LIST OF PROBLEMS

We follow the notation in the page of open questions. For a survey of Galois points and related topics, see [10]. In what follows, comments and references are given in *italic* between ◊ and ◊.

I Characteristic Zero Case

(A) Curve Case

(1) Galois points and Galois groups for plane curves C

(a) Find Galois points and the Galois groups for singular plane curves.
◊ *for smooth curves, the number of Galois points is at most three (resp. four) if it is outer (resp. inner). The Galois group is cyclic.* [23, 34] ◊

(i) How is the Galois group and how many Galois points do there exist ? Is the maximal number three (resp. four) for outer (resp. inner) Galois point?
◊ *for rational curves* [22, 40], *for elliptic curves* [16], *for a curve of prime degree* [4] ◊

(ii) Study the property of singularity when C has a Galois point. In particular if a singular point P is Galois, how is the singularity? Find the characterization of the curve with the maximal number of Galois points.
◊ *for lower degree or rational curves*, [18, 19, 20, 31, 40] ◊

(iii) Study the above in detail for lower degree curves. For example, for a singular plane quintic and $P \in C$, find Galois points and Galois groups. Do there exist a quintic curve with two Galois points such that the group are cyclic group and Klein's four group respectively ?
◊ [31]) ◊

(b) Each element of G induces a birational transformation on C over $\mathbb{P}^1$. When is it extendable to a projective or birational transformation of $\mathbb{P}^2$?
◊*for rational curves*, [21, 41] ◊

(2) non-Galois points for plane curves C
◊ *If P is a general point for C, then the Galois group at P is a full symmetric group of degree d (resp. $d-1$) if it is outer (resp. inner).* [3, 34] ◊

It seems interesting to study these subjects above in the remaining cases, i.e., the case where P is neither general nor Galois.

(a) Find the Galois group and the Galois closure curve for C at P, in particular the genus $g(P)$ of the Galois closure curve.
◊ for quartic curves, [23, 33] ◊

(b) Fixing a smooth curve C, determine the set
$\mathcal{G}(C) = \{ \ g(P) \mid P \in C \ \}$.
In particular, determine $\mathcal{G}(F_d)$, where F_d is the Fermat curve of degree d.
Fixing d, determine the set

$\mathcal{G}(d) = \{\ g(P) \mid C$ is a smooth curve of degree d and $P \in C\}$.
$\Diamond$ for quartic curves, [33] $\Diamond$

(c) Find the number of points at which the Galois group is isomorphic to a fixed group, for example, in the case where the fixed group is an alternating group. Does there exist a curve with an infinitely many points at which the Galois group is isomorphic to an alternating group?
$\Diamond$ [1] $\Diamond$

(d) Study the above in detail for special curves. For example, let F_d be the Fermat curve of degree $d \geq 5$. Suppose that $d-1$ is not a prime number. Then, how is the Galois group at the flex ?
$\Diamond$ [24, 34] $\Diamond$

(e) There exists a smooth projective surface S and a morphism $\varphi : S \longrightarrow C$ satisfying that $\varphi^{-1}(Q) \cong C_Q$, where Q is a general point of C. Study the structure of S and the singular fiber of φ.
$\Diamond$ [37] $\Diamond$

(f) If a point P is close to another one Q, then are the Galois closure curves C_P and C_Q not isomorphic to each other ?
$\Diamond$ [27, 37] $\Diamond$

(g) There exists a smooth projective threefold M and a morphism $\varphi : M \longrightarrow \mathbb{P}^2$ satisfying that $\varphi^{-1}(Q) \cong C_Q$, where Q is a general point of $\mathbb{P}^2 \setminus C$. Study the structure of M and the singular fiber of φ.
$\Diamond$ [28] $\Diamond$

(3) space curve

Study the same subjects as in the plane curves. In particular, study the following.

(a) curves in $\mathbb{P}^3$
- (i) Find Galois lines ℓ for C in two cases $C \cap \ell = \emptyset$ and $C \cap \ell \neq \emptyset$. In particular, find the arrangement of Galois lines.
$\Diamond$ *for a quartic curve*, [5] $\Diamond$
- (ii) Some space curve with a Galois line is obtained as a Galois closure curve of a plane curve. Characterize such a space curve.
$\Diamond$ [38] $\Diamond$
- (iii) Suppose C is a (d_1, d_2)-complete intersection. Then, find the Galois lines and Galois groups. Does there exist any relation with the hypersurfaces of degree d_i $(i = 1, 2)$?

(b) curves in $\mathbb{P}^n$
- (i) Find Galois subspaces for C, or in particular, the arrangement of Galois subspaces.
- (ii) Study the Galois subspaces and Galois group for rational normal curve C. In particular, find the Grassmannian of the subspaces.
- (iii) Study the Galois group when the subspace is not Galois.

(iv) Some space curve with a Galois subspace is obtained as a Galois closure curve of a plane curve. Characterize such a space curve.

(B) Surface Case

Study the same subjects as in the plane curves. In particular study the following.

(1) Galois points and Galois groups
  (a) Find the Galois points and Galois groups for singular hypersurfaces, in particular for normal hypersurfaces.
  ♢ *for smooth hypersurface* [35, 36], *for normal quartic surfaces* [30] ♢
  (b) Characterize a hypersurface with the maximal number of inner Galois points. Does it has a special property?
  ♢ *quartic surface has a special property,* [35] ♢
(2) non-Galois points
  (a) When P is not a Galois point, consider the Galois closure surface.
  ♢ *for the definition of minimal splitting variety, see* [32] ♢
  ♢ *In many cases the Galois closure surfaces* S_P *are of general type. So, we have an interest in the case where* S_P *are not of general type.* [29] ♢

(C) Higher dimensional Case

Study the Galois embeddings for a smooth projective varieties.
♢ [39, 42] ♢

(1) Study the Grassmannian $\{ W \in \mathbb{G}(N-n-1, N) \mid G_W \cong S_d \}$. In particular, is it true that the codimension of the complement of the set is at least two ?
  ♢ [25] ♢
(2) Suppose that $\dim Lin(V) = 0$, W is close to W' (in the Grassmannian) and $W \neq W'$. Let L_W be the Galois closure of the extension determined by the projection. Then is it true that L_W is not isomorphic to $L_{W'}$?
(3) For an embedding (V, D) find the structure of Galois group G_W for each $W \in \mathbb{G}(N-n-1, N)$. In particular, let A be a principally polarized abelian variety with the polarization Θ. Then study the structure of A and the Galois group when $(A, 3\Theta)$ gives a Galois embedding.
(4) Find all the Galois subspaces for one Galois embedding, or find the rule of arrangements of Galois subspaces. Suppose the Kodaira dimension of V is non-negative. Then, is it true that the number of Galois subspaces is finitely many?
  ♢ [38] ♢
(5) Consider the similar subject in the case where $f(V) \cap W \neq \emptyset$.
(6) For abelian, hyperelliptic, $K3$ and Enriques surfaces consider the Galois embeddings, the Galois subspaces and the Galois groups.
  ♢ [39] ♢

(D) Related Topics

(1) In case the curve C is defined over $\mathbb{Q}$, can we develop the similar research? How is the Galois group at rational points? If the "degree of a point" becomes large, then how does the Galois group at it become? How is the reduction to $\mathbb{Z}/p\mathbb{Z}$?
$\diamondsuit$ Theorem 3 in [34] $\diamondsuit$

(2) In case an irreducible curve C is in a ruled surface Σ. Consider the projection $\pi : \Sigma \longrightarrow \Delta$, where Δ is a curve and suppose $\pi(C)$ is not a point. Restricting π to C, we have an extension of function fields $k(C)/\pi^* k(\Delta)$. Do the similar research for this case as in the plane curve case. If C is smooth and $\pi|_C$ is a Galois cover, then is the Galois group cyclic?

(3) Study the same subject as above in a weighted projective space, i.e., consider a weighted projective variety, projection and function field . . .

II Positive Characteristic Case

(A) Throughout the case where $p > 0$: Generalize results obtained in $p = 0$ to arbitrary characteristic $p \geq 0$.

$\diamondsuit$ *We have not checked yet whether a lot of results on Galois points obtained in $p = 0$ hold also in $p > 0$. Therefore, almost all problems in* I *are open also in $p > 0$.* $\diamondsuit$

(B) Curve Case

(1) Galois points and Galois groups for plane curves C

(a) Determine the number of outer (resp. inner) Galois points for smooth plane curves when $d \equiv 0$ (resp. $d \equiv 1$) modulo p.
$\diamondsuit$ *The Galois group has been already classified in* [8], *and the number of outer (resp. inner) Galois points are settled if $d \not\equiv 0$ (resp. $d \not\equiv 1$) or $d = p$ (resp. $d = p + 1$), when C is smooth.* [6, 7, 8, 10, 13] $\diamondsuit$

(b) Find Galois points and Galois groups for singular curves of lower genera.
$\diamondsuit$ Even if the genus (of the smooth model) is 0 or 1, this problem is still open in $p > 0$. $\diamondsuit$

(c) Find new examples of plane curves having many Galois points. Determine the Galois groups for such curves.
$\diamondsuit$ *Hermitian curves are smooth curves having many inner and outer Galois points.* [13]. *Curves having infinitely many Galois points are found in $p > 0$ and classified.* [9, 11] $\diamondsuit$

(d) Find Galois points and Galois groups for special curves in $p > 0$. For example, when C is a strange curve, or a non-reflexive curve.
$\diamondsuit$ *See* [17] *for definition.* $\diamondsuit$

(e) Study the relations between Galois points and rational points when C is defined over a finite field. Do there exist any relations between Galois points and rational points?
$\Diamond$ [13] $\Diamond$

(2) non-Galois points for plane curves C
$\Diamond$ *In $p > 0$, the Galois group G_P at a general point $P \in C$ (resp. $P \notin C$) is not always a symmetric group, even if C is smooth.* [14] $\Diamond$

(a) Is the Galois group G_P at a general point $P \in C$ (resp. $P \notin C$) a symmetric group when C is reflexive?
$\Diamond$ *cf.* [25, 26] $\Diamond$

(b) What group appears as the Galois group G_P at a general point $P \in C$ (resp. $P \notin C$) when C is non-reflexive? What is the genus $g(P)$ of C_P?

(C) Higher dimensional Case

(1) Find Galois points for smooth hypersurfaces.
$\Diamond$ *Few results seem to be obtained for higher-dimensional case in $p > 0$.* $\Diamond$

(2) Find Galois points for a Fermat hypersurface of degree $p^e + 1$. Can we give a new characterization of this Fermat hypersurface by using Galois points?
$\Diamond$ *A Fermat hypersurface of degree $p^e + 1$ seems to have many inner and outer Galois points.* [13]. $\Diamond$

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