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Galois Lines for Space Curves

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Abstract. Let C be a curve, and l, l_0 be lines in the projective three space $\mathbb{P}^3$. Consider a projection $\pi_l : \mathbb{P}^3 \cdots \to l_0$ with center l, where $l \cap l_0 = \emptyset$. Restricting π_l to C, we get a morphism $\pi_l|_C : C \to l_0$ and an extension of fields $(\pi_l|_C)^* : k(l_0) \hookrightarrow k(C)$. We study the algebraic structure of the extension and the geometric structure of C. In particular, we study the structure of the Galois group and the number of Galois lines for some special cases.

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1 Introduction

Let k be the ground field of our discussion and assume it to be an algebraically closed field of characteristic zero. Let C be a smooth irreducible non-degenerate curve of degree d in the projective three space $\mathbb{P}^3$, and l be a line in $\mathbb{P}^3$. Consider a projection $\pi_l : \mathbb{P}^3 \cdots \to l_0$ with center l, where l_0 is a line satisfying $l \cap l_0 = \emptyset$. Restricting π_l to C, we get a morphism $\pi_l|_C : C \to l_0$ and an extension of fields $(\pi_l|_C)^* : k(l_0) \hookrightarrow k(C)$. It is easy to see that the structure of this extension does not depend on the choice of l_0, but on l. So we put $K_l = k(l_0)$. Moreover, we put $K = k(C)$ and let L_l be the Galois closure of K/K_l. We study this extension K/K_l from various points of view. To this aim, we make the following definitions:

Definition 1.1. We call $\mathrm{Gal}(L_l/K_l)$ the *Galois group* for l and denote it by G_l. Moreover, we call l a *Galois line* for C if the extension K/K_l is Galois, or equivalently, if π_l induces a Galois covering $\pi_l : C \to l_0$.

If l is a Galois line, then each element $\sigma \in G_l$ induces an automorphism of C over l_0. We denote it by the same letter σ.

Definition 1.2. The automorphism σ is called an *automorphism associated with the Galois line l*.

A line $l \subset \mathbb{P}^3$ is said to be *skew* for C if $l \cap C = \emptyset$.

Definition 1.3. Let C_l be the non-singular projective model of L_l. We call C_l the *Galois closure curve* for (C, l).

Remark 1.4. (1) If $[K : K_l] \leq 2$, then l is always a Galois line. Such lines are said to be trivial Galois lines, and we will not consider them. Indeed, skew Galois lines will be considered mainly.

(2) The Galois closure curve is called the minimal splitting curve in Tokunaga [9].

For the motivation of this research, see [10], where we considered the Galois points for plane curves.

Naturally, the following problems arise:

(I) Find the structure of the Galois closure curve for (C, l) and the structure of G_l.

(II) Find all Galois lines l for C and their arrangements.

(III) Determine the intermediate fields corresponding to subgroups of G_l.

We originate the study of the curves C, lines l and the extensions of fields K/K_l examining these problems. We use the following notation and convention:

$\mathfrak{S}_n$:	the symmetric group of degree n;
$\mathfrak{A}_n$:	the alternating group of degree n;
D_{2m} :	the dihedral group of order $2m$;
Z_n :	the cyclic group of order n;
$V_4 \cong Z_2 \times Z_2$:	the Klein four group;
$\sim$:	the linear equivalence of divisors;
$g = g(C)$:	the genus of C;
$(x_0, \ldots, x_n)$:	homogeneous coordinates on $\mathbb{P}^n$ (when $n = 2$ or $n = 3$, we use (X, Y, Z) or (X, Y, Z, W) instead, respectively);
v_n :	the Veronese map of degree n, $v_n(x, y) = (x^n, x^{n-1}y, \ldots, y^n)$;
$H_n(\alpha, \beta)$:	the hyperplane in $\mathbb{P}^n$ defined by $\sum_{i=0}^{n} \binom{n}{i} \alpha^{n-i}\beta^i x_i = 0$, where $(\alpha, \beta) \in \mathbb{P}^1$;
$\delta(C)$:	the number of skew Galois lines for C;
ζ_n :	a primitive n-th root of 1;
$i(C, H; P)$:	the intersection number of C and a plane H at P;
$\mathrm{div}(f)$:	the divisor of a function f;
$\mathcal{L}(D)$:	$= \{f \in k(V)^* \mid \mathrm{div}(f) + D \geq 0\} \cup \{0\}$;
$\langle a_1, \ldots, a_n \rangle$:	the subgroup of a group generated by the elements $a_1, \ldots, a_n$;
$\mathrm{diag}\,[a_0, \ldots, a_3]$:	the diagonal matrix with components $a_0, \ldots, a_3$.

Definition 1.5. When l is a Galois line for C and $G_l \cong Z_n$ (resp., $V_4, \ldots$), we call l a Z_n*-line* (resp., V_4*-line*, $\ldots$).

2 Statement of Results (General Line Case)

If l is a general line for C, then we will prove that G_l is the full symmetric group. We state this fact in a more definite form. Let $\mathbb{G} = \mathbb{G}(1,3)$ be the Grassmannian parameterizing lines in $\mathbb{P}^3$, which is realized as a quadric hypersurface in $\mathbb{P}^5$. Let Σ be the set of lines l in $\mathbb{P}^3$ such that there exists a plane $H \supset l$ satisfying the following condition (a) or (b):

(a) There exists a point $P \in C$ such that $i(C,H;P) \geq 3$.
(b) There exist two points P_1 and P_2 on C such that $i(C,H;P_r) \geq 2$ $(r = 1,2)$.

Then we prove the following assertion:

Lemma 2.1. *Σ is contained in a divisor of $\mathbb{G}$.*

As a corollary, we obtain the following assertion:

Theorem 2.2. *There exists a divisor Δ on $\mathbb{G}$ satisfying $G_l \cong \mathfrak{S}_d$ if $l \in \mathbb{G} \backslash \Delta$.*

Indeed, recently we have a stronger assertion than this (see [7]). We infer from Theorem 2.2 the following assertions.

Corollary 2.3. *If l is a general line, then the following assertions hold*:

(1) *There exists no field between K and K_l.*
(2) *The genus of the Galois closure curve for (C,l) is $(g+d-3)d!/2+1$.*

Therefore, we may say that the problems (I) and (III) have been solved for general l.

3 Statement of Results (Galois Line Case)

We are interested in special lines, i.e., the lines corresponding to the points in Δ (which is defined in the previous section), especially Galois lines.

First, we notice the following lemma, whose proof is easy.

Lemma 3.1. *Let l be a Galois line for C. If T is a projective transformation of $\mathbb{P}^3$, then $T(l)$ is a Galois line for $T(C)$.*

Throughout this section, we assume that C is *linearly normal*, i.e., the hyperplanes cut out the complete linear series $|\mathcal{O}_C(1)|$.

Theorem 3.2. *For a skew Galois line l, there is a representation $G = G_l \hookrightarrow PGL(3,k)$ and the exact sequence of groups $1 \to G_1 \to G \to G_2 \to 1$, where G_1 is a cyclic group and G_2 is a subgroup of $\mathrm{Aut}(l) \cong PGL(1,k)$.*

If an automorphism σ of C associated with a Galois line extends to an automorphism of $\mathbb{P}^3$, the extension will also be denoted by σ. In this case, it will be shown that $\sigma(l) = l$.

Remark 3.3. (1) For a smooth plane curve, we have studied similar problems and shown that G_P is a cyclic group for a Galois point P (cf. [10]).

(2) If l is not skew or C is not linearly normal, then it does not necessarily hold true that $G_l \subset PGL(3,k)$ (see Remark 6.4 below).

Example 3.4. There are five possibilities for the structure of G_2; i.e., D_{2m}, Z_n, $\mathfrak{A}_4$, $\mathfrak{S}_4$, $\mathfrak{A}_5$. We can find the examples for the first two cases. In fact, let C be the curve defined by the following equations (1), (2) and (3), respectively, where we assume $m \geq 2$:

(1) $ZW = g_2(X,Y)$ and $f_m(X,Y) + Z^m + W^m = 0$, where g_2 and f_m are forms of degrees 2 and m, respectively, and have no common factors, and $f_m^2 - 4g_2^m$ has no multiple factors;

(2) $Z^2 = XY$ and $(X^{m-1} + Y^{m-1})Z + W^m = 0$;

(3) $W^2 = XZ$, $(X^{m-1} + Y^{m-1})W + Z^m = 0$ and $X^m + XY^{m-1} + Z^{m-1}W = 0$.

Then the line $l : X = Y = 0$ is a Galois line for each curve and $G_l \cong D_{2m}$, Z_{2m} and Z_{2m-1} corresponding to (1), (2) and (3), respectively. Moreover, their genera are $(m-1)^2$, $(m-1)^2$ and $(m-1)(m-2)$, respectively. Therefore, they are extremal curves (for the definition, see [1]). In particular, the curves (1) and (2) are complete intersections of quadrics and surfaces of degree m (cf. [1, p. 119]), hence they are projectively normal (cf. [1, p. 141]). We will prove that the curve (3) is linearly normal.

Next we study the number of Galois lines for C. The first result is the following:

Theorem 3.5. *If $g = g(C) \geq 1$, then there exist finitely many skew Galois lines for C.*

Remark 3.6. As we will see in Proposition 4.1 below, for a rational normal curve, the Galois lines form a two-dimensional locally closed subvariety of $\mathbb{G}$.

How many Galois lines are there, and how are they arranged? Do there exist rules for the arrangements as for Galois points on quartic surfaces (cf. [11])? Generally, it seems quite difficult to determine them (cf. Theorem 4.3 and Remark 6.2 below); however, in the case where d is a prime number, they are simple.

Theorem 3.7. *If $d \geq 5$ is a prime number, then the number of skew Galois lines is at most one.*

Remark 3.8. Under the assumption of Theorem 3.7, let $G_l = \langle \sigma \rangle$ and P be a point on l satisfying $\sigma(P) = P$. Let $\pi_P : \mathbb{P}^3 \cdots \to \mathbb{P}^2$ be a projection with center P. Then the plane curve $\overline{C} = \pi_P(C)$ has an automorphism $\overline{\sigma}$ associated with the outer Galois point $\pi_P(l)$. From this, we can get the defining equation of $\overline{C}$. In fact, by taking suitable coordinates, we can express the defining equation as $h_d(X,Y) + Z^d = 0$, where h_d is a form of degree d and $(0,0,1)$ is the Galois point. Using this equation, we will be able to recover the curve C (see Theorem 4.4 below).

4 Lower Degree Curves

We hope that we can find defining equations for all curves C with Galois lines, and moreover, we can find all the Galois lines. However, if d is not small, it is very hard. Let us examine some cases for $d \leq 6$ hereafter.

[I] ($d = 3$). We consider the case where $d = 3$. Then C is a twisted cubic and can be expressed as the image of the Veronese map $v_3 : \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$, where $v_3(x_0, x_1) = ({x_0}^3 : {x_0}^2 x_1 : x_0 {x_1}^2 : {x_1}^3)$ for $(x_0 : x_1) \in \mathbb{P}^1$. Defining the line $l_{(\alpha\beta)(\alpha'\beta')} = H_3(\alpha, \beta) \cap H_3(\alpha', \beta')$, where $\alpha\beta' - \alpha'\beta \neq 0$, we have the following:

Proposition 4.1. *Suppose that $d = 3$ and C is defined as above. Then each Galois line for C is given by $l_{(\alpha\beta)(\alpha'\beta')}$. Therefore, all the Galois lines form a two-dimensional locally closed subvariety V of $\mathbb{G}$. To be more precise, using the Plücker coordinates, the defining equations of V are given by $9x_{01}x_{23} = {x_{12}}^2$, $9x_{02}x_{13} = x_{12}(9x_{03} + x_{12})$ and $x_{01}{x_{13}}^2 = x_{23}{x_{02}}^2$, where $3x_{03} \neq x_{12}$.*

Of course, except for these lines, we have $G_l \cong \mathfrak{S}_3$.

[II] ($d = 4$). We consider the case where $d = 4$. We have obtained the results for this case (see [3]). For the completeness, we mention them here. Since C is linearly normal, we have $g(C) = 1$. Let E be a Weierstrass model for C, i.e., $y^2 = 4x^3 + px + q = 4(x - e_1)(x - e_2)(x - e_3)$, where we put $x = X/Z$ and $y = Y/Z$.

Definition 4.2. Suppose that C is the curve above. Then the space curve defined by $Z^2 = XY$ and $W^2 = 4YZ + pXZ + qX^2$, where $p^3 + 27q^2 \neq 0$, is called the *standard form* of C. We denote it by C_s.

Theorem 4.3.

(1) *Each curve C with the Weierstrass canonical form E is projectively equivalent to the standard form C_s.*

(2) *There exist just three V_4-lines for each C_s, which are given by the equations $Y + c_i X = 0$ and $Z - e_i X = 0$, where $i = 1, 2, 3$ and $c_i = e_j e_k + {e_i}^2$ satisfying $(i - j)(j - k)(k - i) \neq 0$. Thus, each curve C has just three V_4-lines, which are obtained from the projective transformation of the V_4-lines for C_s.*

(3) *The curve C has Z_4-lines if and only if the J-invariant of E is one, i.e., $p = -1$ and $q = 0$ in the standard form. Moreover, C has four Z_4-lines.*

[III] ($d = 5$). We consider the case where $d = 5$. Since C is linearly normal, we have $g(C) = 2$. Using this fact and Remark 3.8, we obtain the defining equation of $\overline{C}$ as:

(1) $XY^3(Y - \alpha X) + Z^5 = 0$ or

(2) $X^2Y^2(Y - \alpha X) + Z^5 = 0$,

where $\alpha \neq 0$.

We note here that these curves are birationally equivalent to each other. Indeed, the curve (1) is transformed into (2) by the mapping

$$(X, Y, Z) \mapsto (X^2Y^3, -\alpha Z^5, -\alpha XYZ^3).$$

By using those equations, we obtain the following:

Theorem 4.4. *Suppose that $d = 5$ and C has a skew Galois line l. Then there exists another line l' such that the reducible curve $C \cup l'$ can be given by equations*

(1) *$Z^2 - YW = 0$ and $XY(Y - \alpha X) + ZW^2 = 0$, or*

(2) $Z^3 - XYW = 0$ *and* $(Y - \alpha X)Z + W^2 = 0$,
where l' *is given by* $Y = Z = 0$ *or* $Z = W = 0$, *respectively. And the Galois line for each curve is given by* $X = Y = 0$. *Moreover, the defining ideal of each curve is*

$$\left(Z^2 - YW,\, XY(Y - \alpha X) + ZW^2,\, XZ(Y - \alpha X) + W^3\right)$$

or

$$\left(Z^3 - XYW,\, (Y - \alpha X)Z + W^2,\, Z^2W + XY(Y - \alpha X)\right).$$

[IV] ($d = 6$). We consider the case where $d = 6$. We assume that C is not hyperelliptic, so it is a canonical curve of genus 4.

Theorem 4.5. *Suppose that* $d = 6$ *and* C *is not hyperelliptic and has a skew Galois line* l. *Then* $G_l \cong \mathfrak{S}_3$ *or* Z_6, *and we have the following*:

(1) $G_l \cong \mathfrak{S}_3$ *if and only if* C *is the Galois closure curve of a smooth cubic* E *with respect to an outer point* P, *where* P *does not lie on the tangent line to* E *at any flex* (cf. [6, Note 2.2]).

(2) $G_l \cong Z_6$ *if and only if* C *can be expressed as a triple Galois covering of a plane conic or* $C = F \cap Q$, *where* F (*resp.*, Q) *is a hypersurface defined by* $X^3 + Y^3 + Z^3 = 0$ (*resp.*, $W^2 + a(X,Y) = 0$ *or* $W^2 + b(X,Y)Z = 0$) *such that* (i) $a(X,Y)$ *and* $b(X,Y)$ *are quadratic and linear forms, respectively*; (ii) $a(X,Y)$ *and* $b(X,Y)$ *have not common factors with* X^3+Y^3, *respectively*; *and* (iii) $a(X,Y)$ *has no multiple factor.*

Finally, we raise some problems:

Problems. (1) Find the estimate of the number of Galois lines for any d.

(2) In Theorem 3.2, it seems that G_2 cannot be isomorphic to $\mathfrak{A}_4$, $\mathfrak{S}_4$ and $\mathfrak{A}_5$. Is it true?

(3) If C is not normal, what can we say about the structure of G_l and the number of Galois lines?

(4) Consider the same problems for non-skew Galois lines.

(5) If two lines l and l' are near in $\mathbb{G}\backslash\Delta$ and $l \neq l'$, then is it true that L_L is not isomorphic to $L_{l'}$ (cf. [12])?

5 Proofs

First, we prove Theorem 2.2. Referring to [4, Chpt. IV, Theorem 3.10] and its proof, we see that there is a finite union of two-dimensional linear subvarieties $\mathcal{L}$ of $\mathbb{P}^3$ with the following properties: If $\pi_Q : \mathbb{P}^3 \cdots \to H$ is the projection from $Q \in \mathbb{P}^3 \setminus (\mathcal{L} \cup C)$ to a hyperplane $H \cong \mathbb{P}^2$ and $X = \pi_Q(C)$ is the image of C, then X is birational to C and has at most nodes for singularities. Similarly as in the proof of [10, Theorem 1′], there is a finite union of lines $\mathcal{L}'$ on H such that if $R \in H \setminus (\mathcal{L}' \cup X)$, then any line l passing through R has the property

(1) $l \cap X$ has normal crossings, or

(2) $l \cap X = \{R_1, \ldots, R_{d-1}\}$, where l and X has normal crossings at R_i ($1 \leq i \leq d-2$) and $i(l, X; R_{d-1}) = 2$.

Then we consider the projection $\pi_R : H \cdots \to l_0$, where l_0 is a line not passing through R. Let $\widetilde{X} \to X$ be the normalization. Then it is easy to see that the discriminant for the covering $\widetilde{\pi_R} : \widetilde{X} \to l_0$ has only simple roots. Therefore, we infer from [8, Lemma 4.4.4] (see also [10]) that $\mathrm{Gal}(k(X)/k(l_0))$ is a full symmetric group. It is clear that the extension $k(C)/k(l_0)$ is isomorphic to $k(X)/k(l_0)$. Hence, taking the line $l = \pi_Q^{-1}(R)$ as the center of a projection, we get $G_l \cong \mathfrak{S}_d$. Thus, we infer readily Theorem 2 2.

The proof of Corollary 2.3 is clear. Indeed, the group corresponding to $k(C)$ is the symmetric group $\mathfrak{S}_{d-1}$, which is primitive, hence it is a maximal subgroup of $\mathfrak{S}_d$. We have a Galois covering $C_l \to \mathbb{P}^1$ of degree $d!$, which branches at $(2g+d-2)$-pieces of points. Then using the Riemann–Hurwitz formula, we obtain Corollary 2.3.

We go to the proof of Theorem 3.2. First, we fix the projection $\pi_l : \mathbb{P}^3 \cdots \to l_0$ inducing the Galois extension $(\pi_l|_C)^* : k(l_0) \hookrightarrow k(C)$. Then clearly $\sigma \in G_l$ induces an automorphism of C such that $\pi_l \cdot \sigma = \pi_l$. Since $C \cap l = \emptyset$, we infer $\sigma(C \cap H) = C \cap H$ for $H \supset l$. Therefore, σ induces an automorphism on $\mathrm{H}^0(C, \mathcal{O}(1))$. Since C is linearly normal and non-degenerate, we have an isomorphism $\mathrm{H}^0(\mathbb{P}^3, \mathcal{O}(1)) \simeq \mathrm{H}^0(C, \mathcal{O}(1))$, hence σ also induces an automorphism on $\mathrm{H}^0(\mathbb{P}^3, \mathcal{O}(1))$. Thus, we obtain the representation of G_l in $PGL(3,k)$. If H is a general plane such that $H \supset l$, then there are non-collinear points of $C \cap H$. Since σ is a projective transformation, we see $\sigma(H) = H$. This implies $\sigma(l) = l$.

By Lemma 3.1, we may assume that the defining equations of l are $X = Y = 0$. From the definition of Galois lines, we infer that σ has the representation

$$\sigma = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ * & * & b & c \\ * & * & d & e \end{pmatrix}.$$

Consider the homomorphism $\gamma : G_l \to PGL(1,k)$ defined by

$$\gamma(\sigma) = \begin{pmatrix} b & c \\ d & e \end{pmatrix}.$$

This is the restriction of σ to the Galois line l. Let G_1 be the kernel of γ. Then each element σ of G_1 has the representation as $b = e$ and $d = c = 0$ in the above matrix. Therefore, we have a homomorphism $\gamma' : G_1 \to k^*$ defined by $\gamma'(\sigma) = b/a$. Clearly this is injective, hence G_1 is a cyclic group.

Next we prove the assertions in Example 3.4. It is easy to see that each curve (1), (2) or (3) is birational to the plane curve defined by

(1) $g_2^m(X,Y) + f_m(X,Y)Z^m + Z^{2m} = 0$,
(2) $(X^{m-1} + Y^{m-1})^2 XY + Z^{2m} = 0$ or
(3) $(X^{m-1} + Y^{m-1})X^m + Z^{2m-1} = 0$,

respectively. From this, we can calculate the genus of C by the genus formula [5, Theorem 9.1]. So it will be sufficient to prove that the curve C in (3) is linearly normal. Note that C is a smooth curve on the quadric cone $Q : W^2 = XZ$. Let

μ be the blowing-up of $\mathbb{P}^3$ at $P = (0, 1, 0, 0)$ and let S be the proper transform of Q. Then S turns out to be a $\mathbb{P}^1$-bundle $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(2)) \to \mathbb{P}^1$ with fiber F and $C_0 = \mu^{-1}(P)$ is the (-2)-curve on S. Let H be a plane not passing through P and put $\mu^*(H \cdot Q) = H_S$, where $H \cdot Q$ is the intersection divisor on Q. Let C' be the proper transform of C and K_S be the canonical divisor on S. Then it is not difficult to see the following:

Claim 1. $H_S \sim C_0 + 2F$, $K_S \sim -2C_0 - 4F$, and $C' \sim (m-1)C_0 + (2m-1)F$.

Putting $D = H \cdot C$ and $D' = \mu^* D$, we get the exact sequence

$$\begin{aligned} 0 &\to \mathrm{H}^0(S, \mathcal{O}(H_S - C')) \to \mathrm{H}^0(S, \mathcal{O}(H_S)) \\ &\to \mathrm{H}^0(C', \mathcal{O}_{C'}(D')) \to \mathrm{H}^1(S, \mathcal{O}(H_S - C')) \to \cdots . \end{aligned}$$

Put $h^i(\Delta) = \dim \mathrm{H}^i(S, \mathcal{O}(\Delta))$ $(i = 0, 1, 2)$ for a divisor Δ on S. It is sufficient to prove $h^1(H_S - C') = 0$ and $h^0(H_S) = 4$. The following assertion is easy to check:

Claim 2. The divisors $(m-2)C_0 + (2m-3)F$ and $3C_0 + 6F$ are 1-connected.

Note that $C' - H_S \sim (m-2)C_0 + (2m-3)F$. Then by Ramanujan's vanishing theorem (cf. [2, p. 131, (8.2)]), we obtain $h^1(H_S - C') = 0$ if $m \geq 3$. In case $m = 2$, since $h^1(\mathcal{O}) = 0$, we can easily prove $h^1(-F) = 0$ from the exact sequence

$$\begin{aligned} 0 &\to \mathrm{H}^0(S, \mathcal{O}(-F)) \to \mathrm{H}^0(S, \mathcal{O}) \to \mathrm{H}^0(F, \mathcal{O}_F) \\ &\to \mathrm{H}^1(S, \mathcal{O}(-F)) \to \mathrm{H}^1(S, \mathcal{O}) \to \cdots . \end{aligned}$$

Since $H_S - K_S \sim 3C_0 + 6F$, we can similarly prove $h^1(H_S) = 0$ by the Serre duality theorem. By the Riemann–Roch theorem,

$$h^0(H_S) = H_S(H_S - K_S)/2 + h^2(\mathcal{O}) - h^1(\mathcal{O}) + 1 + h^1(H_S) - h^0(K_S - H_S),$$

we have $h^0(H_S) = 4$.

Let us proceed with the proof of Theorem 3.5. Let $\mathcal{S}$ be the set of skew Galois lines for C and let $\mathcal{A}$ be the set of subgroups of $\mathrm{Aut}(C)$, which is the automorphism group of C. Then we have a mapping $\varphi : \mathcal{S} \to \mathcal{A}$ defined by $\varphi(l) = G_l$. If $l \neq l'$, then we can find two planes H and H' satisfying $H \supset l$, $H' \supset l'$, $H \cap C \neq H' \cap C$ and $H \cap H' \cap C \neq \emptyset$, hence $G_l \neq G_{l'}$. This means that φ is injective. If $g(C) \geq 2$, then $\mathrm{Aut}(C)$ is a finite group, so the assertion holds. On the other hand, in case $g = 1$, we have the fact $\delta(C) \leq 7$ (cf. [3] or Theorem 4.3 above). Thus, we complete the proof of Theorem 3.5.

Before the proof of Theorem 3.7, we provide some lemmas.

Lemma 5.1. *Suppose that d is a prime number and σ is an automorphism associated with a skew Galois line l. Then at most two eigenvalues of σ coincide with each other, more precisely, the eigenvalues of σ can be written as $\{\alpha, \alpha, \alpha\zeta, \alpha\zeta^i\}$, where $\zeta = \zeta_d$ and $1 \leq i \leq d-1$.*

Proof. As we have seen in the proof of Theorem 3.2, σ has eigenvalues $\alpha, \alpha, \beta, \gamma$. Suppose $\alpha = \beta$. Since σ has order d, $\gamma = \alpha\zeta$ and the Galois line is not contained in the plane H given by $W = 0$ and σ has a fixed point $P \in l \setminus H$. Let $\pi_P : \mathbb{P}^3 \cdots \to H$ be the projection with center P and put $\pi_P(C) = \overline{C}$ and $\overline{P} = \pi_P(l)$. Let $\pi_{\overline{P}} : H \cdots \to l_0$ be the projection with center $\overline{P}$. Then we have $\pi_l = \pi_{\bar{P}}\pi_P$. Since $\deg \pi_P|_C = d$, we have $\deg \pi_{\overline{P}}|_{\overline{C}} = 1$. Thus, $\overline{C}$ must be a line, which contradicts that C is a non-degenerate curve. $\square$

Let Λ be a linear system on C with $\dim \Lambda = r - 1$ and let $\{a_1, \ldots, a_r\}$ be the Λ-gap sequence at P. The Λ-index $\rho_P(C, \Lambda)$ at P is defined to be $\sum_{i=1}^{r}(a_i - (i-1))$. Then we have the following lemma (cf. [5, p. 222]).

Lemma 5.2. *If Λ is a linear system and $D \in \Lambda$, then*

$$\sum_{P \in C} \rho_P(C, \Lambda) = r \deg D + r(r-1)(g-1).$$

Now we prove Theorem 3.7 step by step.

Claim 3. Let δ_m be the maximal number of skew Galois lines which do not meet one another. If $d \geq 5$ is a prime number, then we have

$$\delta_m \leq \frac{2(d-1)(d+3g-3)}{(d-3)(d+g-1)}.$$

Indeed, since d is prime, G_l is a cyclic group of order d. Let s be the number of ramification points of the Galois covering $\pi_l : C \to l_0 \cong \mathbb{P}^1$. Since d is prime, the ramification index of π_l is d, hence we have $2g - 2 = -2d + (d-1)s$. Let Λ be the linear system on C obtained from restricting planes to C. By the same reason as above, the Λ-gap sequence at each ramification point P is $\{0, 1, \lambda, d\}$, where $1 < \lambda < d$. So we have $\rho_P(C, \Lambda) = \lambda + d - 5 \geq d - 3$. Therefore, we infer from Lemma 5.2 that $(d-3) \cdot s \cdot \delta(C) \leq 4d + 12(g-1)$. Since $s = (2d + 2g - 2)/(d-1)$, we get the inequality.

Claim 4. If $d \geq 5$ is a prime number, then skew Galois lines do not meet one another.

In fact, suppose that there exist two Galois lines l_1 and l_2 meeting at P. Then by Lemma 3.1, we may assume that their defining equations are $X = Y = 0$ and $X = Z = 0$, respectively, thus $P = (0, 0, 0, 1)$. Let $\pi_P : \mathbb{P}^3 \cdots \to H$ be the projection with center P, where H is the plane given by $W = 0$, and put $\overline{C} = \pi_P(C)$. Since $\pi_{l_i} = \pi_{\overline{P_i}} \cdot \pi_P$, where $\overline{P_i} = \pi_P(l_i)$, and d is a prime number, we have $\deg \pi_P|_C = 1$ or d. Suppose $\deg \pi_P|_C = d$. Then $\overline{C}$ must be a line, hence C is contained in a plane, which is a contradiction. Thus, we have $\deg \pi_P|_C = 1$. Let σ_i be a generator of G_{l_i} $(i = 1, 2)$.

Claim 5. The point P is a fixed point for σ_1 and σ_2.

Suppose the contrary. Then we may assume $\sigma_1(P) = P_1$ and $\sigma_1(l_2) = l_3$, where $P_1 \neq P$ and $l_3 \neq l_2$. We note that l_3 becomes a Galois line for C satisfying $G_{l_3} = \langle \sigma_1 \sigma_2 \sigma_1^{-1} \rangle$. Let H be the plane spanned by l_1 and l_2. Since l_1 and l_2 are Galois lines, we have $\sigma_1(H) = H$ and $\sigma_2(H) = H$. Next we consider $\sigma_1^2(P), \ldots, \sigma_1^{d-1}(P)$ similarly. By Lemma 5.1, these points are different from one another. Thus, we can find $(d+1)$-pieces of Galois lines on H. We have the Galois covering $\pi_{l_i} : C \to \mathbb{P}^1$, where $l_{i+2} = {\sigma_1}^i(l_2)$ for each i $(i = 1, \dots, d-1)$. Corresponding these coverings, there are s-pieces of planes H_{ij} satisfying $H_{ij} \supset l_i$ and $i(C, H_{ij}, Q) = d$ for some point $Q \in C$, where $s = (2d + 2g - 2)/(d-1)$. One of the planes H_{ij} happens to coincide with H. At any rate, using Lemma 5.2, we get

$$4d + 12(g-1) \geq (d+1)(s-1)(d-3) + (d-3).$$

This implies

$$2g(d^2 - 8d + 3) + (d-1)(d-2)(d-3) \leq 0.$$

Clearly, we have $d \leq 7$. In the cases where $d = 5$ and $d = 7$, we can prove by considering the following cases (i) and (ii) separately:

(i) $\sigma_2(P) \neq P$. We can find $(2d+1)$-pieces of Galois lines by the similar reason. Thus, we get the inequality $4d + 12(g-1) \geq (2d+1)(s-1)(d-3) + (d-3)$, which is a contradiction.

(ii) $\sigma_2(P) = P$. If $\sigma_2(l_1) \neq l_1$, then we have a contradiction by the same reason as above (i). So we assume $\sigma_2(l_1) = l_1$. Then there exist three fixed points of σ_2 on $l_1 \cup l_2$. Suppose that there exists $Q \in C$ satisfying $i(C, H; Q) = d$. Then σ_2 has a fourth fixed point on H, whence $\sigma_2|_H = \mathrm{id}$. This contradicts Lemma 5.1. Therefore, in this case, there exists no such Q. Hence, we get the inequality

$$4d + 12(g-1) \geq (d+1)s(d-3),$$

which is also a contradiction.

Moreover, we will use the following lemma.

Lemma 5.3. *Let Γ be a (possibly singular) irreducible plane curve of degree $d \geq 3$. Suppose that d is a prime number and Γ has two outer Galois points satisfying $G_{P_i} = \langle \sigma_i \rangle$, where σ_i is assumed to be a restriction of a projective transformation. Then Γ is a Fermat curve.*

Proof. Referring to [10, Theorem 4′ and Proposition 5′], we only need to prove that Γ is smooth. Taking suitable coordinates, $\sigma = \sigma_1$ can be expressed as a diagonal matrix diag $[1, 1, \zeta_d]$. Then the defining equation of Γ has the expression as $h_d(X, Y) + Z^d = 0$. Suppose that Γ has singular points. Then they lie on the line $Z = 0$. Let $\mu : \widetilde{\Gamma} \to \Gamma$ be the resolution of the singularities of Γ, and $\pi_P : \mathbb{P}^2 \cdots \to \mathbb{P}^1$ be the projection with center P. Then $\mu \cdot \pi_P : \widetilde{\Gamma} \to \mathbb{P}^1$ becomes a Galois covering of degree d. Since d is a prime number, we infer that Γ has only one singular point Q and the three points P_1, P_2 and Q are collinear. Suppose $\sigma_1(P_2) \neq P_2$. Then we have $(d+1)$-pieces of Galois points P_1, P_2, $\sigma_1(P_2), \dots, \sigma_1^{d-1}(P_2)$, which lie on the line $\overline{P_1P_2}$. Let the covering $\mu \cdot \pi_P$ have s-pieces of ramification points. Then we have $2g - 2 = -2d + s(d-1)$. Let Λ be a linear system of lines on $\mathbb{P}^2$ and $\widetilde{\Lambda}$ be a linear system on $\widetilde{\Gamma}$ obtained from Λ. We infer from [5, §6.7] that

$$\sum_{P \in \widetilde{\Gamma}} \rho_P(\widetilde{\Gamma}, \widetilde{\Lambda}) = 3d + 6(g-1) - (e + d - 3),$$

where e is the multiplicity of Γ at Q. Since $\mu^{-1}(Q)$ is the common ramification point for each covering $\mu \cdot \pi_{P_i}$, we obtain the inequality

$$3d + 6(g-1) - (e + d - 3) \geq (d+1)(s-1)(d-2).$$

This implies

$$2g\left(\frac{(d+1)(d-2)}{d-1} - 3\right) + (d+1)(d-2) - 2d + e + 3 \leq 0.$$

This inequality cannot hold true, hence we get a contradiction. □

Let us continue the proof of Claim 4. By Claim 5, σ_i induces an automorphism of $\overline{C}$, hence $\overline{P}_1 = \pi_P(l_1) = (0,0,1)$ and $\overline{P}_2 = \pi_P(l_2) = (0,1,0)$ become Galois points for $\overline{C}$. By Lemma 5.3, we see that $\overline{C}$ is a smooth curve. Then we get $g(C) = g(\overline{C}) = (d-1)(d-2)/2$ by the genus formula. On the other hand, we have Castelnuovo's bound $g(C) \leq (d-1)(d-3)/4$. Thus, we get $d = 1$, which is a contradiction. Therefore, we have finished the proof of Claim 4.

Let us resume the proof of Theorem 3.7.

Claim 6. If $d \geq 5$ is a prime number, then each skew Galois line meets another.

Indeed, suppose that there exist two skew Galois lines l and l' which do not meet each other. Then we may assume that the defining equations of l (resp., l') are $X = Y = 0$ (resp., $Z = W = 0$). Let $G_l = \langle \sigma \rangle$. We now prove $\sigma(l') = l'$. Suppose the contrary. Then there exist at least $(d+1)$-pieces of skew Galois lines which do not meet one another by Claim 4. By Claim 3, we have the inequality

$$d + 1 \leq \frac{2(d-1)(d+3g-3)}{(d-3)(d+g-1)}.$$

Clearly, this cannot hold true if $d \geq 5$, thus we have a contradiction. Therefore, we have $\sigma(l') = l'$. In addition, since $\sigma(H \cdot l') = H \cdot l'$, we have $\sigma|_{l'} = \mathrm{id}$. Hence, σ has the representation as

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{pmatrix}.$$

By Lemma 5.1, each eigenvalue of the small matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is not one. Therefore, we infer that the fixed points of σ are contained in $l \cup l'$. Thus, σ has no fixed point on C, which is a contradiction.

Combining Claims 4 and 6, we complete the proof of Theorem 3.7.

The proof of Remark 3.8 is clear, so we skip to the proof of Proposition 4.1. The following lemma is easy to prove (cf. [9, Corollary 3.2]).

Lemma 5.4. *Let $p : C \to \mathbb{P}^1$ be a triple covering and $P_1, \ldots, P_r$ $(r \geq 2)$ be the branch points. If $p^{-1}(P_i)$ $(i = 1, \ldots, r)$ consists of one point, that is, the ramification index of $p^{-1}(P_i)$ is 3, then $p : C \to \mathbb{P}^1$ is a cyclic triple covering.*

From this lemma, we see that a Galois line for C is just the intersection of two planes H_j satisfying $i(C, H_j; P_j) = 3$ for some point $P_j \in C$, whence we infer readily Proposition 4.1.

Next we treat the case where $d = 5$. Since $g(\overline{C}) = 2$, the multiplicity sequences of singular points of $\overline{C}$ are $(3,2)$ or $\{(2,2),(2,2)\}$ (cf. [5, Theorem 9.1]). Thus, by taking suitable coordinates, we have the defining equations of $\overline{C}$. First, we take up the case (1). Put $x = X/Z$ and $y = Y/Z$. Let $\mu : C_0 \to \overline{C}$ be the resolution of singularities. Let φ be the rational map associated with a divisor D on C_0 such that φ gives an embedding of C_0 into $\mathbb{P}^3$. Then clearly $\mathcal{L}(D) \supset \langle 1, \mu^*x, \mu^*y\rangle$. Thus, we may assume $D = \mu^*(\overline{C} \cdot l_Z)$, where l_Z is the line $Z = 0$. Hence, we infer readily that $\mathcal{L}(D) = \langle 1, \mu^*x, \mu^*y, \mu^*(1/y)\rangle$, where $xy^3(y - \alpha x) + 1 = 0$. Therefore, we obtain the defining equations of $C \cup l'$. The case (2) is similarly obtained. In fact, we infer the result from $\mathcal{L}(D) = \langle 1, \mu^*x, \mu^*y, \mu^*(1/xy)\rangle$.

Last, we prove Theorem 4.5. Let l be the line defined by $X = Y = 0$.

(1) If $G_l \cong \mathfrak{S}_3$, then $G_l = \langle \sigma, \tau\rangle$, where $\sigma^3 = \tau^2 = 1$ and $\tau\sigma\tau = \sigma^2$. Then we can take coordinates giving the representation

$$\sigma = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \omega & 0 \\ 0 & 0 & 0 & \omega^2 \end{pmatrix} \quad \text{and} \quad \tau_i = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Let π_P be the projection with center $P = (0,0,1,-1)$. Then $\pi_P(C) = \overline{C} \cong C/\langle\tau\rangle$ is a smooth plane cubic. The point $\overline{P} = \pi_P(l)$ does not lie on $\overline{C}$. Hence, the projection $\pi_{\overline{P}}$ with center $\overline{P}$ defines an extension $k(\overline{C})/k(\mathbb{P}^1)$ of degree 3, which is not a Galois extension. Thus, $\overline{P}$ does not lie on each tangent line to $\overline{C}$ at a flex. Therefore, we see that $k(C)$ is isomorphic to the Galois closure of this extension. The converse assertion is clear (cf. [6, Note 2.2]).

(2) If $G_l \cong Z_6$, then $G_l = \langle\sigma\rangle$, where σ has the representation $\operatorname{diag}[1, 1, \alpha, \beta]$, satisfying $\operatorname{ord}(\alpha) \leq \operatorname{ord}(\beta)$.

Claim 7. The following cases cannot occur:

(i) $\operatorname{ord}(\alpha) = \operatorname{ord}(\beta) = 6$,

(ii) $\alpha = 1$ and $\operatorname{ord}(\beta) = 6$.

Suppose the contrary. Then in the case (i), we may assume $(\alpha, \beta) = (\zeta, \zeta)$ or (ζ, ζ^5), where $\zeta = \zeta_6$. Thus, the fixed locus of G_l is $l \cup l'$, where l and l' are lines defined by $X = Y = 0$ and $Z = W = 0$, respectively. Since l is a skew Galois line, we have $C \cap l = \emptyset$, hence the number of fixed points of G_l on C is at most three, because if it is greater than three, then there exists a morphism from C to $\mathbb{P}^1$ with degree ≤ 2. Since C is not hyperelliptic, this is a contradiction. Thus, there exist at most three ramification points of the covering $\pi_l : C \to C/G_l \cong \mathbb{P}^1$, but we see that such a covering cannot exist by the Hurwitz formula. In the case (ii), we may assume $(\alpha, \beta) = (1, \zeta)$. Let π_P be the projection with center $P = (0,0,0,1)$. Then $\pi_P(C) = \overline{C} = C/\langle\sigma\rangle$ is a smooth rational curve on $\mathbb{P}^2$, hence it is a line or a conic. Then C must be in a plane or $\deg \pi_l = 12$, which is a contradiction.

Therefore, we may assume $(\mathrm{ord}(\alpha), \mathrm{ord}(\beta)) = (3,6)$, $(2,3)$ or $(2,6)$. In the second case, let us interchange α and β. Then in the first two cases, we have $\sigma^3 = \mathrm{diag}\,[1,1,1,-1]$, hence $\overline{C} = C/\langle\sigma^3\rangle$ is obtained from the projection with center $P = (0,0,0,1)$. Clearly, $\overline{C}$ is a smooth plane cubic with an automorphism of order 3, i.e., it can be defined by $h(X,Y) + Z^3 = 0$, where h is a form of degree 3. Therefore, C can be expressed as $Q \cap F$, where F is a cubic surface $f(X,Y) + Z^3 = 0$ in $\mathbb{P}^3$ and Q is a quadric surface. Note that Q is unique (cf. [1, p. 118]), i.e., it is invariant under the action of σ. Hence, the defining equation of Q has the expression as in Theorem 4.5. Moreover, since the plane curve $\overline{C}$ has an automorphism of order 3 with fixed points, $h(X,Y)$ can be represented as $X^3 + Y^3$.

Finally, we consider the third case. We have $\sigma^2 = \mathrm{diag}\,[1,1,1,\omega]$. Similarly, we consider the projection π_P. The curve $\overline{C} = \pi_P(C)$ is a conic defined by

$$h(X,Y) + Z^2 = 0,$$

where h is a form of degree 2. Thus, $\pi_P : C \to \overline{C}$ gives a triple Galois covering. Note that if we take $\sigma^3 = \mathrm{diag}\,[1,1,-1,-1]$, then we get a curve $C/\langle\sigma^3\rangle$ of genus 2, however this curve cannot be embedded into the plane.

6 Appendix

In this section, we consider the case where C is not linearly normal. We give examples of not only skew Galois lines but also non-skew ones l for quartic curves C. These examples suggest the complexity of the general case.

If $g = 0$, then the curves can be obtained as follows. Let φ_λ be the embedding $\mathbb{P}^1 \hookrightarrow \mathbb{P}^3$ defined by $\varphi_\lambda(x_0, x_1) = ({x_0}^4,\ {x_0}^3 x_1 + \lambda {x_0}^2 {x_1}^2,\ x_0 {x_1}^3,\ {x_1}^4)$, where $\lambda \in k$. Referring to [4, Chpt. II, Example 7.8.6], we have that the rational curve C with $d = 4$ is projectively equivalent to $C_\lambda = \varphi_\lambda(\mathbb{P}^1)$ for some $\lambda \in k$.

Example 6.1. Suppose that $d = 4$ and C is defined as above and has Galois lines. Since C_λ is rational, there cannot exist V_4-lines. Thus, they are Z_4- or Z_3-lines which are given as follows.

(1) As for Z_4-lines, if $\lambda \neq 0$, then there exist three lines defined by two equations chosen from the following three ones:

$$\begin{cases} X = 0, \\ 3^4 X + 2^3 3^3 \lambda Y + 2^5 3 \lambda^3 Z + 2^4 \lambda^4 W = 0, \\ W = 0. \end{cases}$$

On the contrary, if $\lambda = 0$, then there exists one Z_4-line $X = W = 0$.

(2) As for Z_3-lines, there exist infinitely many Z_3-lines. To be more precise, we study two cases $\lambda \neq 0$ and $\lambda = 0$ separately.

(2-1) $\lambda \neq 0$.

Let α_i $(i = 1, 2)$ be roots of the equation $3\alpha^2 - 2\lambda\alpha + 3\lambda^2 = 0$.

(2-1-a) For each point $\varphi_\lambda(-\alpha, 1) \in C_\lambda$, there exist three Z_3-lines passing through it except at $\alpha = \alpha_i$ $(i = 1, 2)$. The equations of these lines are given by two equations chosen from the following three ones:

$$\begin{cases} X + (\alpha + 3\beta)Y + \beta^2(3\alpha + \beta)Z + \alpha\beta^3 W = 0, \\ X + (\alpha + 3\beta')Y + {\beta'}^2(3\alpha + \beta')Z + \alpha{\beta'}^3 W = 0, \\ Z + \alpha W = 0, \end{cases}$$

where β and β' are two roots of the equation $3\beta^2 + 3(\alpha - \lambda)\beta - \alpha\lambda = 0$.

(2-1-b) At the point $\varphi_\lambda(-\alpha_i, 1) \in C_\lambda$ $(i = 1, 2)$ or $\varphi_\lambda(1, 0)$, there exists one Z_3-line passing through it. The lines for the former two points are given by the equations in (2-1-a), where the first two equations coincide with each other. The line for the latter one is given by $Y + (\lambda^2/3)Z = 0$ and $W = 0$.

(2-2) $\lambda = 0$.

(2-2-a) At the point $\varphi_0(-\alpha, 1)$, where $\alpha \neq 0$, there exist three Z_3-lines passing through it. The equations of these three lines are given by two equations chosen from the following three ones:

$$\begin{cases} X + \alpha Y = 0, \\ X - 2\alpha Y + 2\alpha^3 Z - \alpha^4 W = 0, \\ Z + \alpha W = 0. \end{cases}$$

(2-2-b) At the point $\varphi_0(0, 1)$ or $\varphi_0(1, 0)$, there exists one Z_3-line passing through it. The equations of these lines are given by $X = Z = 0$ and $Y = W = 0$, respectively.

Remark 6.2. Let $\widetilde{C}$ be the normal rational curve of degree 4 in $\mathbb{P}^4$, which is the image of v_4. The curve C_λ above is obtained by the projection π_λ with center $Q_\lambda = (0, -\lambda, 1, 0, 0)$. The curve $\widetilde{C}$ has one Z_4-plane $H_4(\alpha, \beta) \cap H_4(\alpha', \beta')$, where $\alpha\beta' - \alpha'\beta \neq 0$, while it has infinitely many Z_3-planes. For example, we can obtain Z_3-lines in Example 6.1(2) by the projection of the Z_3-planes with center Q_λ. The other Galois lines are similarly obtained.

If $g(C) = 1$, then we use the same notation as in Section 4. As C is linearly normal, we consider only non-skew Galois lines.

Example 6.3. Suppose that C has a non-skew Galois line l. Then $G_l \cong Z_3$ if and only if $J = 0$. In this case, for each point P on C, there exist three Z_3-lines passing through it. Those lines are given as follows. Suppose that C has a non-skew Galois line l and let σ be a generator of G_l. Then put $l \cap C = \{P\}$ and let $\pi_P : \mathbb{P}^3 \cdots \to H$ be the projection with center P, where H is a plane not containing P. Put $\overline{C} = \pi_P(C)$, then it is a smooth plane cubic with $J = 0$, hence it has three skew Galois points Q'_i $(i = 1, 2, 3)$, which do not lie on $\overline{C}$ (cf. [10]) and one of which is $\pi_P(l)$. Therefore, $(\pi_P)^{-1}(Q_i)$ becomes a Z_3-line for C. Let us present the quartic curve C and Z_3-lines more concretely. Since $J = 0$, the Weierstrass form E is $y^2 = 4x^3 - 1$. Let τ be an automorphism of E given by $\tau(z) = \zeta_3 z$. It acts on $k(x, y)$ as $\tau^*(x) = \zeta_3 x$ and $\tau^*(y) = y$, hence the fixed field $k(x, y)^\tau$ is $k(y)$. Putting $D = 3P_0 + Q$, where $Q = (a, b, 1)$ and $b^2 = 4a^3 - 1$, we have $\mathrm{div}(y) + D \geq 0$. It is easy to see that $\mathcal{L}(D) = \langle 1, x, y, (y+b)/(x-a) \rangle$. Therefore, C can be expressed as $W(X - aZ) = Z(Y + bZ)$ and $W(Y - bZ) = 4(X^2 + aXZ + a^2Z^2)$, where $b^2 = 4a^3 - 1$. Then the Z_3-lines passing through $(0,0,0,1)$ are given by $Y = Z = 0$, $X = Y + \sqrt{3}\,Z = 0$ and $X = Y - \sqrt{3}\,Z = 0$.

Remark 6.4. If C is not linearly normal or a Galois line is not skew, then the automorphism associated with the Galois line is not necessarily extended to a projective transformation. For example:

(i) Suppose that C is the curve C_λ $(\lambda \neq 0)$ in Example 6.1 and take the Galois line defined by $W = 0$ and $3^4X + 2^3 3^3 \lambda Y + 2^5 3 \lambda^3 Z + 2^4 \lambda^4 W = 0$. Then it is easy to see that σ cannot be extended.

(ii) Let C be the curve in Example 6.3 and take $\sigma \in G_l$. Then σ is the restriction of the quadratic transformation:

$$(X, Y, Z, W) \mapsto (\omega X(\omega X - aZ), Y(\omega X - aZ), Z(\omega X - aZ), Z(Y + bZ))\,.$$

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