O}(\alpha^*(TX)). \tag{11}$$

It is clear that an infinitesimal deformation written as an element of $(\operatorname{Im}\mathcal{P} + \operatorname{Im}\mathcal{Q})$ does not change the moduli of the Galois closure curve. The quotient vector space

$$H^0(Y, \mathcal{O}(\alpha^*(TX))) / (\operatorname{Im}\mathcal{P} + \operatorname{Im}\mathcal{Q}) \tag{12}$$

has fundamental importance for our infinitesimal deformation theory.

Results

In the article [1] the author derived the following theorem for a holomorphic map of *general ramification type*, which is defined below.

Definition 2. A branched covering map α is of *general ramification type* if all the ramification points $p_j \in Y$ of the map α are of order two, and there exist no p_i, p_j $(i \neq j)$ satisfying $\alpha(p_i) = \alpha\left(p_j\right)$.

Theorem 1. *If the branched covering map $Y \overset{\alpha}{\to} X$ is of general ramification type, the following sequence is exact:*

$$0 \to H^0(Y, \mathcal{O}(\alpha^*(TX))) \;/\; (\operatorname{Im}\mathcal{P} + \operatorname{Im}\mathcal{Q}) \overset{\text{K-S}}{\to} H^1(Z_\alpha, \mathcal{O}(TZ_\alpha))_G\,, \tag{13}$$

where the homomorphism K-S describes how the infinitesimal deformations of map α cause the infinitesimal change of the moduli of the Galois closure curve Z_α.

Furthermore, if the curve X is the projective space $\mathbf{P}^1$, the exact sequence extends to the following one:

$$0 \to H^0\left(Y, \mathcal{O}\left(\alpha^*\left(T\mathbf{P}^1\right)\right)\right) / (\operatorname{Im}\mathcal{P} + \operatorname{Im}\mathcal{Q}) \overset{\text{K-S}}{\to} H^1(Z_\alpha, \mathcal{O}(TZ_\alpha))_G \overset{\mathcal{I}_Y}{\to} H^1(Y, \mathcal{O}(TY)). \tag{14}$$

Remark 1. We should note that, for a branched covering map α which is *not* of general ramification type, the homomorphism K-S may not be defined on the whole vector space (12).

Example 1. Let X be the projective space $\mathbf{P}^1$ and Y an elliptic curve. The curve Y can be embedded as a degree three smooth projective plane curve $Y \overset{i}{\hookrightarrow} \mathbf{P}^2$. And, at that time, we can correspond each point $p \in \mathbf{P}^2 \setminus Y$ to the linear projection $Y \overset{\alpha_p}{\to} \mathbf{P}^1$ with center $p \in \mathbf{P}^2 \setminus Y$. The mapping degree of these maps α_p is three. Then a translation of the center of the linear projection $p \in \mathbf{P}^2 \setminus Y$ induces the following infinitesimal deformation of the map α_p.

$$T_p\mathbf{P}^2 \xrightarrow{\cong} H^0\left(Y, \mathcal{O}\left(\alpha_p^*\left(T\mathbf{P}^1\right)\right)\right) / (\operatorname{Im}\mathcal{P} + \operatorname{Im}\mathcal{Q}). \tag{15}$$

Elementary computations show that the embedding $Y \overset{i}{\hookrightarrow} \mathbf{P}^2$ has nine flex points, all of which are 1-flex, and that it has corresponding nine multi-tangent lines. (We denote the union of these nine multi-tangent lines by $Y^\sharp$.) For a point $p \in \mathbf{P}^2 \setminus \left(Y \cup Y^\sharp\right)$, the map α_p is of general ramification type and has six ramification points of order two. The genus of the Galois closure curve of such a map α_p, denoted by Z_p, is four. And the Galois group G is the symmetric group of degree three. Then, under the identification (15) we obtain the exact sequence

$$0 \longrightarrow T_p\mathbf{P}^2 \xrightarrow{\text{K-S}} H^1\left(Z_p, \mathcal{O}\left(TZ_p\right)\right)_G \xrightarrow{\mathcal{J}_Y} H^1\left(Y, \mathcal{O}\left(TY\right)\right), \tag{16}$$

by using Theorem 1. The dimensions of the vector spaces above are as follows:

$$\dim T_p\mathbf{P}^2 = 2, \quad \dim H^1\left(Z_p, \mathcal{O}\left(TZ_p\right)\right)_G = 3, \quad \dim H^1\left(Y, \mathcal{O}\left(TY\right)\right) = 1. \tag{17}$$

As a matter of fact, we can construct a fiber space

$$\phi : F \to \mathbf{P}^2, \tag{18}$$

the fiber $\phi^{-1}(p)$ of which is isomorphic to the Galois closure curve Z_p for every point $p \in \mathbf{P}^2 \setminus \left(Y \cup Y^\sharp\right)$. Then, for every point $p \in \mathbf{P}^2 \setminus \left(Y \cup Y^\sharp\right)$, the homomorphism $\mathrm{K-S}$ is the *Kodaira-Spencer map* of the family of the Galois closure curves.

0.1 Outline of proof

In the article [1], the author developed a general framework of the infinitesimal deformations of finite branched Galois covering spaces of complex dimension one. And, by using the framework, he discussed in the article the correspondence between the infinitesimal deformations of a branched covering map and those of the Galois closure curve of the map to derive Theorem 1. We sketch here an outline of the discussion.

For the purpose to answer the question of our primary concern— how the complex structure of the Galois closure curve Z_α varies as we deform the branched covering map α— which we stated above, we investigate the "inverse problem" of this, which we describe below.

For a branched covering map $Y \xrightarrow{\alpha} X$, we construct uniquely up to isomorphism the Galois closure of the branched covering map α, which consists of a Galois covering space $Z \xrightarrow{\pi} X$, the Galois group of which is a finite group G and a Galois covering space $Z \xrightarrow{\beta} Y$, the Galois group of which is a subgroup $H \subset G$, satisfying $\alpha \circ \beta = \pi$. Then the curves X, Y are written as

$$X = Z/G, \quad Y = Z/H. \tag{19}$$

The inverse problem which we mentioned above is as follow. Let Z be a curve endowed with a finite group G-action so that the quotient map

$$\pi : \ Z \to X := Z/G \tag{20}$$

should be a Galois covering space with Galois group G. Then, for an arbitrary subgroup $H \subset G$, the quotient map

$$\beta : \ Z \to Y := Z/H \tag{21}$$

is a Galois covering space with Galois group H. Under the situation above, we can construct a branched covering map $Y \xrightarrow{\alpha} X$ by setting

$$\alpha \circ \beta = \pi. \tag{22}$$

(Note that the Galois covering spaces $Z \xrightarrow{\pi} X$ and $Z \xrightarrow{\beta} Y$ are not supposed to be the Galois closure of the branched covering map α.) The problem is how the branched covering map α varies as we deform the complex structure of the curve Z in a G-invariant way in which the moduli of the curves X, Y should not be changed.

For the purpose to solve the problem, we proceed as follows. Under the identification

$$X = Z/G, \quad Y = Z/H, \tag{23}$$

we observe the phenomena which occurs as we deform the complex structure of the curve Z in a G-invariant way. We denote the almost complex structures of the curves X, Y, Z by J_X, J_Y, J_Z respectively. Let $J_Z(t)$ be a smooth family of G-invariant almost complex structures on the curve Z depending on a real parameter $t \in (-\varepsilon, \varepsilon)$ satisfying the following Condition Σ:

> (Condition Σ)
>
> The initial value satisfies $J_Z(0) = J_Z$ and $J_Z(t)$ identically equals the original almost complex structures J_Z on some neighborhood of the ramification points $r_1, r_2, \ldots, r_N \in Z$ of the Galois covering space $Z \xrightarrow{\pi} X$.

Since every almost complex structure is integrable on a complex manifold of dimension one, and therefore, since it defines a complex structure of the manifold, the smooth family of G-invariant almost complex structures $J_Z(t)$ above defines a *deformation of the complex structure* of the curve Z. We denote the curve Z with the almost complex structure $J_Z(t)$ by $Z(t)$.

For a point $p \in Z$ which is not a ramification point of the Galois covering space $Z \xrightarrow{\pi} X$, the homomorphisms of tangent spaces below are isomorphisms:

$$d\pi : \ T_pZ \xrightarrow{\cong} T_{\pi(p)}X, \tag{24}$$

$$d\beta : \ T_pZ \xrightarrow{\cong} T_{\beta(p)}Y. \tag{25}$$

Therefore we can regard $J_Z(t)$ also as a smooth family of almost complex structures $J_X(t)$ on the curve X by setting

$$J_X(t) := d\pi \circ J_Z(t) \circ (d\pi)^{-1} \quad \text{on } T_{\pi(p)}X. \tag{26}$$

Since $J_X(t)$ defined above identically equals the original almost complex structure J_X on some neighborhood of the branch points $b_i \in X$ of the Galois covering space $Z \xrightarrow{\pi} X$ by the property of Condition Σ, it defines a smooth family of almost complex structures on the curve X. The similar argument shows that we can regard $J_Z(t)$ also as a smooth family of almost complex structures $J_Y(t)$ on the curve Y by setting

$$J_Y(t) := d\beta \circ J_Z(t) \circ (d\beta)^{-1} \quad \text{on } T_{\beta(p)}Y. \tag{27}$$

In this way a G-invariant deformation of the complex structure of the curve Z satisfying Condition Σ induces deformations of those of the curves X, Y.

Next we set

$$\eta_Z := \frac{d}{dt} J_Z(t)\Big|_{t=0} \tag{28}$$

and we can regard η_Z as an G-invariant Dolbaut form $\eta_Z \in \Lambda^{0,1}(Z, TZ)_G$, and we see that it satisfies the following Condition $\tilde{\Sigma}$:

(Condition $\tilde{\Sigma}$)

η_Z vanishes on some neighborhood of the ramification points $r_1, r_2, \ldots, r_N \in Z$ of the Galois covering space $Z \xrightarrow{\pi} X$.

Needless to say, the homomorphisms of cotangent space are also isomorphisms for a point $p \in Z$ which is not a ramification point,

$$d\pi^* : T^*_{\pi(p)}X \xrightarrow{\cong} T^*_p Z, \tag{29}$$

$$d\beta^* : T^*_{\beta(p)}Y \xrightarrow{\cong} T^*_p Z. \tag{30}$$

Therefore we can apply a similar argument as with $J_Z(t)$ to the Dolbaut form η_Z satisfying Condition $\tilde{\Sigma}$, to see that we can regard η_Z also as an element of $\eta_X \in \Lambda^{0,1}(X, TX)$, and as an element of $\eta_Y \in \Lambda^{0,1}(X, TY)$ by using the isomorphisms of the tangent spaces and those of cotangent spaces. Then it is not difficult to check that the following identities hold:

$$\frac{d}{dt} J_X(t)\Big|_{t=0} = \eta_X, \quad \frac{d}{dt} J_Y(t)\Big|_{t=0} = \eta_Y. \tag{31}$$

We should note that, for any $\eta_Z \in \Lambda^{0,1}(Z, TZ)_G$ satisfying Condition $\tilde{\Sigma}$, we can construct a smooth family of G-invariant almost complex structures $J_Z(t)$ depending a real parameter $t \in (-\varepsilon, \varepsilon)$ satisfying Condition Σ. Furthermore, as is shown in the article [1], that, for any class $a \in H^1(Z, \mathcal{O}(TZ))_G$, we can find a G-invariant Dolbaut form $\eta_Z \in \Lambda^{0,1}(Z, TZ)_G$ which satisfies

$$[\eta_Z] = a \tag{32}$$

and which satisfies Condition $\tilde{\Sigma}$.

As we see in the article [1], we can construct a homomorphism of cohomology groups

$$H^1(Z, \mathcal{O}(TZ))_G \xrightarrow{\mathcal{I}_X} H^1(X, \mathcal{O}(TX)) \tag{33}$$

so that the following diagram should commute:

$$\begin{array}{ccc} H^1(Z, \mathcal{O}(TZ))_G & \xrightarrow{\mathcal{I}_X} & H^1(X, \mathcal{O}(TX)) \\ \uparrow & & \uparrow \\ \eta_Z \in \Lambda^{0,1}(Z, TZ)_G & \longrightarrow & \eta_X \in \Lambda^{0,1}(X, TX) \end{array} \tag{34}$$

The diagram describes how the complex structure of the curve X varies via the correspondence $\eta_Z \to \eta_X$ which we defined above. And in a similar way, we can construct a homomorphism of cohomology groups

$$H^1(Z, \mathcal{O}(TZ))_G \xrightarrow{\mathfrak{I}_Y} H^1(Y, \mathcal{O}(TY)) \tag{35}$$

so that the following diagram should commute:

$$\begin{array}{ccc} H^1(Z, \mathcal{O}(TZ))_G & \xrightarrow{\mathfrak{I}_Y} & H^1(Y, \mathcal{O}(TY)) \\ \uparrow & & \uparrow \\ \eta_Z \in \Lambda^{0,1}(Z, TZ)_G & \longrightarrow & \eta_Y \in \Lambda^{0,1}(Y, TY) \end{array} \tag{36}$$

The diagram describes how the complex structure of the curve Y varies via the correspondence $\eta_Z \to \eta_Y$ which we defined above.

For the purpose to obtain the infinitesimal deformations of the complex structure of Z which do not change the moduli of X, Y, we observe the kernels of the homomorphisms (33) and (35). We set subspaces $K_X \subset H^1(Z, \mathcal{O}(TZ))_G$ and $K_Y \subset H^1(Z, \mathcal{O}(TZ))_G$ as follows:

$$K_X := \mathrm{Ker}\left(H^1(Z, \mathcal{O}(TZ))_G \xrightarrow{\mathfrak{I}_X} H^1(X, \mathcal{O}(TX))\right), \tag{37}$$

$$K_Y := \mathrm{Ker}\left(H^1(Z, \mathcal{O}(TZ))_G \xrightarrow{\mathfrak{I}_Y} H^1(Y, \mathcal{O}(TY))\right). \tag{38}$$

Under the definition abovethe subspace

$$K_X \cap K_Y \subset H^1(Z, \mathcal{O}(TZ))_G \tag{39}$$

corresponds to the infinitesimal deformations of the branched covering map $Y \xrightarrow{\alpha} X$ as we see below.

Since the smooth family of almost complex structures $J_X(t)$ of the curve X equivalent as the original almost complex structure J_X there exists a smooth family of C^∞-automorphisms $f_t: X \to X$ which satisfies

$$J_X(t) = (f_t)_*^{-1} \circ J_X \circ (f_t)_* . \tag{40}$$

And the similar argument shows that there exists a smooth family of C^∞-automorphisms g_t : $Y \to Y$ which satisfies

$$J_Y(t) = (g_t)_*^{-1} \circ J_Y \circ (g_t)_* . \tag{41}$$

Under the situation above, for a given $a \in K_X \cap K_Y$, we can find $J_Z(t)$, a smooth family of G-invariant almost complex structures on the curve Z depending on a real parameter $t \in (-\varepsilon, \varepsilon)$ satisfying Condition Σ and satisfying that the cohomology class of the differential

$$\eta_Z := \left.\frac{d}{dt} J(t)\right|_{t=0} \tag{42}$$

coincides with $a \in K_X \cap K_Y$, and we can construct a smooth family of holomorphic Galois covering spaces $\pi_t :\ Z(t) \to X$ with Galois group G by setting

$$\pi_t := f_t \circ \pi, \tag{43}$$

and we can construct a smooth family of holomorphic Galois covering spaces $\beta_t :\ Z(t) \to Y$ with Galois group H, by setting

$$\beta_t := g_t \circ \beta. \tag{44}$$

Note that the complex structures of the curves X, Y are unchanged above.

Then the composite of maps

$$\alpha_t := \pi_t \circ \beta_t^{-1} \quad \text{(well-defined)} \tag{45}$$

is a smooth family of branched covering maps $Y \xrightarrow{\alpha_t} X$ with initial value $\alpha_0 = \alpha$.

The minus of the differential of the family α_t at $t = 0$

$$u := - \frac{d}{dt}\bigg|_{t=0} \alpha_t \tag{46}$$

defines a holomorphic section of the pull-back bundle $\alpha^* (TX)$. Namely

$$u \in H^0 (Y, \mathcal{O} (\alpha^* (TX))) . \tag{47}$$

Then the equivalence class of u

$$[u] \in H^0 (Y, \mathcal{O} (\alpha^* (TX))) \ / \ (\mathrm{Im}\, \mathcal{P} + \mathrm{Im}\, \mathcal{Q}) \tag{48}$$

defines a equivalence class of infinitesimal deformations of the branched covering map α.

As we state in the following Proposition 1, which is the main result in the article [1], that the equivalence class (48) does not depend on the particular choice of the families $J_Z(t), \pi_t, \beta_t$, but only depends on the element

$$a \in K_X \cap K_Y \tag{49}$$

which we took first.

Proposition 1. *We can define a homomorphism*

$$K_X \cap K_Y \to H^0 (Y, \mathcal{O} (\alpha^* (TX))) \ / \ (\mathrm{Im}\, \mathcal{P} + \mathrm{Im}\, \mathcal{Q}) \tag{50}$$

so that the following diagram should commute:

$$\begin{array}{ccc} K_X \cap K_Y & \longrightarrow & H^0 (Y, \mathcal{O} (\alpha^* (TX))) \ / \ (\mathrm{Im}\, \mathcal{P} + \mathrm{Im}\, \mathcal{Q}) \\ \uparrow & & \uparrow \\ \eta_Z \in \Lambda^{0,1} (Z, TZ)_G & \longrightarrow & u \in H^0 (Y, \mathcal{O} (\alpha^* (TX))) \end{array} \tag{51}$$

Furthermore, if the Galois covering spaces $Z \xrightarrow{\pi} X, Z \xrightarrow{\beta} Y$ *are the Galois closure of the branched covering map* $Y \xrightarrow{\alpha} X$*, the homomorphism* (50) *above is injective*

Moreover, under the assumptions above, if the branched covering map $Y \xrightarrow{\alpha} X$ *is of general ramification type, the injective homomorphism* (50) *is surjective and, consequently, it is an isomorphism*

We see that Proposition 1 can do solve "the inverse problem"—how the branched covering map α varies as we deform the complex structure of the curve Z in a G-invariant way in which the moduli of the curves X, Y should not be changed—which we mentioned above. Then, by taking the inverse mapping of homomorphism (50) in Proposition 1, we obtain the following corollary:

Corollary 1. *If the branched covering map* $Y \xrightarrow{\alpha} X$ *is of general ramification type, the Kodaira-Spencer map of our primary concern is the minus of the inverse map the homomorphism* (50)

$$H^0 (Y, \mathcal{O} (\alpha^* (TX))) \ / \ (\mathrm{Im}\, \mathcal{P} + \mathrm{Im}\, \mathcal{Q}) \xrightarrow{\cong} K_X \cap K_Y \subset H^1 (Z, \mathcal{O} (TZ))_G \tag{52}$$

which is an isomorphism. In particular the Kodaira-Spencer map is injective.

Just as in Example 1, we often investigate cases where the curve $X = \mathbf{P}^1$. In those cases, we have

$$H^1 \left(\mathbf{P}^1, \mathcal{O} \left(T\mathbf{P}^1\right)\right) = 0, \tag{53}$$

therefore we finally obtain Theorem 1 by using Corollary 1.

Remark 2. As we noted in Remark 1 that, for a branched covering map α which is *not* of general ramification type, the corresponding homomorphism (50) may not necessarily be surjective. Therefore, in those cases, the inverse map of the homomorphism (50) may not be defined on the whole vector space

$$H^0 \left(Y, \mathcal{O} \left(\alpha^* \left(T\mathbf{P}^1\right)\right)\right) / \left(\mathrm{Im}\, \mathcal{P} + \mathrm{Im}\, \mathcal{Q}\right). \tag{54}$$

In the case we discussed in Example 1, the linear projections with center $p \in Y^\sharp$ are not of general ramification type. The genera of the Galois closure curves for those maps are strictly less than four. Although the identification

$$T_p\mathbf{P}^2 \cong H^0 \left(Y, \mathcal{O} \left(\alpha_p^* \left(T\mathbf{P}^1\right)\right)\right) / \left(\mathrm{Im}\, \mathcal{P} + \mathrm{Im}\, \mathcal{Q}\right) \tag{55}$$

still holds for such points $p \in Y^\sharp$, the Kodaira-Spencer map

$$T_p\mathbf{P}^2 \to \text{tangent space of moduli} \tag{56}$$

can not be defined.

References

[1] Hiroyuki Sakai, *Infinitesimal deformation of Galois covering space and its application to Galois closure curves*, Nihonkai Math. J. **14(2)** (2003), 133–177.